THE VALUE OF MORTGAGE PREPAYMENT AND DEFAULT OPTIONS

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We use an implicit alternating direction numerical procedure to estimate the value of a fixed-rate mortgage (FRM) with embedded default and prepayment options. The value of FRMs depends on interest rates, the house value, and mortgage maturity. Our numerical results suggest that the joint option value of prepayment and default is considerably high, even at loan origination. We extend the model to include prepayment penalties in FRM valuation. © 2009 Wiley Periodicals, Inc. Jrl Fut Mark 29:840–861, 2009

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INTRODUCTION

We apply an option-based approach to estimate the value of a fixed-rate mortgage (FRM). Specifically, we consider embedded prepayment and default options in FRMs. Using numerical procedures, we demonstrate how risk factors impact the value of an FRM and discuss pricing, prepayment, and default issues in mortgage markets. These risk factors include interest rate volatility, house value return volatility, the loan-to-value (LTV) ratio, and prepayment penalties. Our model incorporates an implicit finite difference method with a modified low-upper (LU) triangular decomposition algorithm as in SIAM (1999).

The valuation of mortgage debt requires modelling the optimal prepayment and default behavior of the borrower. The view of prepayment and default as options held by the borrower has emerged over the last three decades. The decision to terminate the mortgage either by prepayment or default is considered as endogenous. The option-based approach to mortgage valuation leads to the solving of a partial differential equation (PDE) with two state variables, the interest rate and the house price. Prior contributions in this line of research are Kau, Keenan, Muller, and Epperson (1992, 1993), Kau and Keenan (1995), and Deng, Quigley, and Van Order (2000), among others. These two-factor models recognize the interest rate and the underlying house value as the main sources of risk and determinants of mortgage termination, either by default or prepayment. Two-factor models have the advantage of being more general, but they raise the problem of solving a nonlinear PDE with two state variables. More recently, Kariya, Pliska, and Ushiyama (2002) and Downing, Stanton, and Wallace (2005) focus on prepayment risk taking into account the underlying house value.

An alternative to the option-based approach is the econometric approach, where default and prepayment are not determined endogenously, but modelled using empirical data (see Kau, Keenan, & Smurov, 2004; Schwartz & Torous, 1989; Titman & Torous, 1989). However, microeconomic factors that drive the mortgage origination market can change dramatically. As Kalotay, Yang, and Fabozzi point out: "Econometric prepayment models have consistently failed during fast prepayment periods. Although mortgage-backed security analysts continually update their prepayment models, their models will always lag behind shifts in the microeconomic structure of the mortgage market" (Kalotay, Yang, & Fabozzi, 2000, p. 954).

The option-based approach is derived from the contingent claims analysis of Cox, Ingersoll, and Ross (1985a), which models derivative securities based on a PDE. The PDE modelling of mortgage valuation with options to prepay or default has two state variables: the interest rate and the value of the underlying house. Because a closed-form solution to the PDE is often unfeasible, numerical methods are generally used. Due to the free-boundary problem posed by the early-exercise property of the American-style options to prepay and default, explicit finite difference methods are one way to effectively estimate the value of mortgage-related derivatives (Azevedo-Pereira, Newton, & Paxson, 2000; Kau et al., 1992, 1993; Newton, Azevedo-Pereira, & Paxson, 2002).

The implicit alternating direction (IAD) method we propose in this study differs from the previous work in several aspects. First, we propose an implicit finite difference method to solve the PDE. Unlike the explicit finite difference method, the implicit method does not suffer from instability due to restrictions on time steps. The steps of the state variables—interest rate and house value—can be small and the time step large without instability problems. The IAD method is stable for both small and large time steps and the residual error term is smaller, e.g., in the order of $O(\Delta t, \Delta r^2, \Delta H^2)$ compared to $O(\Delta t, \Delta r^2, \Delta H^2)$, where Δ is the grid size of the explicit method (Wilmott, 1998).

Second, we use a modified LU triangular decomposition algorithm (SIAM, 1999) to improve the computational efficiency. The LU decomposition algorithm is a well-developed mathematical method to solve sparse linear systems of equations, relieving memory space and reducing rounding errors. We further modify the backward substitution procedure of LU decomposition to impose the early-exercise constraint of an American-style option.

Finally, we extend the model to include transaction costs to the borrower of exercising the prepayment or default options. The transaction cost is assumed to be a constant portion of the unpaid balance on prepayment termination or a constant portion of the value of the underlying house on default termination, as in Stanton (1995) and Kalotay, Williams, and Fabozzi (2004).

We examine the theoretical value of an FRM. In our numerical approximation, we first consider a callable but non-defaultable FRM, and then the value of a callable and defaultable FRM. The numerical illustration with benchmark parameters similar to Kau et al. (1993) indicates that the value of embedded prepayment and default options in a conventional mortgage is about 6.8% of the value of an equivalent option-free mortgage, even at mortgage origination.

We contribute to the finance literature in two distinct and important ways. First, we demonstrate the application of the IAD method and its effectiveness in pricing mortgage-backed securities with embedded options. The IAD method with the LU decomposition algorithm improves upon both the accuracy and the convergence stability compared to previously used numerical methods. Second, we document a significant value of the prepayment option and the default option present in most conventional mortgages, which have been traditionally understated or overlooked. The joint option value is significant not only at loan origination but also for the duration of the mortgage. Further, we illustrate the optimal default boundary for an amortizing FRM and discuss its implications for the current sub-prime mortgage markets in the United States.

The remainder of this article is organized as follows: the second section sets up a theoretical framework for valuing the holding of a mortgage in the presence of prepayment and default options. The third section transforms the underlying PDE to obtain a numerical solution. The fourth section discusses the boundary conditions for parameters. The fifth section outlines procedures in the free-boundary IAD method. The sixth section presents numerical results and their implications. Finally, the last section concludes.

VALUATION FRAMEWORK

Fixed-Rate Scheduled Payments in Continuous Time

Although payments from an FRM occur at discrete intervals, most traded mortgage-backed and fixed-income securities accrue interest daily. Thus, the assumption of continuous payment is a convenient means of approximating the way in which FRM securities are actually valued. Consequently, the option to default on a mortgage may be treated as an American-style option rather than a European-style option with multiple stand-alone exercise dates, which are the scheduled mortgage payment dates.

