S&P 500 INDEX OPTION TESTS OF JARROW AND RUDD'S APPROXIMATE OPTION VALUATION FORMULA

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INTRODUCTION

The Black–Scholes (1973) option pricing model is a universal standard among option valuation models. Despite its widespread popularity, however, the model has some known deficiencies in actual applications. For example, when calibrated to accurately price at-the-money options, the Black–Scholes model frequently misprices deep in-the-money and deep out-of-the-money options. Pricing biases associated with the Black–Scholes option pricing model are well documented. Well-known studies by Black (1975), Emanuel and MacBeth (1982), MacBeth and Merville (1979), and Rubinstein (1985, 1994) all report that the Black–Scholes model tends to systematically misprice in-the-money and out-of-the-money options. Rubinstein (1994) also points out that strike price biases associated with the Black–Scholes model have been especially severe for S&P 500 index options since the October 1987 market crash and that

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the magnitude of these biases has tended to increase since the 1987 crash.

These observed mispricing patterns are generally thought to result from the parsimonious assumptions used to derive the Black–Scholes model. In particular, the Black–Scholes model assumes that stock log-prices follow a constant variance diffusion process, which, in turn, implies that over finite intervals stock prices are lognormally distributed. There exists considerable evidence indicating, however, that the lognormal distribution is valid only as an approximate description of the distribution of stock prices. Hull (1993) and Nattenburg (1994) discuss the implications of deviations from lognormality for options pricing.

Jarrow and Rudd (1982) proposed an extension to the Black–Scholes model designed to overcome most of its limitations. Essentially, they adopt the Black–Scholes formula as the core component of an expanded formula that accounts for skewness and kurtosis deviations from lognormality in stock price distributions. When skewness and kurtosis correspond to a lognormal distribution, the Jarrow–Rudd formula collapses to the Black–Scholes formula.

This study applies Jarrow and Rudd's extended formula to S&P 500 index options and estimates coefficients of skewness and kurtosis for the S&P 500 index implied by index option prices. This methodology extends the widely used procedure of obtaining implied standard deviations (*ISD*) to also include procedures to simultaneously obtain implied skewness (*ISK*) and implied kurtosis (*IKT*) coefficients.

The following section briefly reviews the structure of the skewness- and kurtosis-adjusted Black–Scholes option price formula proposed by Jarrow and Rudd (1982). The data sources are then described. In the subsequent empirical section, the performance of the Black–Scholes model is compared to that of Jarrow and Rudd's extension to the Black–Scholes model. The final section summarizes and concludes the article.

JARROW AND RUDD'S APPROXIMATE OPTION VALUATION FORMULA

Jarrow and Rudd (1982) proposed a method to value European style options when the underlying security price at option expiration follows a distribution, F, known only through its moments. They derived an option pricing formula from an Edgeworth series expansion of the distribution F about an approximating distribution, A. Their analysis yielded several variations, but the simplest is the following expression for an approximate option price.

$$C(F) = C(A) - e^{-rt}\frac{\kappa_3(F) - \kappa_3(A)}{3!}\frac{da(K)}{dS_t} + e^{-rt}\frac{\kappa_4(F) - \kappa_4(A)}{4!}\frac{d^2a(K)}{dS_t^2} + \varepsilon(K) \quad (1)$$

The left-hand term, $C(F)$, in eq. (1) denotes a call option price based on the stock price distribution, F. The first right-hand term, $C(A)$, is a call price based on a known distribution, A, followed by adjustment terms based on cumulants, $\kappa_j(F)$ and $\kappa_j(A)$, of the distributions, F and A, respectively. The relationship between the cumulants of a distribution and its moments are

$$\kappa_2(F) = \mu_2(F) \qquad \kappa_3(F) = \mu_3(F) \qquad \kappa_4(F) = \mu_4(F) - 3\mu_2^2(F)$$

where μ_2, μ_3, and μ_4 are second, third, and fourth central moments, respectively [Stuart and Ord (1987, p. 87)]. The density of A is denoted by $a(S_t)$, where S_t is a random stock price at option expiration. Derivatives of $a(S_t)$ are evaluated at the strike price K. The remainder $\varepsilon(K)$ continues the Edgeworth series with terms based on higher order cumulants and derivatives. Jarrow and Rudd suggest that with a good choice for the distribution, A, higher order terms in the remainder are likely to be negligible. Because of its preeminence in option pricing theory and practice, Jarrow and Rudd suggest the lognormal distribution as a good approximating distribution. When the distribution, A, is lognormal, $C(A)$ becomes the familiar Black–Scholes call price formula.

