A NOTE ON THE DERIVATION OF BLACK-SCHOLES HEDGE RATIOS

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An option hedge ratio is the sensitivity of an option price with respect to price changes in the underlying stock. It measures the number of shares of stocks to hedge an option position. This article presents a simple derivation of the hedge ratios under the Black-Scholes option-pricing framework. The proof is succinct and easy to follow. © 2003 Wiley Periodicals, Inc. Jrl Fut Mark 23:1119–1122, 2003

INTRODUCTION

An option hedge ratio is the sensitivity of an option price with respect to price changes in the underlying security. It measures how much the option price changes for a small change in the underlying security. Hedge ratios are widely used in risk management. They can be used to specify the number of option contracts needed to neutralize price movements in a stock portfolio.

Due to its elegance and simplicity, the Black-Scholes (1973) option-pricing formula has been well accepted by many finance professionals. Even though a number of empirical studies have shown that the

I would like to thank John Howe for helpful comments. All remaining errors are my own responsibility.

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Received February 2003; Accepted February 2003

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Black-Scholes model produces certain systematic pricing errors, finance academics and practitioners generally agree that the Black-Scholes model provides a simple and effective way to price and hedge stock and stock index options.

This article provides a simple derivation for the hedge ratios of call and put options within the Black-Scholes option-pricing framework. Mathematically, a hedge ratio is the partial derivative of the option price with respect to the price of the underlying security. However, because of complication in taking partial derivatives in a Black-Scholes option price, existing derivations of hedge ratios are mathematically involved and difficult to follow. This article provides a succinct proof to the hedge ratios of Black-Scholes call and put options.

The article is organized as follows. The next section presents the Black-Scholes (1973) option-pricing model, and then I present a simple derivation of the hedge ratios of Black-Scholes call and put options. The final section concludes the article.

HEDGE RATIO DERIVATION

The Black-Scholes model is given by the following standard notation:

\[
C = S_0 N(d_1) - X e^{-rt} N(d_2); \quad P = X e^{-rt} N(-d_2) - S_0 N(-d_1)
\]

where

\[
d_1 = \frac{\ln(S_0/X) + (r + \sigma^2/2)t}{\sigma \sqrt{t}}
\]

and

\[
d_2 = d_1 - \sigma \sqrt{t}
\]

where \(C\) and \(P\) are the Black-Scholes call and put option premiums, respectively; \(S_0\) is the current price of the underlying stock; \(X\) is the option’s exercise price; \(r\) is the annualized risk free rate of interest; \(t\) is the time to option’s maturity; \(\sigma\) is the standard deviation of the returns of the underlying stock; and \(N(d)\) is the cumulative distribution function of the standard normal distribution, whose probability density function is \(n(d)\). By definition,

\[
\frac{\partial N(d)}{\partial d} = n(d) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{d^2}{2}\right)
\]
The derivation of the hedge ratios requires the proof of the following property:

\[ S_0 n(d_1) = X e^{-r} n(d_2) \]  \hfill (1)

Consider \( d_1^2 - d_2^2 \):

\[
d_1^2 - d_2^2 = (d_1 - d_2)(d_1 + d_2)
\]
\[
= \sigma \sqrt{t} \frac{(\ln(S_0/X) + (r + \sigma^2/2)t) + (\ln(S_0/X) + (r - \sigma^2/2)t)}{\sigma \sqrt{t}}
\]
\[
= 2(\ln(S_0/X) + rt)
\]
\[
\therefore d_1^2 = d_2^2 + 2 \ln(S_0/X) + 2rt \tag{2}
\]

\[
S_0 n(d_1) = S_0 \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{d_1^2}{2} \right) \quad \text{[apply Equation (2)]}
\]
\[
= S_0 \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{d_2^2 + 2 \ln(S_0/X) + 2rt}{2} \right)
\]
\[
= S_0 \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{d_2^2}{2} \right) \exp(-\ln(S_0/X) - rt)
\]
\[
= S_0 n(d_2)X e^{-r}/S_0
\]
\[
= X e^{-r} n(d_2)
\]

Now that Equation (1) is established, the hedge ratio of a Black-Scholes call option can be easily computed as follows:

Call hedge ratio

\[
\frac{\partial C}{\partial S_0} = \frac{\partial (S_0 n(d_1) - X e^{-r} n(d_2))}{\partial S_0}
\]
\[
= N(d_1) + S_0 n(d_1) \frac{\partial d_1}{\partial S_0} - X e^{-r} n(d_2) \frac{\partial d_2}{\partial S_0}
\]
\[
= N(d_1) + S_0 n(d_1) \frac{\partial (d_1 - d_2)}{\partial S_0}
\]
\[
= N(d_1) + S_0 n(d_1) \frac{\partial \sigma \sqrt{t}}{\partial S_0}
\]
\[
= N(d_1) + S_0 n(d_1) \times 0 = N(d_1)
\]
To derive the hedge ratio of a Black-Scholes put option, I use the put-call parity relation for European style options: \( S_0 + P = X e^{-r_t} + C \)

\[
P = X e^{-r_t} + C - S_0
\]

Put hedge ratio \( \frac{\partial P}{\partial S_0} = \frac{\partial (X e^{-r_t} + C - S_0)}{\partial S_0} = N(d_1) - 1 \)

Note that the put hedge ratio is always nonpositive, between \(-1\) and zero.

**CONCLUSION**

In summary, this article provides a simple derivation of hedge ratios for call and put options within the Black-Scholes option-pricing framework. The proof is succinct and easy to follow.

**BIBLIOGRAPHY**