The following notation describes the scheduled continuous payment and the components of mortgage value:

M The initial mortgage balance, which is the mortgage loan amount at origination.

T The scheduled mortgage maturity.

t The mortgage time into term, $0 \leq t \leq T$, with t equal to zero at mortgage origination and equal to T at maturity.

R_0 The annualized effective mortgage contract rate at loan origination. This interest rate is used to determine the scheduled payment rate and the unpaid balance during the remaining life of the mortgage.

m The scheduled rate of instantaneous continuous payment, m, is determined by contract maturity T, the mortgage contract rate R_0, and the par value of the mortgage M. The payment over time interval Δt will be $m\Delta t$.

$M(R_0, t)$ The mortgage balance at time t, the present value of remaining payment stream discounted at R_0.

$r(t)$ The instantaneous spot rate at time t driven by the single factor Cox, Ingersoll, and Ross (1985b) (CIR) process. It varies through time and is further discussed below.

$A[r(t)]$ The present value of future scheduled payments in the absence of prepayment and default options. This present value of remaining scheduled payments is discounted by the current term structure of interest rates. Note that $A[r(t)]$ is not a function of the house value. It is purely determined by the current term structure of interest rates, represented by $r(t)$, the remaining scheduled payments m, and the remaining life of the mortgage $T - t$. We call it the value of the option-free mortgage.

$H(t)$ The market value of the house at time t.

$V_B[r(t), H(t)]$ The present value of future payments with prepayment and default options. We call this the value of the mortgage to the borrower. It is a function of the term structure of interest rates $r(t)$, remaining mortgage payments $M(R_0, t)$, and the house value $H(t)$. At any time, the borrower must make the decision to prepay, default, or service the mortgage. When the interest rate $r(t)$ and the underlying house value $H(t)$ are high, it is optimal for the borrower to continue servicing the mortgage. This domain defines a continuation region G, where the PDE implied by Cox et al. (1985a) is satisfied. In the prepayment and default region, $\overline{G}$, it is optimal to terminate the mortgage by either prepayment or default. In the prepayment region, the mortgage value is replaced by $M(R_0, t)$, and in the default region by $H(t)$. In fact, in the prepayment and default region, the mortgage does not exist any more. Before the state variables reach the termination boundary ∂G, the borrower will have already terminated the mortgage by either prepayment or default. It may be more explicitly expressed as $V_B[r(t), H(t), M(R_0, t), m]$.

$J[r(t), H(t)]$ The value of the joint prepayment and default option is a function of the term structure of interest rates and the underlying house value, i.e., $J[r(t), H(t)] = A[r(t)] - V_B[r(t), H(t)]$.

π The penalty for prepayment or refinancing fees. We assume that the penalty is a constant fraction of the payoff, i.e., $\pi = \xi M(R_0, t)$, where ξ is a fraction of the payoff at prepayment.

Based on the above set of definitions, we have the following relations:

$$M = M(R_0, 0) = \int_0^T e^{-R_0 t} m dt \quad \text{and} \quad m = \frac{M}{\int_0^T e^{-R_0 t} dt} = M \frac{R_0}{1 - e^{-R_0 T}} \tag{1}$$

$$M(R_0, t) = \int_0^{T-t} e^{-R_0 u} m du = m \int_0^{T-t} e^{-R_0 u} du = M \frac{1 - e^{-R_0(T-t)}}{1 - e^{-R_0 T}}. \tag{2}$$

$A[r(t)]$ is defined so as not to allow for prepayment or default. The value of the option-free mortgage is a function of current term structure of interest rates:

$$A[r(t)] = \int_0^{T-t} e^{-r(u)u} m du. \quad (3)$$

The Joint Prepayment and Default Option Value

The value of mortgage debt to the lender is reduced by the option to prepay or to default because the borrower may prepay or default, but not both, at any time. Furthermore, when default occurs, the borrower not only surrenders the house but also terminates the prepayment option, which has value. Similarly, by prepaying, the borrower foregoes the default option. Therefore, the default and prepayment options should not be valued independently.

We refer to the value of default and prepayment options as the value of the joint option. Let C represent the value of the option to prepay for a non-defaultable mortgage, and D represent the value of the option to default for a non-callable mortgage. The value of the joint option, J, is a non-decreasing function of C and D; otherwise, there is an arbitrage opportunity. It follows that

$$J[r(t), H(t)] = \text{Max}\{C[r(t), H(t)], D[r(t), H(t)]\}. \quad (4)$$

For convenience, we refer to the value of mortgage debt to the borrower, V_B, as the mortgage value. This value is the value of holding a mortgage with embedded default and prepayment options:

$$V_B[r(t), H(t)] = A[r(t)] - J[r(t), H(t)]. \quad (5)$$

Note that the value of the mortgage to the borrower reflects a combination of a liability in the amount of the option-free mortgage and a long position in the joint option to default or prepay.

We also need to point out that in computing the value of the mortgage to the borrower, we assume that markets are efficient and that the borrowers rationally and optimally exercise the default and prepayment options. However, in reality many borrowers may sub-optimally exercise default or prepayment options due to personal or non-financial reasons, such as job relocation, death, marital status change, or moving. Personal and non-financial reasons may induce or force sub-optimal early exercises of default or prepayment options. Consequently, the market value of a mortgage will reflect both optimal and sub-optimal default and prepayment from various borrowers. Some mortgage contracts are designed to appeal to different clienteles with different private information about the likelihood of moving, for example. Such a contract may offer high points with low rates to attract borrowers who tend to keep the mortgage longer, and low points with high rates to appeal to potential movers.

We price mortgages assuming optimal prepayment and default. Such mortgages carry a zero NPV to borrower who optimally exercises prepayment and default options, and carry a negative NPV to borrower who sub-optimally exercises these options.

The Economic Environment

The borrower faces uncertainty in the spot interest rate and the house value. To describe the diffusion process of interest rates, we introduce additional notation. The term structure of interest rates is assumed to be generated from the CIR process describing the spot interest rate $r(t)$ as a mean-reverting square root diffusion process:

$$dr = \kappa(\theta - r)dt + \sigma_r\sqrt{r}dz_r \tag{6}$$

where the parameters are:

κ The speed of adjustment in the mean-reverting process.
θ The long-term average spot interest rate.
σ_r The instantaneous standard deviation of spot interest rate.
z_r A standardized Wiener process of the interest rate.