The call price in eq. (1) corresponds to Jarrow and Rudd's (1982, p. 354) first option price approximation method. This method selects an approximating distribution that equates second cumulants of F and A, i.e., $\kappa_2(F) = \kappa_2(A)$. Econometric reasons discussed later in the empirical results section of this article justify a preference for this first approximation method.

Dropping the remainder term, $\varepsilon(K)$, the option price in eq. (1) can be compactly restated in this form

$$C(F) = C(A) + \lambda_1 Q_3 + \lambda_2 Q_4 \quad (2)$$

where the terms, λ_j and Q_j, above are defined as follows.

$$\lambda_1 = \gamma_1(F) - \gamma_1(A) \qquad Q_3 = -(S_0 e^{rt})^3 (e^{\sigma^2 t} - 1)^{3/2}\frac{e^{-rt}}{3!}\frac{da(K)}{dS_t} \quad (3a)$$

$$\lambda_2 = \gamma_2(F) - \gamma_2(A) \qquad Q_4 = (S_0 e_{rt})^4 (e^{\sigma^2 t} - 1)^2 \frac{e^{-rt}}{4!}\frac{d^2a(K)}{dS_t^2} \quad (3b)$$

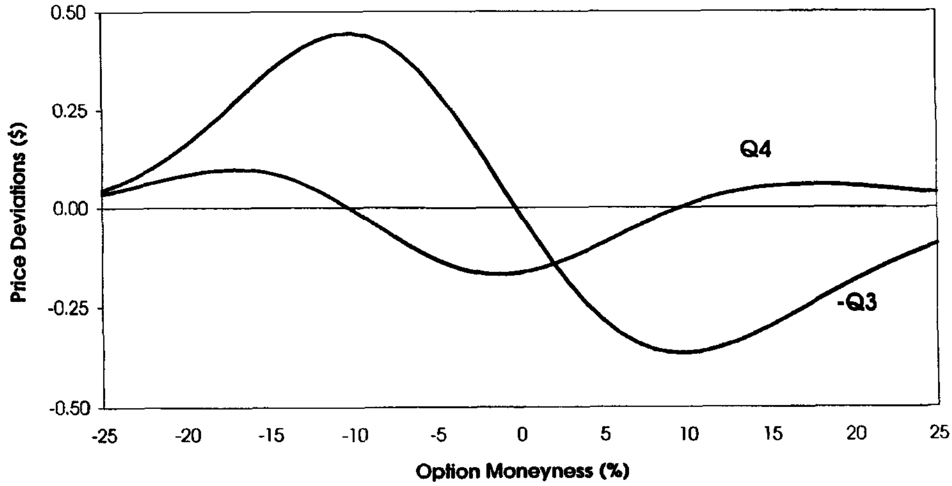

FIGURE 1
Skewness ($-Q_3$) and kurtosis (Q_4) deviations.

In eq. (3), $\gamma_1(F)$ and $\gamma_1(A)$ are skewness coefficients for the distributions, F and A, respectively. Similarly, $\gamma_2(F)$ and $\gamma_2(A)$ are excess kurtosis coefficients. Skewness and excess kurtosis coefficients are defined in terms of cumulants as follows [Stuart and Ord (1987, p. 107)].

$$\gamma_1(F) = \frac{\kappa_3(F)}{\kappa_2^{3/2}(F)} \qquad \gamma_2(F) = \frac{\kappa_4(F)}{\kappa_2^2(F)} \tag{4}$$

In eq. (3) above, the terms, Q_3 and Q_4, represent skewness and kurtosis deviations from lognormality. To graphically assess the nature of these deviations from lognormality, Figure 1 plots $-Q_3$ and Q_4 in an example where $S_0 = 100$, $\sigma = .15$, $t = 1/4$, $r = 0.05$, and K varies from 75 to 125. The horizontal axis measures option moneyness, defined as the percentage difference between a discounted strike price and a stock price, i.e.,

$$\text{Moneyness}(\%) = \frac{Ke^{-rt} - S_0}{Ke^{-rt}} \times 100$$

and the vertical axis measures values for $-Q_3$ and Q_4. The most telling observation from Figure 1 is that negative skewness deviations from lognormality ($-Q_3$) cause the Black–Scholes formula to underprice in-the-money call options and overprice out-of-the-money call options.

DATA SOURCES

This study is based on the CBOE market for S&P 500 (SPX) index options. Rubinstein (1994) argues that this market best approximates conditions required for the Black–Scholes model. Nevertheless, Jarrow and Rudd (1982, 1983) point out that a stock index distribution is a convolution of its component distributions. Therefore, when the Black–Scholes model is the correct model for individual stocks, it is an approximation in the case of index options.

