Model-Free Boundaries of Option Time Value and Early Exercise Premium

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Abstract

Based on option put-call parity relation, we derive model-free boundary conditions of option time value and option early exercise premium with presence of cash dividends on the underlying stock. The paper produces four main results. (1) For European options, the difference in time value between a call option and a put option is the discount interest earned on the exercise price less the present value of cash dividends to be paid before the option expiration. (2) For American options, the difference in time value between a call option and a put option is bounded between the negative amount of the present value of dividends and the discount interest earned on the exercise price. (3) The early exercise premium of an American put option is bounded between zero and the discount interest earned on the exercise price. (4) The early exercise premium of an American call option is bounded between zero and the present value of cash dividends to be paid before the option expiration. Based on results (3) and (4), the difference in early exercise premiums between an American put option and a call option is bounded below by the negative amount of the present value of cash dividends and bounded above by the discount interest earned on the exercise price. We numerically test these results in the Black-Scholes and binomial tree models. This paper contributes to the finance literature. It extends the understanding of option time value and option early exercise premium, provides boundary conditions for option-pricing model calibration, and indirectly helps enhance market efficiency and make optimal option early exercise decisions especially when the underlying stock pays cash dividends.

I. Introduction

Option time value is known as speculative value of an option. It measures the value of the option buyer's ability to exercise the option at any time prior to the option's expiration, instead of exercising the option immediately. Early exercise premium of an American option is defined as the difference between the option premium of an American option and that of an otherwise identical European option. It measures the value of the ability to optimally early exercise an option instead of holding the option until its maturity. Clearly, option early exercise premium is closely related to option time value. Option time value captures the ability to wait to exercise the option later while the option early exercise premium captures the ability to exercise the option early and not to wait until later. Option early exercise premium is embedded in the option time value if the option is an American option.

Mathematically, option time value is defined as the difference between an option premium and the option's intrinsic value. An option's intrinsic value is readily available for a given underlying stock price and the option's exercise price, which are both directly observable. Option pricing is essentially time value pricing. Option's time value reflects the speculative value, and the value of the "option to wait" for the exercise decision.

Even though the importance of option time value has been well recognized, current finance literature contains little in-depth discussion on option time value. Topics related to option time value are included in early studies in options, *e.g.*, Black-Scholes (1973), Merton (1973), Smith (1976), Cox and Ross (1976), and Cox and Rubinstein (1985), and essentially all derivative textbooks, *e.g.*, Hull (2006), Jarrow and Turnbull (2000), and Chance and Brooks (2007), among others. Papers and textbooks in the current finance literature do not go beyond the most basic properties of option time value, *e.g.*, monotonicity of time value of American options in various

underlying option components, such as remaining time to maturity and stock return volatility. There is much insight on option time value that has been left uncovered.

It is generally more difficult to value American options because of their early exercise feature. Closed-form solutions to American options, especially put options, are particularly challenging without many additional assumptions about the underlying stock return process. Binominal tree and Monte Carlo simulation methods are commonly used to numerically solve for an American option price. However, market efficiency requires that option prices generated from mathematical models satisfy no-arbitrage parity and boundary conditions, which are crucial in model calibration.

Some research explores analytic approximations of the value of American options (Cox *et al*. 1979, and Geske and Johnson, 1984). Zivney (1991) reports that previously developed analytic models for American options do not completely capture the market value of early exercise. He suggests using empirical method to estimate option's early exercise premium. Zivney (1991) develops a regression model using S&P 100 options to show that the average value of early exercise is about 3.5% for call options and 10% for put options. This result is consistent with that in Engstrom and Norden (2000)'s study, which estimates that the value of the early exercise premium is around 9% of the average market price for the put options using Swedish equity options data.

Unlike previous studies, which develop empirical models to estimate options early exercise premiums, this paper derives arbitrage-driven model-free boundary conditions for both option time value and early exercise premium. Our analysis is based on only option put-call parity and the assumption that markets are perfectly competitive and efficient. It is totally independent of any assumptions of stock return distributions or processes. Consequently, the results from this

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paper apply in any efficient market. Furthermore, we relax the assumption of no cash dividend distributions from the underlying stock, and specifically consider the impact of cash dividends on option time value and early exercise premium.

We produce four major results and a few derivatives of these conclusions. There are two results related to option time value: (1) For European options, the difference in time value between a call option and a put option is the discount interest earned on the exercise price less the present value of cash dividends to be paid before the option expiration. (2) For American options, the difference in time value between a call option and a put option is bounded between the negative amount of the present value of dividends and the discount interest earned on the exercise price. There are two results related to option early exercise premium: (3) The early exercise premium of an American put option is bounded between zero and the discount interest earned on the exercise price. (4) The early exercise premium of an American call option is bounded between zero and the present value of cash dividends to be paid before option expiration. Based on these results, we further derive boundary conditions on option premiums, time value, and early exercise premium. We test these boundary conditions using option prices calculated from binomial tree and Black-Scholes models. The numerical results are consistent with what we have derived.

This paper contributes to the literature by providing model-free boundary conditions on option time value and early exercise premium with presence of cash dividends in the underlying stock. To our best knowledge, the results presented in this paper have not been documented in previous research. Our conclusions extend the understanding of option time value and early exercise premium, provide boundary conditions for option model calibration, and indirectly help enhance market efficiency and make optimal option early exercise decisions especially when the underlying stock pays dividends.

The paper is organized as follows. The next section presents notations used in this paper. The third section demonstrates the relations of time value of European and American options. The fourth section examines the bounds of early exercise premium of American put and call options. The final section concludes the paper.

II. Notations

To begin our discussion of the boundaries of option time value and early exercise premium, we define the following standard notation:

C^E and P^E are European call and put option premiums, respectively;

C^A and P^A are American call and put option premiums, respectively;

S_0 is the current price of the underlying stock;

X is the options' exercise price;

r is the annualized continuously compounded risk free rate of interest;

T is the remaining time to options' maturity;

We further define $IV(C)$ and $IV(P)$ as the intrinsic value of a call and a put option, respectively, $TV(C)$ and $TV(P)$ as the time value of a call and a put option, respectively, and $EEP(C^A)$ and $EEP(P^A)$ as the early exercise premium of an American call and an American put option, respectively. By definition, the following equations always hold:

$$IV(C) = max(S_0 - X, 0); \quad IV(P) = max(X - S_0, 0);$$

$$TV(C) = C - IV(C); \quad TV(P) = P - IV(P);$$

$$EEP(C^A) = C^A - C^E; \quad EEP(P^A) = P^A - P^E.$$

Clearly, when an option is out-of-the-money, its intrinsic value is zero and the whole option premium is the option time value. Because the expressions derived in this paper are based

on the put-call parity relation and are free of any particular model specification, *e.g.*, the Black-Scholes option-pricing model or the binomial tree option pricing model, the conclusions we derive hold for all options in general in an efficient market.

III. Boundary conditions of option time value

We derive relations of option time value based on the put-call parity. In our derivation, we find that time value of a call option and that of a put option are closely related. Consequently, we examine the difference in time value of a call option and its corresponding put option. In appendices A and B to this paper, we mathematically prove the following properties related to option time value:

(1) The difference in time value of European call and put options is the amount of discount interest earned on the exercise price less the present value of all the cash dividends to be paid before option expiration, *i.e.*,

$$TV(C^E) - TV(P^E) = X(1 - e^{-rT}) - PV(div).$$

(2) The difference in time value of American call and put options is bounded below of the negative amount of the present value of cash dividends to be paid before option expiration and bounded above the amount of discount interest earned on the exercise price, *i.e.,*

$$-PV(div) \leq TV(C^A) - TV(P^A) \leq X(1 - e^{-rT})$$

The following two sections, III.a and III.b, discuss above two results in more detail.