Equation (6) has the advantage over the Vasicek (1977) model in allowing only a positive nominal interest rate. The slope of the term structure of interest rates depends on the parameters of the model. Provided that r_0 is less than θ, the CIR model suggests that the slope of the term structure of interest rates will be positive and interest rates will converge to the mean value in the long run. This property is convenient for us to analyze the mortgage value in different economic environments.

We treat the house value, $H(t)$, as a lognormal diffusion process. Because the homeowner receives benefits from living in the house—saving the rent paid to live in a comparable property—we deduct the service flow, δ, provided by the house as a continuous cash dividend being paid to the homeowner:

$$\frac{dH}{H} = (\mu - \delta)dt + \sigma_H dz_H \tag{7}$$

where the parameters are:

μ The instantaneous rate of house value appreciation.
δ The continuously compounded service flow provided by the house.
σ_H The instantaneous standard deviation of returns on the house value.
z_H A standardized Wiener process of the house value.

It is assumed that houses are traded at no risk premium, which allows μ to be replaced by r. The correlation between the two processes is indicated by

$$dz_r(t)dz_H(t) = \rho dt$$

where ρ is the instantaneous correlation between the two Wiener processes.

The CIR framework provides a general methodology for the valuation of contingent claims. We may write the PDE of the mortgage value, $V_B[r(t), H(t)]$, as a function of the state variables, the house value, H, and the spot interest rate, r:

$$\frac{1}{2}H^2\sigma_H^2\frac{\partial^2 V_B}{\partial H^2} + \rho H\sqrt{r}\sigma_H\sigma_r\frac{\partial^2 V_B}{\partial H\partial r} + \frac{1}{2}r\sigma_r^2\frac{\partial^2 V_B}{\partial r^2} + \kappa(\theta - r)\frac{\partial V_B}{\partial r} + (r - \delta)H\frac{\partial V_B}{\partial H} + \frac{\partial V_B}{\partial t} - rV_B + m = 0. \quad (8)$$

No analytic solution to the PDE is available; therefore, a numerical solution must be sought. We solve the system by working backward in time, using a finite difference mesh, identifying points at which early exercise is optimal.

TRANSFORMATION OF THE PDE

We solve the PDE numerically by transforming the variables, namely, converting the coefficients of the PDE to constants, yielding a linear form. The boundary conditions are then more accurately and easily applied. From the PDE equation (8), any derivative with respect to H has a factor H multiplying it. This pattern suggests the natural logarithmic transformation $q = \ln(H)$. Because the house value follows a lognormal random walk, we transform H into its natural logarithm value in the grid. The logarithmic transformation means that many grid points are spread around low values of the house where the default option is more likely to be in-the-money and thus exercised. The transformation is

$$q = \ln(H). \quad (9)$$

The function q has an infinite domain $(-\infty, +\infty)$ because the natural boundaries for the house value are 0 and ∞. In any practical application, the state variable will remain finite. A "large" house value, $H = \overline{H}$, can thus approximate infinity, providing a boundary condition where the default option is worthless. In practice, the upper bound, $\overline{H}$, does not have to be too large, typically three or four times the exercise price (Wilmott, 1998). For the lower bound, we ignore the negative log value of the house value. Consider, for example a house value of \$1; that is, $H = 1$. The transformed house value variable is thus $q \in [0, \ln(\overline{H})]$.

Natural boundaries for the interest rate grid are 0 and ∞. Rather than solving Equation (8) directly, we use the transformation

$$y = \frac{1}{1 + r\beta}. \quad (10)$$

For some constant $\beta > 0$, we map the infinite range $[0, \infty)$ of r into the finite range $[0, 1]$ for y. The factor β is chosen so as to center the range of most concern for the state variable in the grid. Equation (8) is a backward parabolic PDE. We transform it into a forward equation by reversing the time dimension:

$$\tau = T - t. \tag{11}$$

These transformations convert the original PDE into the following equation whose value is a function of the new variables τ, q, and y:

$$\frac{1}{2}\sigma_H^2\frac{\partial^2 V_B}{\partial q^2} + \left(r - \delta - \frac{1}{2}\delta_H^2\right)\frac{\partial V_B}{\partial q} + \frac{1}{2}r\sigma_r^2\beta^2 y^4\frac{\partial^2 V_B}{\partial y^2} + [\sigma_r^2\beta^2 r y^3 - \beta\kappa(\theta - r)y^2]\frac{\partial V_B}{\partial r} - \frac{\partial V_B}{\partial \tau} - rV_B + m = 0. \tag{12}$$

With the transformed variables τ, q, and y, we now have a three-dimensional box defined in the space $[0, T] \times [0, \ln(\overline{H})] \times [0, 1]$.

THE BOUNDARY CONDITIONS

To solve Equation (12), not only do we need to apply the finite difference approximations to the first and second derivatives within the lattice but also we must satisfy all boundary conditions. The boundary conditions can be identified at the faces and edges of the three-dimensional box. We will consider each face in turn, followed by the edges where the faces intersect.

The Log Value of the House Value is Zero

When the log value of the house value is zero, i.e., $H(t) = 1$, the value of the mortgage debt owed by the borrower cannot be more than the house value. The borrower's rational behavior is to default. Consequently, the prepayment option is worthless. The mortgage value is then equal to \$1, the house value:

$$V_B[r(t), H(t)] = H(t) = 1. \tag{13}$$

In fact, the homeowner will have already defaulted for a long time before the log value of the house value approaches zero because the borrower holds an American-style default option. We assign the mortgage value to be the option payoff in the prepayment and default regions, where it is optimal for the borrower to exercise either of the two options. This assignment provides us a boundary condition so that the mortgage value can be defined in the entire

$[r(t), H(t)]$ space. At this boundary where $H(t) = 1$, the value of the joint option equals the value of the option-free mortgage minus the house value:

$$J[r(t), H(t)] = A[r(t)] - 1. \tag{14}$$

The Interest Rate is Zero

When the interest rate is zero, there is no discounting. The boundary condition is then either in a prepayment region or in a default region (Kau et al., 1992). In other words, the borrower will either prepay or default with certainty:

$$dr_t = \kappa\theta dt \quad \text{and} \quad V_B[r(t), H(t)] = \text{Min}[H(t), M(R_0, t)]. \tag{15}$$