Intraday price data for this study come from the Berkeley Options Data Base of Chicago Board Options Exchange (CBOE) options trading. Index levels, strike prices, and option maturities also come from the Berkeley data base. To avoid bid–ask bounce problems in transaction prices, option prices are taken as midpoints of CBOE dealers' bid–ask price quotations. The risk free rate of interest is measured as U.S. Treasury bill rates with maturities closest to option contract expirations. Interest rate information is culled from the *Wall Street Journal.* Since S&P 500 index options are European style, Black's (1975) method is used to adjust index levels by subtracting present values of dividend payments made before each option's expiration date. Daily S&P 500 index dividends are gathered from the *S&P 500 Information Bulletin.*

Following data screening procedures in Barone-Adesi and Whaley (1986), all option prices less than \$0.125 and all transactions occurring before 9:00 A.M. are deleted. Obvious outliers are also purged from the sample; including recorded option prices lying outside well-known no-arbitrage option price boundaries [Merton (1973)]. The reported results obtained from option price quotations for contracts traded in March, 1990.

ESTIMATION PROCEDURES AND EMPIRICAL RESULTS

The first set of estimation procedures assesses the out-of-sample performance of the Black–Scholes option pricing model. In these procedures, implied standard deviations (*ISD*) are estimated on a daily basis for call options on the S&P 500 index. Specifically, on the day prior to a given current day for a given option maturity class, a unique *ISD* is obtained from all bid–ask price midpoints using Whaley's (1982) simultaneous equations procedure. This prior-day, out-of-sample *ISD* becomes an input used to calculate current-day theoretical Black–Scholes option prices for all price observations within the same maturity class. These theoretical

Black–Scholes prices are then compared to their corresponding market-observed prices.

The second set of estimation procedures assesses the out-of-sample performance of the skewness- and kurtosis-adjusted Black–Scholes option pricing formula proposed by Jarrow and Rudd (1982). On the day prior to a given current day, implied standard deviation (*ISD*), implied skewness (*ISK*), and implied kurtosis (*IKT*) parameters are simultaneously estimated using all bid–ask midpoints for a given option maturity class. These prior-day, out-of-sample parameter estimates provide inputs used to calculate current-day, theoretical option prices for all options within the same maturity class. Theoretical skewness, and kurtosis-adjusted Black–Scholes option prices are then compared to their corresponding market-observed prices.

The Black–Scholes Option Pricing Model

The Black–Scholes formula requires five inputs: a security price, a strike price, a risk-free interest rate, an option maturity, and a return standard deviation. The first four inputs come directly from market data. Only the return standard deviation is not directly observable. Return standard deviations are estimated from values implied by options using the simultaneous equations procedure suggested by Whaley (1982). A simultaneous equations estimate is the value of the argument *BSISD* that minimizes the following sum of squares.

$$\min_{BSISD} \sum_{j=1}^{N} [C_{OBS,j} - C_{BS,j}(BSISD)]^2 \tag{5}$$

In eq. (5) above, N is the number of bid–ask price quotations available on a given day for a given maturity class, C_{OBS} is a market-observed call price, and C_{BS} (*BSISD*) is a theoretical Black–Scholes call price calculated using the standard deviation parameter *BSISD*. Using the prior-day value of *BSISD* that minimizes the sum of squared errors in eq. (5), theoretical Black–Scholes option prices are calculated for all options in a current-day sample within the same maturity class. These theoretical Black–Scholes option prices are then compared to their corresponding market-observed prices.

Table I summarizes calculations for S&P 500 index call option prices observed during March, 1990 for options maturing in April, 1990. Column 1 lists sampling dates within the month and column 2 lists the number of price quotations available on each date. To maintain table

TABLE I

Comparison of Black–Scholes Prices and Observed Prices of S&P 500 Index Options (SPX)

Date	*Number of Price Observations*	*Implied Standard Deviation (%)*	*Proportion of Theoretical Prices Outside Bid–Ask Spreads*	*Average Deviation of Theoretical Price From Spread Boundaries ($)*	*Average Call Price ($)*	*Average Bid–Ask Spread ($)*
90/3/2	706	19.25	0.85	0.78	19.85	0.59
90/3/6	560	18.73	0.93	0.80	21.64	0.62
90/3/8	813	17.39	0.96	0.81	23.43	0.47
90/3/12	773	16.98	0.95	0.88	20.76	0.44
90/3/14	673	17.82	0.95	0.65	20.02	0.45
90/3/16	756	16.20	0.79	0.65	19.12	0.42
90/3/20	1108	15.71	0.81	0.40	19.49	0.44
90/3/22	1224	16.80	0.91	0.61	17.83	0.45
90/3/26	856	18.16	0.91	0.53	22.26	0.47
90/3/28	861	19.37	0.86	0.62	19.92	0.42
90/3/30	970	17.10	0.82	0.36	18.07	0.43
Average	845	17.59	0.88	0.64	20.22	0.47