III.a. Time value of European options

This section focuses on time value of European options. It starts with a discussion of the relation of option time value when the underlying stock pays dividends. A numerical example

developed using binomial tree models is presented to demonstrate the relation. It follows with a discussion of the relation of option time value in the absence of dividends. The relation is illustrated using a numerical example developed in the Black-Scholes framework.

The relation of time value of European call and put options sheds light on characteristics of option time value. The difference in the time value between a European call and put option is $X(1 - e^{-rT}) - PV(div)$, which is *insensitive* to the identity of the underlying stock, levels of stock price (or moneyness of the options), and stock return volatility changes.

This conclusion is a very strong result on European options. It holds across different stocks, options with different time to maturity and moneyness, and different stock return volatilities. The difference in time value between a deep in-the-money call option and a correspondingly deep out-of-the-money put option is the same as the difference in time value between an at-the-money call and put options, as long as these options have the same exercise price and time to maturity.

It may be counter-intuitive that the difference in time value between call and put options is *insensitive* to changes in stock return volatility. It is because high time value in a call option induced by high stock return volatility is offset by the high time value in the corresponding put option induced by the same high stock return volatility. Consequently, the difference in time value between the call and put option remains constant despite of changes in the underlying stock return volatility. We further address this point later in this section.

Table 1 provides a numerical example of the relation of time value of European call and put options calculated from a 10-step binomial tree. Parameters used in the binomial tree model are: X = \$100, T = 0.5 years, r = 10%, and σ = 0.60. Dividends are paid at 1% of the stock price at the end of each of the 10 time steps. Column 1 of the table contains the underlying stock prices that

range from $10 to $200. Columns 2 to 5 report the premium and time value of call and put options. Column 6 computes the difference of time value of European call and put options (difference of columns 4 and 5). Column 7 is a constant which equals to the discount interest earned on the strike price. The last column reports the present value of dividends. Column 6 is equal to the difference between columns 7 and 8.

Figure 1(a) displays the relation of option time value graphically. The solid-dotted line represents the difference of the time value of call (solid line) and put (dotted line) options, measured by $X(1 - e^{-rT}) - PV(div)$.

Above discussion considers the impact of dividends in the relations of options time value. In the absence of dividends, time value of a European call option is always higher than that of an otherwise identical put option by the discount interest earned on the exercise price, $i.e.$, $TV(C^E) - TV(P^E) = X(1 - e^{-rT})$. This relation immediately delivers two direct conclusions on options and their time value.

First, in the absence of dividends, call options are more valuable alive than dead. The time value of an in-the-money call option has a floor of $X(1 - e^{-rT})$ because the corresponding out-of-the-money put option time value is equal to the entire option premium, which is nonnegative. The time value of an out-of-the-money call option is the entire call option premium, which is nonnegative as well. Therefore, call option time value is always nonnegative before the option's maturity.

Second, early exercise of an American put option may be optimal. Because the time value of a European put option is the time value of an otherwise identical call option less the discount interest earned on the exercise price, the time value of a European style put option can become negative when the put option is sufficiently deep in-the-money. Negative put option time value

occurs when a small time value of a deep out-of-the-money call option is less than the discount interest earned on the exercise price. Consider an extreme situation where the underlying stock price drops to zero, the time value of an out-of-the-money call option becomes zero[1]. Based on the put-call parity, the value of the European put option becomes Xe^{-rT}, which is less than the value of the payoff of the option exercise price X if the put option is exercised immediately. When the time value of a European put option is negative, a European put option with a shorter time to maturity is worth more than an otherwise identical option with a longer time to maturity.

We further demonstrate above inferences using a numerical example in the Black-Scholes framework. The Black-Scholes option-pricing model has been widely accepted as valuation tools of European options. We set parameters in the Black-Scholes model as the same as those used in the binomial tree model except that there are no dividend payments: $X = \$100$, $T = 0.5$ years, $r =$ 10%, and $\sigma = 0.60$. The Black-Scholes option premium and option time value are presented in Table 2. Column 1 of Table 2 contains the underlying stock prices that range from $10 to $200, corresponding to in-the-money and out-of-the-money call/put options. Columns 2 and 3 compute the Black-Scholes call and put option premiums, respectively. Column 4 records time value of the call options. Column 5 reports the time value of put options. Columns 6 and 7 report the difference in time value of European call and put options, which is the difference between columns 4 and 5. Note that the last column is a constant which equals to the discount interest earned on the options' exercise price, $X(1 - e^{-rT})$.

Figure 2 provides a graphical description of results in Table 2. It plots the time value of European call and put options and their differences based on computed Black-Scholes call and put option premiums. The solid line represents the time value of European call options. The dashed

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[1] Note that call option price is bounded above by the underlying stock price. When the underlying stock price goes to zero, its call option value converges to zero.

line represents the time value of the corresponding put options. The dash-dotted line is the difference between time value of the call and put options, measured by $X(1 - e^{-rT})$. Note that the dash-dotted line is flat and insensitive to changes in the underlying stock price. It is interesting to point out that the slope of call option time value line matches exactly that of the put option time value line at every point in the curves, *i.e.*, the two lines are perfectly parallel to each other at any given stock price level.

Figure 2 also graphically demonstrates the three inferences we conclude above: (1) The time value of a European call option is non-negative and bounded below by the discount interest earned on its exercise price when the call option is in-the-money. (2) Deep in-the-money European put options may acquire negative time value. (3) The difference in time value of a European call option and put option is constant across different underlying stock prices.

We have demonstrated that the difference in time value between European call and put options is insensitive to the underlying stock price and stock return volatility. However, time value of both call and put options clearly varies across different levels of underlying stock price and stock return volatility. We examine sensitivities of option time value with respect to the underlying stock price and return volatility below.

The time value of a European call option and that of a European put option change at exactly the same rate with respect to the underlying stock price changes because the difference in time value of a European call and put options is independent of the underlying stock price.

When the underlying stock price is greater than the options' exercise price, *i.e.*, $S_0 \geq X$, the put option is out-of-the-money and the put option premium consists entirely of its time value, *i.e.*, $P^E = TV(P^E)$. The time value of call option and that of a put option share a common negative instantaneous slope of $\frac{\partial TV(C^E)}{\partial S_0} = \frac{\partial TV(P^E)}{\partial S_0} = \frac{\partial P^E}{\partial S_0} = delta(P^E) < 0$, which is quantitatively the

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same as the delta of the put option. In the Black-Scholes framework, the slope is $N(d_1) - 1$, where $N(.)$ is the cumulative distribution function of the standard normal distribution and d_1 follows the Black-Scholes definition.

Conversely, when the underlying stock price is less than the options' exercise price, *i.e.*, $S_0 < X$, the call option is out-of-the-money and the call option premium consists of entirely its time value, *i.e.*, $C^E = TV(C^E)$. The time value of a call option and that of a put option share a common negative instantaneous slope of $\frac{\partial TV(P^E)}{\partial S_0} = \frac{\partial TV(C^E)}{\partial S_0} = \frac{\partial C^E}{\partial S_0} = delta(C^E) > 0$, which is quantitatively the same as the delta of the call option. In the Black-Scholes framework, the slope equals $N(d_1)$.