The House Value Becomes Very Large

As the house value tends to infinity, the value of the default option approaches zero. This corresponds to one face of the box where the house value has an extreme value as indicated in the following equation:

$$\lim_{H\to\infty} D[r(t), H(t)] = 0. \tag{16}$$

Because the value of the default option tends to zero, the mortgage value at this extreme is given by

$$\lim_{H\to\infty} V_B[r(t), H(t)] = A[r(t)] - \lim_{H\to\infty} C[r(t), H(t)]. \tag{17}$$

At this face of the box, the mortgage contract is equivalent to a callable but non-defaultable one, and must therefore satisfy the degenerate form:

$$\frac{\partial V_B}{\partial t} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 V_B}{\partial r^2} + \kappa(\theta - r)\frac{\partial V_B}{\partial r} - rV_B + m = 0. \tag{18}$$

Before addressing the PDE of Equation (12), we must solve Equation (18). In the absence of the house value in Equation (18), we may solve for it at every interest rate step at each point in time. As the interest rate approaches infinity, the value of a callable mortgage approaches zero. As the interest rate approaches zero at the other extreme, the borrower will prepay with certainty, with the mortgage value equalling the unpaid balance $M(R_0, t)$.

The Interest Rate Becomes Very Large

As interest rates approach infinity, the present value of any future payments approaches zero. At this boundary, the present value of the option-free mortgage is worthless. The borrower cannot further minimize the mortgage value by defaulting or prepaying. We can therefore write the following:

$$\lim_{r\to\infty} A[r(t)] = 0 \quad \text{and} \quad \lim_{r\to\infty} V_B[r(t), H(t)] = 0. \tag{19}$$

Both *H* and *r* Have Extreme Values

Next, we consider the edges of the box where the extreme values of H and r occur simultaneously. First, we take $r = 0$ and consider the two extremes of the house value, $H = 1$ and H approaching infinity:

$$V_B[r(t), H(t)] = H(t) = 1 \tag{20}$$

and

$$\lim_{H\to\infty} V_B[r(t), H(t)] = M(R_0, t). \tag{21}$$

In Equation (20), the mortgage value is replaced with the house value at default as the borrower will default with certainty at this boundary. When the interest rate and the house value approach zero, the payoff from default dominates that from prepayment. The borrower will default instead of prepay. Equation (21) suggests that the borrower will prepay with certainty as the interest rate approaches zero and the house value tends to infinity.

Finally, when r tends to infinity, the conditions on the rest of face H versus time are given by the following equations:

$$\lim_{\substack{r\to\infty \\ H\to 0}} V_B[r(t), H(t)] = 0 \tag{22}$$

$$\lim_{\substack{r\to\infty \\ H\to\infty}} V_B[r(t), H(t)] = 0 \tag{23}$$

At this boundary, as r goes to infinity, $A[r(t)]$, the value of the option-free mortgage, approaches zero. Because $0 \le V_B \le A[r(t)]$, the mortgage value V_B approaches zero, default occurring.

The Initial Condition at Maturity

At maturity, the default option, the prepayment option, and the value of the mortgage are worthless:

$$V_B[r(T), H(T)] = 0 \tag{24}$$

$$D[r(T), H(T)] = 0 \quad (25)$$

and

$$C[r(T), H(T)] = 0 \quad (26)$$

The mortgage is fully amortized; thus, the value of prepayment and default options vanishes.

THE TREATMENT OF FREE BOUNDARY USING THE IAD METHOD

Because the principles of the finite difference method are well known in the mortgage valuation literature, only an outline of the particular technique is provided here. We focus on the treatment of the free-boundary problem in an implicit numerical scheme with a modified LU decomposition algorithm.

In the finite difference method of solution to the PDE, small but finite changes in each dimension of the two state variables, i.e., interest rate and house value, as well as time are considered. The value of the mortgage to the borrower is then computed. Terms in the PDE are approximated by linear slopes across the grid. The idea behind the finite difference method is to approximate the first and second derivatives in the PDE with discrete small increments in each dimension of the state variables in the lattice while moving forward in time. Knowing terminal conditions, it is possible to work backward in time, valuing the derivative at each point on the grid, until the initial values are obtained. The new values are found using existing values at the previous time step, working backward in time.

In an explicit scheme, each new value is calculated, respectively, using several existing values. The explicit method calculates each value individually. The optimal early-exercise boundary can be found by examining the difference between the exercise payoff and the calculated new mortgage value. The difference will approach zero at the optimal early-exercise boundary. Thus, the optimal prepayment boundary can be determined by moving backward in time from $t = T$ to $t = 0$. At each time step, the interest rate dimension is moved from infinity to zero. Although the explicit method has the advantage of simplicity, it raises problems of stability and speed of solution. To overcome these problems, the size of the time step has to be sufficiently small. For example, previous work adopted a time step size of $\frac{1}{60}$ month, such as in Newton et al. (2002) and Kau et al. (1992, 1993). Even if the time step is small, the rounding errors in computation will tend to be considerably large.

The alternative implicit method is more complicated in calculating all the new values at the same time using all known values at the previous time step.

The implicit method requires solving a linear system of equations, and the new values are determined by the solution of the linear system of equations, linking the values to each other.

We use the LU triangular decomposition algorithm (SIAM, 1999) to solve the linear system of equations. The LU decomposition algorithm is a well-developed mathematical method to solve a sparse linear system of equations, which can relieve memory space and reduce rounding errors.

In numerical computation, we further modify the backward substitution procedure of LU decomposition by applying the free-boundary constraints. With the modified backward substitution procedure, we check whether exercise is optimal at the same time the simultaneous equations are solved. This algorithm for accommodating the early-exercise feature of an American-style option is proposed by Brennan and Schwartz (1977) and further analyzed in Jaillet, Lamberton, and Lapeyre (1990) and Ikonen and Toivanen (2004).