The above analysis also demonstrates that time value of both call and put options is the highest when the options are exactly at-the-money.

Figure 2 presents time value of European call and put options as a function of underlying non-dividend paying stock. The solid line represents time value of a call option and the dashed line represents time value of a corresponding put option. Note that when the underlying stock price is less than the option exercise price of \$100, the slopes of both time value curves for call and put options are positive and equal to $N(d_1)$ in the Black-Scholes world. When the underlying stock price is greater than the option exercise price of \$100, the slopes of both time value curves for call and put options are negative and equal to $N(d_1) - 1$ in the Black-Scholes framework. Option time value peaks at the options' exercise price, where both options are exactly at-the-money.

In the absence of dividends, the time value of European call and put options also shares the same sensitivity with respect to changes in underlying stock return volatility, which is quantitatively the same as option vega. In the Black-Scholes framework, the call and put option vega is $S_0 n(d_1)\sqrt{T}$:

$$\frac{\partial TV(C^E)}{\partial \sigma} = \frac{\partial TV(P^E)}{\partial \sigma} = \frac{\partial P^E}{\partial \sigma} = \frac{\partial C^E}{\partial \sigma} = vega = S_0 n(d_1)\sqrt{T} > 0$$

where σ is the standard deviation of the annualized returns of the underlying stock and $n(.)$ is the probability density function of the standard normal distribution. Because an option vega is always positive, the time value of the call and put options increases monotonically as the return volatility of the underlying stock increases. Stock return volatility does not directly affect option intrinsic value. Its impact on option time value translates to its impact on option value. The higher the volatility, the higher is the option time value, which leads to higher option value. This conclusion coincides with intuition.

III.b. Time value of American options

Time value of American options is no less than that of its European counterpart. Because of the early exercise feature of American options, computation of option premium, and consequently, time value of American options, is more difficult. We demonstrate in Appendix B that the difference in time value between an American call option and a put option must satisfy the following boundary conditions:

$$-PV(div) \leq TV(C^A) - TV(P^A) \leq X(1 - e^{-rT})$$

It may seem counter-intuitive that the width of the boundary interval, which is the difference between upper and lower bounds, is insensitive to either the underlying stock return volatility or option moneyness, even though the time value itself is a function of both the underlying stock return volatility and option moneyness. Specifically, we may have two American call and put option pairs on two different underlying securities. As long as these options share the same time to maturity, cash dividends, and exercise price, the differences in time value are subject to the same upper and lower bounds.

A numerical example is presented in Table 3 to illustrate the boundaries of time value difference of American call and put options. Parameters used in this table are the same as those used in Table 1. Column 1 of the table contains the underlying stock prices that range from $10 to $200. Columns 2 to 5 report the premium and time value of call and put options. Column 6 computes the difference of time value of American call and put options. The lower bound of the time value difference of American call and put options, as shown in Column 7, equals to the negative amount of the present value of cash dividends to be paid before option expiration. The upper bound, as shown in last column, is the amount of discount interest earned on the exercise price. Note that Column 6 is bounded by columns 7 and 8.

This result can also be observed in Figure 1(b), where the difference of option time value (solid-dotted line) is within the strip confined by the lower bound (dash-dotted line), measured by $-PV(div)$, and the upper bound (dashed line), measured by $X(1 - e^{-rT})$.

It is worth pointing out that in the absence of dividends, the lower bound of difference in time value between American call and put options is zero. It indicates that, in the absence of dividends, time value of an American call option dominates that of an American put option. An investor who early exercises an American call option on a non-dividend paying stock loses positive time value. A call option, when held instead of the stock, can be viewed as an insurance policy against the stock price falling below the exercise price. If the investor early exercises the call option and plans to hold the stock till the expiration of the call option, the insurance coverage from the option vanishes. The investor may suffer a loss if the stock price falls below exercise price on or before option expiration date. The result suggests that a rational investor should not early exercise an American call option on a non-dividend paying stock, and that in the absence of dividends, the early exercise premium of an American call option is zero, $i.e.$, $EEP(C^A) = 0$.

Consequently, an American call option and an otherwise identical European call option must carry exactly the same option premium if the underlying stock does not pay cash dividends prior to the options' expiration, *i.e.*, $C^A = C^E$.

IV. Boundary conditions of option early exercise premium

So far in this paper, we have discussed the relations of option time value. In this section, a detailed analysis of the boundary conditions of option early exercise premium (EEP) yields interesting results below.

Results in Appendix C state that the early exercise premium of the American put, $EEP(P^A)$, which is the difference between an American put option and a European put option, is bounded below zero and bounded above by the amount of discount interest earned on the exercise price, *i.e.*,

$$0 \leq EEP(P^A) = P^A - P^E = TV(P^A) - TV(P^E) \leq X(1 - e^{-rT})$$

The above inequalities are important because they provide an upper bound of the early exercise premium of American put options, which is $X(1 - e^{-rT})$. With this upper bound, we state that the value of an American put option is no greater than the sum of the value of an otherwise identical European put option and the discount interest earned on the exercise price, *i.e.*, $P^A \leq P^E + X(1 - e^{-rT})$. This expression is valid whether or not there are future cash dividends prior to the option expiration.

Table 4 provides an example of the EEP of American put option calculated from a 10-step binomial tree. Parameters used in this table are the same as those used in Table 1. Column 1 of the table contains the underlying stock prices that range from $10 to $200. American and European put option premium are reported in columns 2 and 3, respectively. Column 4 computes the EEP of

American put option, which equals to the difference in option premium of an American put option and a corresponding European put option. The last column documents the upper bound of the EEP of American put options, which equals to the amount of discount interest earned on the exercise price. Note that Column 4 stays below the upper bound defined by the last column.

Figure 3 (a) illustrates the results in Table 4 visually. The solid line is the EEP of American put option. The dashed line is the upper bound of the EEP, measured by $X(1 - e^{-rT})$. It can be observed that the EEP of American put option is located between zero (lower bound) and its upper bound.

The non-negative EEP of American put option suggests that early exercise of American put options may be optimal, A put option, when held with a stock, can be viewed as providing insurance against the underlying stock price falling below the exercise price. However, when the stock price is sufficiently low, an investor may rationally forgo this insurance coverage and realize the gain immediately because the underlying stock price cannot be negative even if the investor waits till the expiration of the option.

In Appendix D, we examine the early exercise premium of an American call option, which is the difference in time value between an American call option and an otherwise identical European call option. Appendix D derives its upper bound to be the present value of cash dividends paid before the option's expiration:

$$0 \leq TV(C^A) - TV(C^E) = C^A - C^E = EEP(C^A) \leq PV(div)$$

The above inequalities states that the value of an American call option is no greater than the sum of the value of an otherwise identical European call option and the present value of dividends paid before the option expiration.

We further illustrate above results using a numerical example in Table 5. Parameters used

in Table 5 are the same as those used in Table 1. Column 1 of the table contains the underlying

stock prices that range from $10 to $200. American and European call option premium are

reported columns 2 and 3, respectively. Column 4 computes the EEP of American call option. The

last column documents the upper bound of the EEP of American call options, which equals to the

present value of all future cash dividends. Note that Column 4 stays below the upper bound

defined by the last column.

Figure 3(b) is a graphical description of results in Table 5. There are two lines in Figure

3(b). The solid line is the EEP of American call option. The dashed line is the upper bound of the

EEP, measured by $PV(div)$. Note that EEP of American call option is located within its lower

bound (zero) and its upper bound.

Transformations of the inequalities in Appendices C and D further reveal lower and upper

bounds of the difference in early exercise premium of American put and call options:

$$-PV(div) \leq EEP(P^A) - EEP(C^A) \leq X(1 - e^{-rT})$$

Note that without dividends, the early exercise premium of an American call option is zero,

while the early exercise premium of an American put option is bounded between zero and the

discount interest earned on the exercise price. When the underlying stock makes cash dividend

payments, the early exercise premium of the American call option acquires positive value.

However, the difference in early exercise premium between American put and call options is

bounded below by the negative amount of the present value of the cash dividends to be paid before

option expiration and bounded above by the discount interest earned on the exercise price. Table 6

provides a summary of all boundary conditions we derive in this paper. The results are separated

into two categories based on whether or not the underlying stock makes cash dividend payments

before option expiration.

V. Conclusions

This paper examines the time value and early exercise premium of call and put options and their properties. Our derivations are based on the put-call parity relationship of European and American options. Under the assumptions of perfectly competitive and efficient market, our results are totally arbitrage-driven and model-free, which should apply in any options market in the absence of "free lunch".

We provide simple expressions for the differences in time value between call options and put options. For European options, the difference in time value between a call option and a put option is the discount interest earned on the exercise price less the present value of cash dividends to be paid before option expiration. For American options, the difference in time value between a call option and a put option is bounded between zero and the discount interest earned on the exercise price when there are no dividends, and bounded between the negative amount of the present value of dividends and the same upper bound when there are cash dividends paid on the underlying stock before option expiration.

We further derive three additional conclusions on the early exercise premium of American options: (1) The early exercise premium of an American put option is bounded above by the discount interest earned on the exercise price. (2) The early exercise premium of an American call option is bounded above by the present value of cash dividends on the underlying stock to be paid before option expiration. (3) The difference in the early exercise premium between the American put and call options is bounded below by the negative amount of the present value of cash dividends to be paid before option expiration and bounded above by the discount interest earned on the exercise price. Based on these properties, we present additional inferences on the time value and option premium of American options. We numerically test these results in the Black-Scholes

and binomial tree models. The testing results are consistent with the conclusions we have derived.

We contribute to the finance literature by using a model-free approach and clearly extending the understanding of option time value and option early exercise premium, which are crucial in option pricing. To our best knowledge, previous research has not documented results that are similar to the ones we report in this paper. Our paper directly provides additional boundary conditions for option pricing model calibration and indirectly helps enhance market efficiency and make optimal early exercise decisions especially when the underlying stock pays cash dividends.

Appendix A:

In this appendix, we prove that the difference in time value of European call and put options is the amount of discount interest earned on the exercise price less the present value of cash dividends to be paid before option expiration, *i.e.*,

$$TV(C^E) - TV(P^E) = X(1 - e^{-rT}) - PV(div)$$

Proof: European options satisfy European put-call parity:

$$C^E + Xe^{-rT} = P^E + S_0$$

If the underlying stock pays dividends before the option expiration, the dividend-adjusted European put-call parity becomes:

$$C^E + Xe^{-rT} = P^E + S_0 - PV(div)$$

Rearrange the above equation, we have:

$$C^E - S_0 + X - P^E = X(1 - e^{-rT}) - PV(div)$$

when $S_0 \geq X$:

$$TV(C^E) - TV(P^E)$$

$$= [C^E - max(S_0 - X, 0)] - [P^E - max(X - S_0, 0)]$$

$$= C^E - S_0 + X - P^E$$

$$= X(1 - e^{-rT}) - PV(div)$$

when $S_0 < X$:

$$TV(C^E) - TV(P^E)$$

$$= [C^E - max(S_0 - X, 0)] - [P^E - max(X - S_0, 0)]$$

$$= C^E - P^E + X - S_0$$

$$= X(1 - e^{-rT}) - PV(div)$$

In both cases, $TV(C^E) - TV(P^E) = X(1 - e^{-rT}) - PV(div)$. We conclude that the difference in

time value of a European call option and that of a European put option is the discount interest earned on the exercise price less the present value of cash dividends to be paid before option expiration.

Note that in the absence of cash dividends, the above relation simplifies to:

$$TV(C^E) - TV(P^E) = X(1 - e^{-rT}).$$

Appendix B:

In this appendix, we prove that the difference in time value of American call and put options is bounded between the negative amount of the present value of cash dividends to be paid before option expiration and the amount of discount interest earned on the exercise price, *i.e.*,

$$-PV(div) \leq TV(C^A) - TV(P^A) \leq X(1 - e^{-rT})$$

Proof: American options on non-dividend paying stocks satisfy the American put-call parity:

$$C^A + Xe^{-rT} \leq P^A + S_0 \leq C^A + X$$

If the underlying stock pays dividends before the option expiration, the put-call parity relationship for American options satisfies the following inequalities [Hull (2006): p219, Guo and Su (2006)]:

$$C^A + Xe^{-rT} \leq P^A + S_0 \leq C^A + X + PV(div)$$

From the first inequality

$$C^A + Xe^{-rT} \leq P^A + S_0$$

We have

$$C^A - S_0 + X - P^A \leq X(1 - e^{-rT})$$

From the second inequality

$$P^A + S_0 \leq C^A + X + PV(div)$$

We have

$$-PV(div) \leq C^A - S_0 + X - P^A$$

Combining the two inequalities, we obtain:

$$-PV(div) \leq C^A - S_0 + X - P^A \leq X(1 - e^{-rT})$$

From Appendix A, we infer:

$$TV(C^A) - TV(P^A) = C^A - S_0 + X - P^A$$

Consequently, we conclude:

$$-PV(div) \leq TV(C^A) - TV(P^A) \leq X(1 - e^{-rT})$$

The difference in time value of American call and put options is bounded below by the negative

amount of the present value of cash dividends and bounded above by the amount of discount

interest earned on the exercise price.

Note that in the absence of cash dividends, the above relation simplifies to:

$$0 \leq TV(C^A) - TV(P^A) \leq X(1 - e^{-rT}).$$

Appendix C:

In this appendix, we prove that the early exercise premium of an American put option is no

greater than the amount of discount interest earned on the exercise price, *i.e.*

$$0 \leq EEP(P^A) = P^A - P^E \leq X(1 - e^{-rT}).$$

Proof: Use proof by contradiction. Suppose $P^A > P^E + X(1 - e^{-rT})$, then we buy

$P^E + X(1 - e^{-rT})$ and sell P^A to capture a positive initial cash flow.

If the buyer of the American put option does not early exercise P^A, we will hold our

portfolio $P^E + X(1 - e^{-rT}) - P^A$ to options' maturity to yield $X(e^{rT} - 1)$.