We first consider a callable but non-defaultable FRM denoted by $V_B^{ND}[r(t)]$. We use the Crank–Nicholson algorithm (Crank & Nicholson, 1947), which has desirable stability and convergence properties, to solve Equation (18). For convenience, we use a time interval of $\frac{1}{16}$ of a month, with mortgage maturity of 300 months or 25 years, yielding a total of 4,800 time intervals. The prepayment decision is not simply triggered when the value of an equivalent option-free mortgage $A[r(t)]$ exceeds the unpaid balance $M(R_0, t)$. Instead, prepayment occurs when the mortgage value $V_B^{ND}[r(t)]$ exceeds the unpaid balance. Therefore, the boundary condition for a callable but non-defaultable mortgage, $V_B^{ND}[r(t)]$, is

$$V_B^{ND}[r(t)] \leq M(R_0, t). \tag{27}$$

To identify the prepayment boundary, we employ an explicit finite difference scheme to make Equation (18) discrete and utilize the smooth-pasting condition (Barone-Adesi, 2005) of prepayment boundary. See Appendix A for details.

Note that the house value $H(t)$ vanishes in Equation (27) because it is irrelevant to the valuation of a callable but non-defaultable mortgage $V_B^{ND}[r(t)]$. To include transaction costs, i.e., prepayment penalty or refinancing fees, associated with prepayment, this boundary condition can be modified as $V_B^{ND}[r(t)] \leq M(R_0, t)(1 + \xi)$, where ξ is the constant portion of the payoff amount when prepayment occurs. The transaction cost can be a prepayment penalty, a refinancing fee, or implicitly the difficulty of qualifying for a new mortgage loan. For illustration purposes, we include an example of a prepayment penalty with ξ equal to 3% of the payoff in the sixth section. This choice is motivated by the current practice of mortgage markets. The amount of prepayment penalty varies across different countries and different types of loan arrangements. In the United States, ". . . 70% of sub-prime loans have such (prepayment) penalties, . . . typically involving six months of interest" (Morgenson, 2007).

We then value an FRM with options to prepay and default. Note that the default decision is not simply triggered when the value of an equivalent option-free mortgage $A[r(t)]$ exceeds the house value $H(t)$. Instead, default occurs when the mortgage value, V_B, exceeds the house value $H(t)$. For a callable and defaultable FRM, early termination either by prepayment or default occurs when the mortgage value exceeds the minimum of the callable but non-defaultable mortgage and the house value. The boundary conditions impose the following constraint:

$$V_B[r(t), H(t)] \le \text{Min}\{H(t), V_B^{ND}[r(t)]\}. \tag{28}$$

With all the known boundaries and initial conditions, we can solve PDE (12) implicitly in one factor and explicitly in the other factor at each time step by using the IAD method. We incorporate Equation (28) in the numerical procedure by modifying the LU decomposition algorithm to determine the optimal early-exercise boundary. Because only one direction is implicit, the solution by IAD method is no more difficult than in the one-factor solution (Wilmott, 1998). With the modified LU decomposition algorithm, the free-boundary conditions are incorporated into the process to solve Equation (12).

NUMERICAL RESULTS

Benchmark Parameters

This section presents and discusses the numerical results provided by the model. Figure 1 and Tables I and II illustrate the mortgage value to the borrower in the $[r(t), H(t)]$ space for different macroeconomic parameter specifications. Computations are based on the following benchmark parameters: σ_r = 7% p.a., σ_H = 10% p.a., θ = 10% p.a., κ = 25% p.a., δ = 7.5% p.a., r_0 = 8% p.a., and $\rho = 0$. Mortgage contract parameters are: T = 300 months, R_0 = 9%, M = \$100,000, and LTV ratio = 80%.

The parameter values are within the ranges considered in the literature (e.g., Kau et al., 1993; Titman & Torous, 1989) and they represent typical economic parameters presented in the U.S. mortgage markets. Note also that the term structure is chosen to be upward sloping.

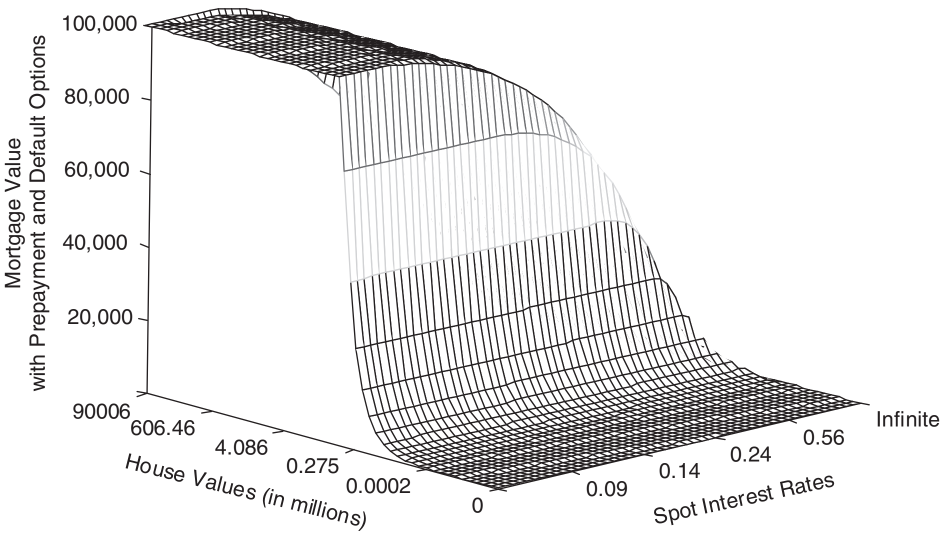

FIGURE 1

Mortgage value to the borrower $V_B[r(t), H(t)]$ at mortgage origination with prepayment and default options. Figure 1 displays the contour of mortgage value as a function of underlying house value and spot interest rates. Computations are based on the following benchmark parameters: $\sigma_r = 7\%$ p.a., $\sigma_H = 10\%$ p.a., $\theta = 10\%$ p.a., $\kappa = 25\%$ p.a., $\delta = 7.5\%$ p.a., $r_0 = 8\%$ p.a., and $\rho = 0$. Mortgage contract parameters are: $T = 300$ months, $R_0 = 9\%$, $M = \$100{,}000$, and LTV ratio = 80%. We use logarithmic scaling for the house value axis and label it with the actual house value. At low house values, the house value itself dominates other factors. The mortgage value equals the house value because the borrower will default with certainty, which is characterized by a linear relation between the house value and the mortgage value. At higher house values, changes in the house value have little effect on the mortgage value and the prepayment option assumes greater significance. At low spot rates, changes in spot rates have little effect because the borrower will default at a low house value or otherwise prepay.