Note that P^A can be early exercised at anytime. If the buyer of the American put option

early exercises P^A at time t, where $t \in [0, T]$, we will have a portfolio of

$$P_t^E + X(1 - e^{-rT})e^{rt} - (X - S_t) = P_t^E + X(e^{rt} - 1) + S_t - Xe^{-r(T-t)} > P_t^E + S_t - Xe^{-r(T-t)}$$

Based on put-call parity: $C_t^E + Xe^{-r(T-t)} = P_t^E + S_t - PV_t(div)$, above expression can be rearranged as

$$P_t^E + S_t - Xe^{-r(T-t)} = C_t^E + PV_t(div) \geq 0$$

The above set of transactions produces an arbitrage in the absence of transaction costs. If markets are perfectly competitive and efficient, such arbitrage opportunities should not exist, which implies:

$$0 \leq EEP(P^A) \leq X(1 - e^{-rT})$$

We conclude that the early exercise premium of an American put option is no greater than the amount of discount interest earned on the exercise price.

Appendix D:

In this appendix, we prove that the early exercise premium of an American call option is no greater than the present value of all future cash dividends paid by the underlying stock before the option expiration, *i.e.* $TV(C^A) - TV(C^E) = C^A - C^E = EEP(C^A) \leq PV(div)$.

Proof: If the underlying stock does not pay any dividends before the option expiration, then the above statement clearly holds:

$$TV(C^A) - TV(C^E) = C^A - C^E = EEP(C^A) = 0 \leq PV(div) = 0.$$

Without loss of generality, we assume that the underlying stock pays only one cash dividend in the amount of *div* at time t, where $t \in [0, T]$.

Use proof by contradiction. Suppose $C^A > C^E + PV(div)$, then we buy $C^E + PV(div)$ and sell C^A to capture a positive initial cash flow.

Note that the only potential optimal early exercise instant occurs right before the underlying stock goes ex-dividend at time t.

At time t, if the buyer of the American call option does not early exercise C^A at time t, we will hold our portfolio $C^E + div - C^A$ to maturity to yield $FV(div)$.

If the buyer of the American call option early exercises C^A at time t, we will have a portfolio of $C^E + div - \left(S_t^{cum-div} - X\right) = C^E - \left(S_t^{cum-div} - div - X\right) = C^E - \left(S_t^{ex-div} - X\right)$

$$C^E - \left(S_t^{ex-div} - X\right) \geq C^E - IV(C^E) = TV(C^E) \geq 0$$

The above set of transactions produces an arbitrage in the absence of transaction costs. If markets are perfectly competitive and efficient, such arbitrage opportunities should not exist, which implies:

$$0 \leq EEP(C^A) \leq PV(div)$$

We conclude that the early exercise premium of an American call option is no greater than the present value of cash dividends paid before option expiration.

Note that based on Appendices C and D, the difference in early exercise premium between an American put option and an otherwise identical call option is bounded below by the negative amount of the present value of cash dividends and bounded above by the discount interest earned on the exercise price.

$$-PV(div) \leq EEP(P^A) - EEP(C^A) \leq X(1 - e^{-rT})$$

References

Black, F. and M. S. Scholes, 1973. The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81, 637-654.

Blomeyer, E. C., and H. Johnson, 1988. An Empirical Examination of The Pricing of American Put Options, *Journal of Financial and Quantitative Analysis* 23, 13-22.

Chance, D. and R. Brooks, 2007. An Introduction to Derivatives and Risk Management, seventh edition, Thomson, South-Western.

Cox, J, and S. Ross, 1976. A Survey of Some New Results in Financial Option Pricing Theory, *Journal of Finance* 31, 383-402.

Cox, J., Ross, S., Rubinstein, M., 1979. Option Pricing: A Simplified Approach, *Journal of Financial Economics* 13, 145–166.

Cox, J. and M. Rubinstein, 1985. Options Markets, Prentice-Hall.

Engstrom, M., and L. Norden, 2000. The Early Exercise Premium in American Put Option Prices, *Journal Of Multinational Financial Management* 10, 461-479.

Geske, R., and Johnson, H., 1984. The American put valued analytically, *Journal of Finance* 39, 1511–1524.

Guo, W. and T. Su, 2006. Option Put-Call Parity Relations When the Underlying Security Pays Dividends, *International Journal of Business and Economics* 3, 225-230.

Hull, J., 2006. Options, Futures, and Other Derivatives, sixth edition, Prentice Hall.

Jarrow, R. and S. Turnbull, 2000. Derivatives Securities, second edition, South-Western College Publishing.

Merton, R.C., 1973. Theory of Rational Option Pricing, *Bell Journal of Economics and Management Science* 4, 141-183.

Smith Jr., C.W., 1976. Option Pricing: A Review, *Journal of Financial Economics* 3, 3-51.

Zivney, T. L., 1991. The Value of Early Exercise in Option Prices: An Empirical Investigation, *Journal of Financial and Quantitative Analysis* 26, 129-138.

Table 1. Difference in time value of European options, implemented using the binomial tree model.

The difference in time value of European call and put options is the discount interest earned on the options' exercise price less the present value of dividends paid by the underlying stock before the option expiration, *i.e.*,

$$TV(C^E) - TV(P^E) = X(1 - e^{-rT}) - PV(div)$$

      This table provides an example of the relations of time value of European call and put options calculated from a 10-step binomial tree. Parameters used in the binomial tree model are: X = $100, T = 0.5 years, r = 10%, and σ = 0.60. Dividends are paid at 1% of the stock price at the end of each of the 10 time steps. Column 1 of the table contains the underlying stock prices that range from $10 to $200. Columns 2 to 5 report the premium and time value of call and put options. Column 6 computes the difference of time value of European call and put options (difference of columns 4 and 5). Column 7 is a constant which equals to the discount interest earned on the strike price. The last column reports the present value of dividends. Column 6 is equal to the difference between columns 7 and 8, subject to rounding errors. All measurements are quoted in dollars ($).

Stock price	Call option premium	Put option premium	Call option time value	Put option time value	Difference of time value	Discount interest	PV(div)
10.00	0.00	86.08	0.00	-3.92	3.92	4.88	0.96
20.00	0.00	77.04	0.00	-2.96	2.96	4.88	1.91
30.00	0.00	67.99	0.00	-2.01	2.01	4.88	2.87
40.00	0.07	59.02	0.07	-0.98	1.05	4.88	3.82
50.00	0.33	50.23	0.33	0.23	0.10	4.88	4.78
60.00	1.25	42.11	1.25	2.11	-0.86	4.88	5.74
70.00	2.98	34.80	2.98	4.80	-1.82	4.88	6.69
80.00	5.43	28.20	5.43	8.20	-2.77	4.88	7.65
90.00	9.05	22.78	9.05	12.78	-3.73	4.88	8.61
100.00	13.66	18.35	13.66	18.35	-4.68	4.88	9.56
110.00	18.27	13.91	8.27	13.91	-5.64	4.88	10.52
120.00	24.87	11.46	4.87	11.46	-6.60	4.88	11.47
130.00	31.58	9.14	1.58	9.14	-7.55	4.88	12.43
140.00	38.30	6.81	-1.70	6.81	-8.51	4.88	13.39
150.00	45.79	5.25	-4.21	5.25	-9.47	4.88	14.34
160.00	53.93	4.35	-6.07	4.35	-10.42	4.88	15.30
170.00	62.07	3.45	-7.93	3.45	-11.38	4.88	16.26
180.00	70.21	2.55	-9.79	2.55	-12.33	4.88	17.21
190.00	78.42	1.71	-11.58	1.71	-13.29	4.88	18.17
200.00	87.22	1.47	-12.78	1.47	-14.25	4.88	19.12

Table 2. Relations of European option time value in the Black-Scholes framework in the absence of dividends.

In the absence of dividends, the difference in time value of European call and put options is the discount interest earned on the options' exercise price, *i.e.*,

$$TV(C^E) - TV(P^E) = X(1 - e^{-rT})$$

This table provides a numerical example of the relations of Black-Scholes option time value. Parameters in the Black-Scholes option-pricing model are the same as those used in the binomial tree as in Table 1 except that there are no dividend payments: $X = \$100$, $T = 0.5$ years, $r = 10\%$, and $\sigma = 0.60$. Column 1 of the table contains the underlying stock prices that range from \$10 to \$200. Columns 2 and 3 compute the Black-Scholes call and put option premiums, respectively. Columns 4 and 5 document time value of the call and put options, respectively. The last column reports the difference in time value of European call and put options, which is the difference between columns 4 and 5. Note that the last column is a constant which equals to the discount interest earned on the options' exercise price. All measurements are quoted in dollars (\$).

Stock price	Call option premium	Put option premium	Call option time value	Put option time value	Difference of time value
10.00	0.00	85.12	0.00	-4.88	4.88
20.00	0.00	75.12	0.00	-4.88	4.88
30.00	0.02	65.15	0.02	-4.85	4.88
40.00	0.19	55.32	0.19	-4.68	4.88
50.00	0.81	45.94	0.81	-4.06	4.88
60.00	2.23	37.35	2.23	-2.65	4.88
70.00	4.70	29.82	4.70	-0.18	4.88
80.00	8.347	23.46	8.34	3.46	4.88
90.00	13.12	18.24	13.12	8.24	4.88
100.00	18.94	14.06	18.94	14.06	4.88
110.00	25.65	10.78	15.65	10.78	4.88
120.00	33.10	8.22	13.10	8.22	4.88
130.00	41.13	6.26	11.13	6.26	4.88
140.00	49.63	4.75	9.63	4.75	4.88
150.00	58.49	3.61	8.49	3.61	4.88
160.00	67.62	2.74	7.62	2.74	4.88
170.00	76.96	2.09	6.96	2.09	4.88
180.00	86.47	1.59	6.47	1.59	4.88
190.00	96.09	1.21	6.09	1.21	4.88
200.00	105.80	0.93	5.80	0.93	4.88

Table 3. Difference of time value of American options, implemented using the binomial tree model.

The difference in time value of American call and put options is bounded between the negative amount of the present value of cash dividends to be paid before option expiration and the amount of discount interest earned on the exercise price, $i.e.$,

$$-PV(div) \leq TV(C^A) - TV(P^A) \leq X(1 - e^{-rT})$$

This table provides an example of the relations of time value of American call and put options calculated from a 10-step binomial tree. Parameters used in the binomial tree model are: $X =$ $100, $T = 0.5$ years, $r = 10\%$, and $\sigma = 0.60$. Dividends are paid at 1% of the stock price at the end of each of the 10 time steps. Column 1 of the table contains the underlying stock prices that range from $10 to $200. Columns 2 to 5 report the premium and time value of call and put options. Column 6 computes the difference of time value of American call and put options (difference of columns 4 and 5). The lower bound of the time value difference of American call and put options, as shown in Column 7, equals to the negative amount of the present value of cash dividends to be paid before option expiration. The upper bound, as shown in last column, is the amount of discount interest earned on the exercise price. Note that Column 6 is bounded by columns 7 and 8. All measurements are quoted in dollars ($).

Stock price	Call option premium	Put option premium	Call option time value	Put option time value	Difference of time value	-PV(div)	Discount interest
10.00	0.00	90.00	0.00	0.00	0.00	-0.96	4.88
20.00	0.00	80.00	0.00	0.00	0.00	-1.91	4.88
30.00	0.00	70.00	0.00	0.00	0.00	-2.87	4.88
40.00	0.08	60.02	0.08	0.02	0.06	-3.82	4.88
50.00	0.38	50.70	0.38	0.70	-0.31	-4.78	4.88
60.00	1.32	42.33	1.32	2.33	-1.00	-5.74	4.88
70.00	3.26	34.90	3.26	4.90	-1.64	-6.69	4.88
80.00	5.81	28.25	5.81	8.25	-2.44	-7.65	4.88
90.00	9.93	22.80	9.93	12.80	-2.87	-8.61	4.88
100.00	14.82	18.36	14.82	18.36	-3.54	-9.56	4.88
110.00	20.34	13.92	10.34	13.92	-3.57	-10.52	4.88
120.00	27.54	11.47	7.54	11.47	-3.93	-11.47	4.88
130.00	35.08	9.14	5.08	9.14	-4.06	-12.43	4.88
140.00	43.15	6.81	3.15	6.81	-3.66	-13.39	4.88
150.00	52.18	5.25	2.18	5.25	-3.07	-14.34	4.88
160.00	61.47	4.35	1.47	4.35	-2.88	-15.30	4.88
170.00	70.91	3.45	0.91	3.45	-2.54	-16.26	4.88
180.00	80.50	2.55	0.50	2.55	-2.05	-17.21	4.88
190.00	90.50	1.71	0.50	1.71	-1.21	-18.17	4.88
200.00	100.50	1.47	0.50	1.47	-0.97	-19.12	4.88

Table 4. Early exercise premium (EEP) of American put option, implemented using the binomial tree model.

The early exercise premium (EEP) of an American put option is no greater than the amount of discount interest earned on the exercise price, *i.e.*,

$$0 \leq EEP(P^A) \leq X(1 - e^{-rT})$$

This table provides an example of the EEP of American put option calculated from a 10-step binomial tree. Parameters used in the binomial tree model are: $X = \$100$, $T = 0.5$ years, $r = 10\%$, and $\sigma = 0.60$. Dividends are paid at 1% of the stock price at the end of each of the 10 time steps. Column 1 of the table contains the underlying stock prices that range from $10 to $200. American and European put option premium are reported in columns 2 and 3, respectively. Column 4 computes the EEP of American put option (difference of columns 2 and 3). The last column documents the upper bound of the EEP of American put options, which equals to the amount of discount interest earned on the exercise price. Note that Column 4 stays below the upper bound defined by the last column. All measurements are quoted in dollars ($).

Stock price	American put option premium	European put option premium	$EEP(P^A)$	Discount interest
10.00	90.00	86.08	3.92	4.88
20.00	80.00	77.04	2.96	4.88
30.00	70.00	67.99	2.01	4.88
40.00	60.02	59.02	1.00	4.88
50.00	50.70	50.23	0.47	4.88
60.00	42.33	42.11	0.22	4.88
70.00	34.90	34.80	0.10	4.88
80.00	28.25	28.20	0.05	4.88
90.00	22.80	22.78	0.02	4.88
100.00	18.36	18.35	0.01	4.88
110.00	13.92	13.91	0.01	4.88
120.00	11.47	11.46	0.00	4.88
130.00	9.14	9.14	0.00	4.88
140.00	6.81	6.81	0.00	4.88
150.00	5.25	5.25	0.00	4.88
160.00	4.35	4.35	0.00	4.88
170.00	3.45	3.45	0.00	4.88
180.00	2.55	2.55	0.00	4.88
190.00	1.71	1.71	0.00	4.88
200.00	1.47	1.47	0.00	4.88

Table 5. Early exercise premium (EEP) of American call option, implemented using the binomial tree model.