Illustrative Results

Figure 1 presents the contour of V_B, mortgage value to the borrower, at mortgage origination as a function of the underlying house value and spot interest rates. We assume that the FRM carries an initial mortgage balance of \$100,000 with a mortgage term of 25 years. Note that from Equation (5), $V_B[r(t), H(t)] = A[r(t)] - J[r(t), H(t)]$, the mortgage value to the borrower can be viewed as the difference between an otherwise-equivalent option-free mortgage value $A[r(t)]$ and the joint prepayment and default option premium $J[r(t), H(t)]$. $A[r(t)]$ is independent of the house value because it is free of default risk. It reflects only the present value of a stream of prepayment-risk-free mortgage payments discounted at the current term-structure-implied spot rates. $J[r(t), H(t)]$ considers both prepayment risk, affected mostly by changes in term structure of interest rates, and default risk, affected by both term structure of interest rates and the underlying house value.

TABLE I

Mortgage Value and Joint Option Value as a Function of Interest Rate Volatility at Mortgage Origination

Standard Deviation of Spot Interest Rates σ_r	*Option-Free Mortgage Value* $A[r(t)]$ (\$)	*The Joint Option Value* $J[r(t), H(t)]$ (\$)	*Mortgage Value* $V_B[r(t), H(t)]$ *Without Prepayment Penalty* (\$)	*Mortgage Value* $V_B[r(t), H(t)]$ *With a 3% Prepayment Penalty* (\$)
0.04	102,327	2,429	99,898	102,290
0.07	106,658	6,838	99,820	102,311
0.10	116,179	17,137	99,042	101,687
0.15	131,426	32,868	98,558	101,277

Note. This table reports mortgage value $V_B[r(t), H(t)]$ with and without prepayment penalty and joint options value for different levels of interest rate volatility. Computations are based on the following benchmark parameters: $\sigma_H = 10\%$ p.a., $\theta = 10\%$ p.a., $\kappa = 25\%$ p.a., $\delta = 7.5\%$ p.a., $r_0 = 8\%$ p.a., and $\rho = 0$. Mortgage contract parameters are: $T = 300$ months, $R_0 = 9\%$, $M = \$100{,}000$, and loan-to-value (LTV) ratio = 80%. Note that $V_B[r(t), H(t)] = A[r(t)] - J[r(t), H(t)]$, where $A[r(t)]$ is the mortgage value without default and prepayment options and $J[r(t), H(t)]$ is the joint default and prepayment option value. As interest rate volatility increases, the value of the option-free mortgage $A[r(t)]$ increases due to Jensen's inequality. At the same time, the joint default and prepayment option value $J[r(t), H(t)]$ also increases because both the default option and the prepayment option have a higher value as interest rate volatility increases. This table demonstrates that the mortgage value is a decreasing function in interest rate volatility. Higher interest rate volatility is associated with lower mortgage values. The statement holds regardless of prepayment penalties. The prepayment penalty has a significant effect on the value of mortgage. When a 3% prepayment penalty is imposed, the value of the mortgage increases about 1%. Furthermore, for the benchmark case where the standard deviation of interest rates is 0.07, the joint option value is \$6,838, or about 6.8% of the loan size.

TABLE II

House Value Return Volatility and Mortgage Value $V_B[r(t), H(t)]$ at Mortgage Origination Under Different Loan-to-Value (LTV) Ratios

Standard Deviation of House Value Returns σ_H	*Mortgage Value* $V_B[r(t), H(t)]$			
	LTV = 70% (\$)	*LTV = 80% (\$)*	*LTV = 90% (\$)*	*LTV = 100% (\$)*
0.07	99,934	99,923	99,833	98,708
0.10	99,904	99,820	99,610	98,587
0.15	99,819	99,421	98,696	97,486
0.20	99,540	98,517	96,626	94,524

Note. This table illustrates the mortgage value for different levels of house value return volatility and loan-to-value (LTV) ratios. Computations are based on the following benchmark parameters: $\sigma_r = 7\%$ p.a., $\sigma_H = 10\%$ p.a., $\theta = 10\%$ p.a., $\kappa = 25\%$ p.a., $\delta = 7.5\%$ p.a., $r_0 = 8\%$ p.a., and $\rho = 0$. Mortgage contract parameters are: $T = 300$ months, $R_0 = 9\%$, and $M = \$100{,}000$. The higher the house value volatility, the higher the value of the default option premium, and consequently the lower the mortgage value. Furthermore, higher LTV ratios are associated with higher default risk and lower mortgage value for all different levels of house value return volatility.

It can be observed from Figure 1 that at low house values, the house value itself dominates other factors. When the house value is sufficiently low, the borrower defaults with certainty, regardless of the level of spot interest rates. At higher house values, changes in the house value have little effect on the mortgage value. The default option is deep out-of-the-money and is essentially almost worthless. However, the prepayment option assumes greater significance.

It is worthwhile to point out our assumptions to reach the above simulation results. We assume the correlation of interest rates and house value to be zero. However, empirical results indicate that the correlation is generally negative (Schwartz & Torous, 1989). In Figure 1, the mortgage value drops significantly as interest rates increase and simultaneously the house value falls.

The values of both the option-free mortgage $A[r(t)]$ and the prepayment option $C[r(t), H(t)]$ vary inversely with the interest rate. For a very large house value, when the interest rate is high, the mortgage value is a decreasing convex function of the interest rate. On the other hand, when the interest rate is relatively low, the prepayment option becomes more valuable and the mortgage value becomes a decreasing concave function of the interest rate. Negative convexity is the driving factor of this phenomenon. The behavior of the mortgage value to the borrower is much like that of a callable bond value. When the bond yield is high, the callable bond exhibits positive duration and positive convexity. However, when the bond yield is sufficiently low, the callable bond exhibits positive duration and negative convexity. In this case, the prepayment option is in-the-money.