The early exercise premium (EEP) of an American call option is no greater than the present value of all future cash dividends paid by the underlying stock before the option expiration, *i.e.*

$$0 \leq EEP(C^A) \leq PV(div)$$

This table provides an example of the EEP of American call option calculated from a 10-step binomial tree. Parameters used in the binomial tree model are: $X = \$100$, $T = 0.5$ years, $r = 10\%$, and $\sigma = 0.60$. Dividends are paid at 1% of the stock price at the end of each of the 10 time steps. Column 1 of the table contains the underlying stock prices that range from $10 to $200. American and European call option premium are reported in columns 2 and 3, respectively. Column 4 computes the EEP of American call option (difference of columns 2 and 3). The last column documents the upper bound of the EEP of American call options, which equals to the present value of all future cash dividends. Note that Column 4 stays below the upper bound defined by the last column. All measurements are quoted in dollars ($).

Stock price	American call option premium	European call option premium	$EEP(C^A)$	PV(div)
10.00	0.00	0.00	0.00	0.96
20.00	0.00	0.00	0.00	1.91
30.00	0.00	0.00	0.00	2.87
40.00	0.08	0.07	0.01	3.82
50.00	0.38	0.33	0.06	4.78
60.00	1.32	1.25	0.08	5.74
70.00	3.26	2.98	0.28	6.69
80.00	5.81	5.43	0.39	7.65
90.00	9.93	9.05	0.88	8.61
100.00	14.82	13.66	1.16	9.56
110.00	20.34	18.27	2.07	10.52
120.00	27.54	24.87	2.67	11.47
130.00	35.08	31.58	3.50	12.43
140.00	43.15	38.30	4.85	13.39
150.00	52.18	45.79	6.40	14.34
160.00	61.47	53.93	7.54	15.30
170.00	70.91	62.07	8.84	16.26
180.00	80.50	70.21	10.28	17.21
190.00	90.50	78.42	12.08	18.17
200.00	100.50	87.22	13.28	19.12

Table 6. Summary of boundary conditions on option time value and option early exercise premium.

This table provides a summary of all upper and lower bounds derived in this paper. The results are separated into two categories based on whether or not the underlying stock makes cash dividend payments before option expiration.

	Without Dividends	**With Dividends**

Differences in time value (TV):

European options:

$$TV(C^E) - TV(P^E) = X(1 - e^{-rT}) \qquad\qquad TV(C^E) - TV(P^E) = X(1 - e^{-rT}) - PV(div)$$

American options:

$$0 \le TV(C^A) - TV(P^A) \le X(1 - e^{-rT}) \qquad -PV(div) \le TV(C^A) - TV(P^A) \le X(1 - e^{-rT})$$

Differences in early exercise premium (EEP):

$$0 \le EEP(P^A) - EEP(C^A) \le X(1 - e^{-rT}) \qquad -PV(div) \le EEP(P^A) - EEP(C^A) \le X(1 - e^{-rT})$$

Bounds on early exercise premium (EEP), time value (TV), and option premium:

Put options

$$0 \le EEP(P^A) \le X(1 - e^{-rT})$$
$$0 \le TV(P^A) - TV(P^E) \le X(1 - e^{-rT})$$
$$0 \le P^A - P^E \le X(1 - e^{-rT})$$

Call options

$EEP(C^A) = 0$	$0 \le EEP(C^A) \le PV(div)$
$TV(C^A) - TV(C^E) = 0$	$0 \le TV(C^A) - TV(C^E) \le PV(div)$
$C^A - C^E = 0$	$0 \le C^A - C^E \le PV(div)$

Figure 1(a). Relations of time value of European options; (b). Relations of time value of American options, implemented using the binomial tree model.
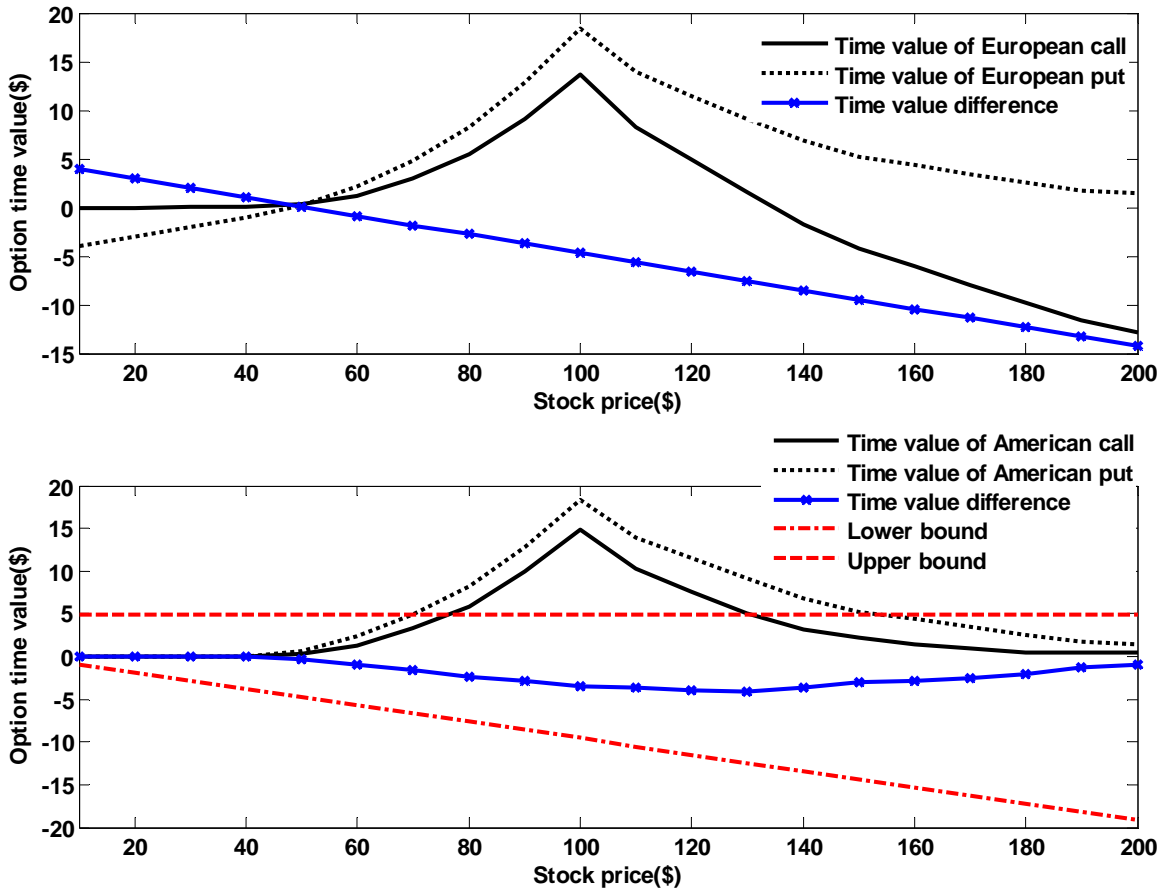

These two figures, 1(a) and 1(b), graphically present relations of option time value. Parameters used in these figures are the same as those used in the binomial tree model in Table 1.