In Table I we report mortgage value V_B with and without prepayment penalties as a function of interest rate volatility. Note that $V_B[r(t), H(t)] = A[r(t)] - J[r(t), H(t)]$, where $A[r(t)]$ is the value of an equivalent option-free mortgage and $J[r(t), H(t)]$ is the joint default and prepayment option value. To separate the effects of the joint default and prepayment option from the effects of term structure of interest rates, we calculate the value of an equivalent option-free mortgage $A[r(t)]$. $A[r(t)]$ is computed as the sum of the value of zero-coupon bonds equal to the payment stream discounted at risk-free spot interest rates (Hull, 2000, p. 570). As interest rate volatility increases, the value of the option-free mortgage value $A[r(t)]$ increases due to Jensen's inequality. At the same time, the joint default and prepayment option value $J[r(t), H(t)]$ also increases because both the default option and the prepayment option assume a higher value as interest rate volatility increases. Consequently, the mortgage value $V_B[r(t), H(t)]$ may either increase or decrease. However, numerical analysis suggests that the mortgage value is a decreasing function of interest rate volatility, as suggested in Table I. Higher interest rate volatility is associated with lower mortgage value. The statement holds regardless of the presence of prepayment penalties.

In fact, we see that the 3% prepayment penalty has a significant effect on the value of mortgage. When a 3% prepayment penalty is imposed, the value of the mortgage increases by more than 1%. For example, in the base case in the second row of Table II, where we set $\sigma_r = 0.07$, the option-free mortgage value $A[r(t)]$ is \$106,658. The mortgage value to the borrower $V_B[r(t), H(t)]$ is

\$99,820, which implies that the value of the joint default and prepayment option $J[r(t), H(t)]$ is \$6,838. After we impose a 3% prepayment penalty, the mortgage value to the borrower increases to \$102,311. The value of joint default and prepayment option is \$6,838, representing about 6.8% of the mortgage balance at origination, an economically significant amount. It suggests the importance of embedded options in conventional FRMs.

In Table II, the volatility of the rate of return on the house, measured by its standard deviation, has a negative effect on mortgage value. This is explained by the fact that an increase in the volatility of the rate of return of the house creates a relatively higher likelihood for the contract to reach the default region; therefore, the default option has a greater value. Consequently, the value of holding the mortgage $V_B[r(t), H(t)]$ falls.

Table II also demonstrates that the LTV ratio has similar, but more significant effects than house value return volatility on mortgage value $V_B[r(t), H(t)]$. For a given house value return volatility and term structure of interest rates, the expected value of prepayment and default increases as the LTV ratio increases. Consequently, the mortgage value decreases accordingly.

Figure 2 demonstrates the borrower's default strategy for an FRM with LTV = 80% and mortgage initial balance = \$100,000 at two years after mortgage origination. The optimal default boundary is identified at the same time that the mortgage value is calculated by imposing the constraint indicated in Equation (28). The borrower will only default if the underlying house value falls far enough below the unpaid balance. We hold the benchmark interest constant at $r_0 = 8\%$, but let the house value and the mortgage value vary. It is optimal for the borrower to default when the house value falls to \$92,500, 5.3% below the then-current outstanding mortgage balance of $M(R_0, t = 2 \text{ years})$, which is \$97,664.

CONCLUSION

Using the contingent claims analysis of CIR (Cox et al., 1985a), we model endogenous prepayment and default under dynamic economic circumstances. A numerical example extends the model to include prepayment penalties. The results show that the mortgage value is lower to the lender and greater to the borrower than the value of an equivalent option-free mortgage, even at origination. Using a well-accepted set of parameter specifications, we find that the value of joint default and prepayment option is about 6.8% of the unpaid outstanding balance at origination, assuming a down-payment of 20%. Lowering the down-payment percentage further reduces the mortgage value to the borrower, implying higher default risk for lending banks.

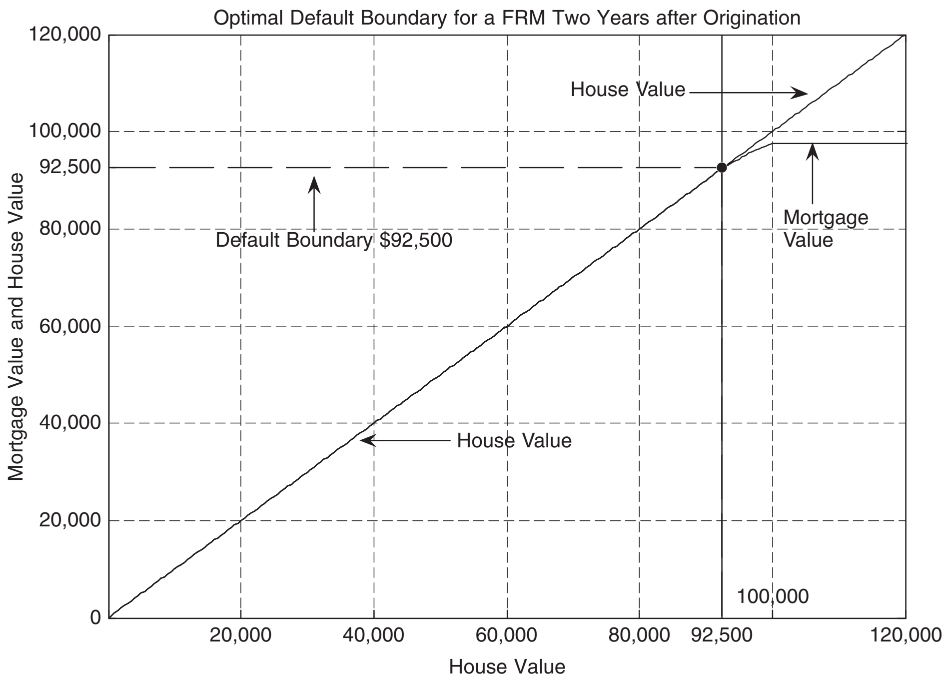

FIGURE 2

The optimal default boundary for an FRM two years after mortgage origination. This figure demonstrates the borrower's optimal strategy to default at two years after mortgage origination. Computations are based on the following benchmark parameters: $\sigma_r = 7\%$ p.a., $\sigma_H = 10\%$ p.a., $\theta = 10\%$ p.a., $\kappa = 25\%$ p.a., $\delta = 7.5\%$ p.a., $r_0 = 8\%$ p.a., and $\rho = 0$. Mortgage contract parameters are: $T = 300$ months, $R_0 = 9\%$, $M = \$100{,}000$, and LTV = 80%. The vertical dimension is the mortgage value $V_B[r(t), H(t)]$, which is a function of mortgage term t, current interest rate $r(t)$, and house value $H(t)$. We hold the benchmark interest rate constant at $r_0 = 8\%$, but let the house value and the mortgage value vary, i.e., we hold $r(t) = 8\%$ and $t = 2$ years in the figure. The mortgage value is only a function of the house value. The house value and the mortgage value converge at the optimal default boundary. Default occurs at a low house value with mortgage value equal to $H(t)$. However, when the house value is high, the default option is deep out-of-the-money and would not be exercised. With the parameters we specified, it is optimal for the borrower to default when the house value falls to about \$92,500, which is about 5.3% below the then-current outstanding mortgage balance of $M(R_0, t = 2$ years$) = \$97{,}664$.