Figure 1(a) plots the European call and put options and the difference in option time value. There are three lines in this figure. The solid-dotted line is the difference of the time value of call (solid line) and put (dotted line) options, measured by $X(1 - e^{-rT}) - PV(div)$.

Figure 1(b) plots the American call and put options and the boundaries of the time value difference of call and put options. The solid line is the time value of American call option. The dotted line is the time value of American put option. It can be observed directly from the figure that the time value difference of American put and call options (solid-dotted line) is within the strip confined by the lower bound (dash-dotted line), measured by $-PV(div)$, and the upper bound (dashed line), measured by $X(1 - e^{-rT})$.

Figure 2. Relations of option time value in the Black-Scholes framework in the absence of dividends.
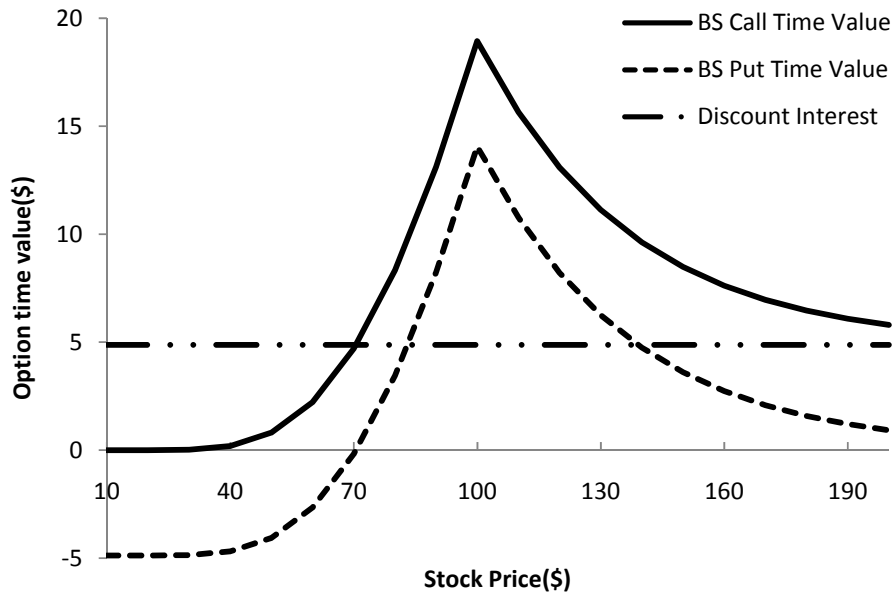

In the absence of dividends, the difference in time value of European call and put options is the discount interest earned on the option's exercise price, *i.e.*,

$$TV(C^E) - TV(P^E) = X(1 - e^{-rT})$$

This figure plots the Black-Scholes time value of European call and put options and difference in time value. Parameters in the Black-Scholes option-pricing model are the same as those used in the binomial tree model in Table 1, except that there are no dividend payments: $X = \$100$, $T = 0.5$ years, $r = 10\%$, and $\sigma = 0.60$. There are three lines in this figure. The solid line represents the Black-Scholes time value of a European call option. The dashed line represents the Black-Scholes time value of the corresponding put option. The flat dash-dotted line is the difference between the time value of call and put options, measured by $X(1 - e^{-rT})$.

Figure 3(a). Early exercise premium (EEP) of American put option; (b). Early exercise premium (EEP) of American call option, implemented using the binomial tree model.
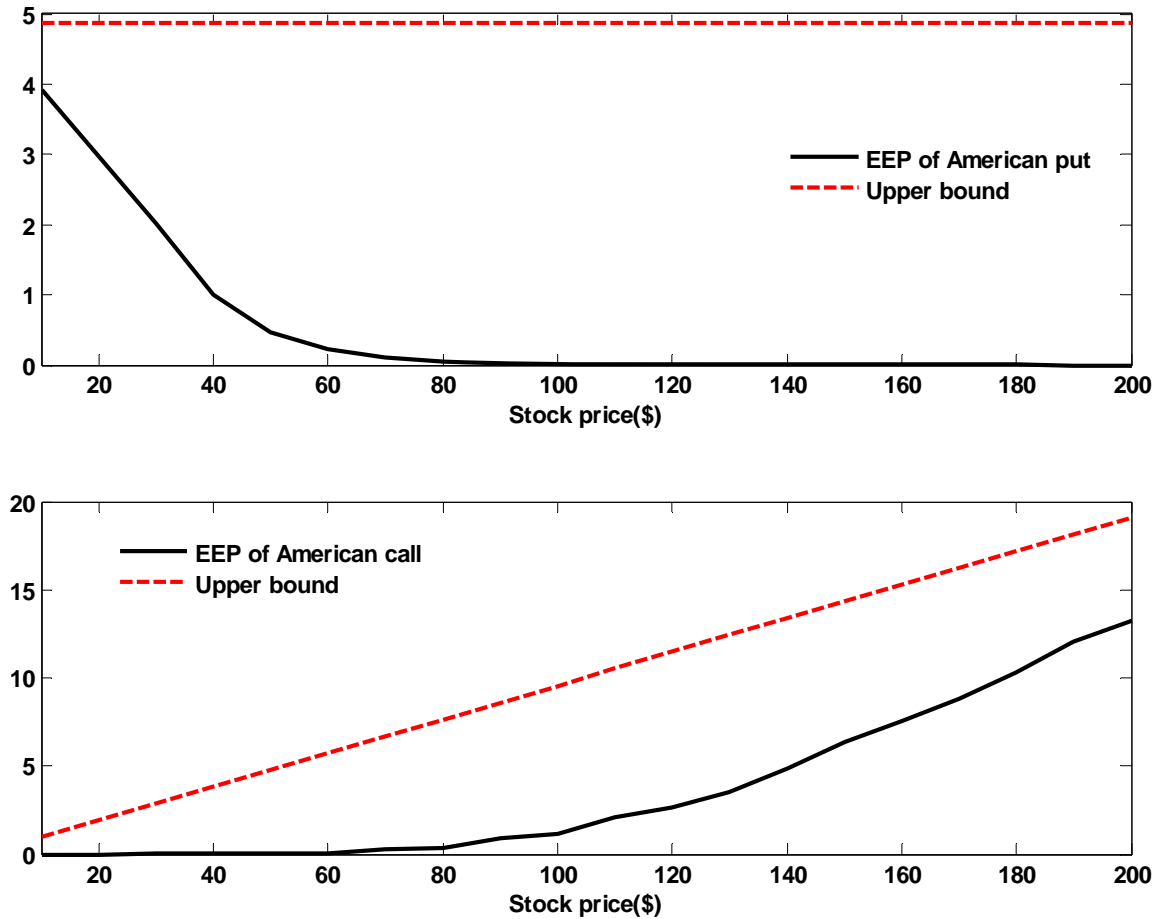
These two figures, 3(a) and 3(b), display boundary conditions of early exercise premium (EEP) of American call and put options. Parameters used in these two figures are the same as those used in the binomial tree model in Table 1.

There are two lines in Figure 3(a). The solid line is the EEP of an American put option. The dashed line is the upper bound of the EEP, measured by $X(1 - e^{-rT})$. It can be observed that the EEP of American put option is bounded between zero (lower bound) and dashed line (upper bound).

There are two lines in Figure 3(b). The solid line is the EEP of American call option. The dashed line is the upper bound of the EEP, measured by $PV(div)$. Note that EEP of American call option is bounded between zero (lower bound) and dashed line (upper bound).