We contribute to the finance literature in two ways. First, we demonstrate the application of the IAD method and its effectiveness in pricing mortgage-based securities with embedded options to default or prepay. The IAD method with an LU decomposition algorithm improves upon both the accuracy and convergence stability compared to alternative numerical methods. For example, we adopt a time step size of $\frac{1}{16}$ month without instability problems, compared to that of $\frac{1}{60}$ month in the previous literature. Second, we document an economically significant value of the joint prepayment and default option present in conventional mortgages.

APPENDIX A: THE PREPAYMENT BOUNDARY

For simplification, we rewrite Equation (18) as

$$\frac{\partial V_B}{\partial t} + L = 0 \quad \text{with} \quad L = \frac{1}{2}\sigma_r^2 r \frac{\partial^2 V_B}{\partial r^2} + \kappa(\theta - r)\frac{\partial V_B}{\partial r} - rV_B + m. \qquad \text{(A1)}$$

We need to address the free-boundary problem in solving Equation (A1). Let $r^*(t)$ denote the prepayment boundary at time t. We apply the "smooth-pasting conditions" at $r(t) = r^*(t)$:

$$V_B[r(t)] = M(R_0, t) \qquad \text{(A2)}$$

$$\frac{\partial V_B[r(t)]}{\partial t} = \frac{\partial M(R_0, t)}{\partial t} = -me^{-R_0(T-t)} \qquad \text{(A3)}$$

$$\frac{\partial V_B[r(t)]}{\partial r} = \frac{\partial M(R_0, t)}{\partial r} = 0. \qquad \text{(A4)}$$

The value matching condition equation (A2) implies that the mortgage value is continuous across the prepayment boundary. It makes no difference for the borrower to prepay or not to do so. The smooth-pasting condition (Barone-Adesi, 2005) (3) and (4) implies that the slope is continuous at the prepayment boundary (Azevedo-Pereira et al., 2000).

We consider the discrete model of Equation (A1). For convenience, we use $V_B[r, t]$ for the mortgage value to the borrower variable in the discrete model. Let Δr and Δt be the size of interest rate steps and time steps, respectively. Equation (A1) can be written as

$$\frac{V_B[r, t + \Delta t] - V_B[r, t]}{\Delta t} + DL = 0 \qquad \text{(A5)}$$

where DL is the discrete expression of L, an algebra expression of $V_B[r - \Delta r, t + \Delta t]$, $V_B[r, t + \Delta t]$, and $V_B[r + \Delta r, t + \Delta t]$. Knowing the mortgage value at time $t + \Delta t$, we can calculate the mortgage value at time t using Equation (A5). We move backward in time from $t = T$ to $t = 0$. At each time step $t + \Delta t$, we find a boundary condition for the next time step t, $V_B[r^*(t + \Delta t), t]$. Note that $V_B[r^*(t + \Delta t), t]$ is the mortgage value at time t given $r(t) = r^*(t + \Delta t)$.

As the prepayment option has no time value at maturity, the optimal strategy is to prepay immediately when the interest rate falls below the contract rate R_0. It follows that $r^*(T) + R_0$. Moving backward in time from $t = T$ to $t = T - \Delta t$, we can calculate $V_B[r(T), T - \Delta t]$ using Equation (A5). Simultaneously, we impose the early-exercise constraint $V_B[r^*(T), T - \Delta t] \leq M(R_0, T - \Delta t)$. At time $t = T - \Delta t$, with the boundary condition at $r(T - \Delta t) = r^*(T) = R_0$ and $r(T - \Delta t) = \infty$, we can use the Crank–Nicholson algorithm to solve Equation (A1) for $r(T - \Delta t) \in [r^*(T), \infty)$.

However, at each time step t, Equation (A1) is not solved for $r(t) \in [0, r^*(t + \Delta t)]$. To achieve this, we make use of the "smooth-pasting conditions" indicated by Equations (A2)–(A4). Substituting Equations (A2)–(A4) into Equation (A1), we have

$$\frac{\partial^2 V_B[r(t)]}{\partial r^2} = \frac{2m[1 - e^{-R_0(T-t)}](r - R_0)}{\sigma_r^2 r R_0}. \tag{A6}$$

It is noteworthy that at the prepayment boundary the second partial difference is negative as the interest rate must be less than the contract rate on mortgage prepayment. This negative convexity reflects the impact of the embedded prepayment option.

To find the prepayment boundary at time t with known $V_B[r^*(t + \Delta t), t]$, we use a Taylor expansion to $V_B[r^*(t + \Delta t), t]$ at $r(t) = r^*(t)$, omitting higher-order terms:

$$V_B[r^*(t + \Delta t), t] = V_B[r^*(t), t] + \frac{\partial V_B[r^*(t), t]}{\partial r}[r^*(t + \Delta t) - r^*(t)] + \frac{\partial^2 V_B[r^*(t), t]}{2\partial r^2}[r^*(t + \Delta t) - r^*(t)]^2. \tag{A7}$$

Using Equations (A2)–(A4) and (A6), we solve Equation (A7) for $r^*(t)$. For $r(t) \in [0, r^*(t)]$, we use the payoff on prepayment for the mortgage value, i.e., $V_B[r, t] = M(R_0, t)\ \forall r(t) \in [0, r^*(t)]$.

This procedure is iterated from $t = T$ to $t = 0$.

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