On each day indicated, a Black–Scholes implied standard deviation (*BSISD*) is estimated from current price observations. Theoretical Black–Scholes option prices are then calculated using BSISD. All observations correspond to call options traded in March, 1990 and expiring in April, 1990.

compactness, results are reported for only the even-numbered dates within the month. To assess out-of-sample forecasting power of the Black–Scholes model, implied standard deviations (*BSISD*) reported in column 3 are estimated from prices observed on trading days immediately prior to dates listed in column 1. For example, the first row of Table II lists the date March 2, 1990, but column 3 reports standard deviation estimates obtained from March 1 prices. Thus, *BSISD* values reported in column 3 are prior-day estimates. These prior-day *BSISD* values are used to calculate theoretical Black–Scholes option prices for all price observations on even-numbered dates listed in column 1. To help assess the economic significance of differences between theoretical and observed prices, column 4 lists the proportion of theoretical Black–Scholes option prices lying outside their corresponding bid–ask spreads, either below a bid price or above an asked price. In addition, column 5 lists average absolute deviations of theoretical prices from bid–ask spread boundaries for only those theoretical prices lying outside their respective bid–ask spreads. Specifically, for each theoretical option price lying outside its respective bid–ask spread, an absolute deviation is calculated using the following formula.

TABLE II

Comparison of Skewness- and Kurtosis-Adjusted Black–Scholes Prices and Observed Prices of S&P 500 Index Options (SPX)

Date	*Number of Price Observations*	*Implied Standard Deviation (%)*	*Implied Skewness (ISK)*	*Implied Kurtosis (IKT)*	*Proportion of Theoretical Prices Outside the Bid–Ask Spread*	*Average Deviation of Theoretical Prices From Spread Boundaries ($)*
90/3/2	706	17.96	−1.29	3.53	0.48	0.25
90/3/6	560	17.63	−1.38	3.91	0.42	0.17
90/3/8	813	16.38	−1.38	3.80	0.53	0.22
90/3/12	773	16.44	−1.38	3.75	0.69	0.28
90/3/14	673	16.59	−1.35	3.42	0.41	0.16
90/3/16	756	16.16	−1.26	4.57	0.79	0.51
90/3/20	1108	15.49	−1.10	4.60	0.58	0.23
90/3/22	1224	16.24	−0.93	4.45	0.79	0.54
90/3/26	856	17.65	−0.84	4.53	0.65	0.26
90/3/28	861	18.95	−1.08	6.15	0.76	0.33
90/3/30	970	17.21	−0.83	4.90	0.57	0.20
Average	845	16.97	−1.16	4.33	0.61	0.29

On each day indicated, implied standard deviation (*ISD*), skewness (*ISK*), and kurtosis (*IKT*) parameters are estimated from one-day lagged price observations. Theoretical option prices are then calculated using these out-of-sample implied parameters. All observations correspond to call options traded in March, 1990 and expiring in April, 1990.

$$\max(C_{BS}(BSISD) - Ask, Bid - C_{BS}(BSISD))$$

This absolute deviation statistic measures the magnitude of deviations of theoretical option prices from CBOE dealers' bid–ask spreads. Finally, column 6 lists day-by-day averages of observed call prices and column 7 lists day-by-day averages of observed bid–ask spreads.

The bottom row of Table I reports column averages for all variables. For example, the average number of daily price observations is 845 (column 2), with an average option price of $20.22 (column 6), and an average bid–ask spread of $0.47 (column 7). The average implied standard deviation is 17.59% (column 3). Regarding the ability of the Black–Scholes model to mimic CBOE dealers' prices, the average proportion of theoretical Black–Scholes prices lying outside their respective bid–ask spreads is 88% (column 4) with an average deviation of $0.64 for those theoretical prices actually lying outside a spread boundary.

The significance of an average deviation of $0.64 is placed in perspective by recalling that the multiplier for S&P 500 (SPX) index options is 100. Thus, a deviation of $0.64 on a quoted price is a deviation of $64 on a single contract, which is not trivial to CBOE dealers making markets in SPX options. Furthermore, broker commissions on retail option trans-

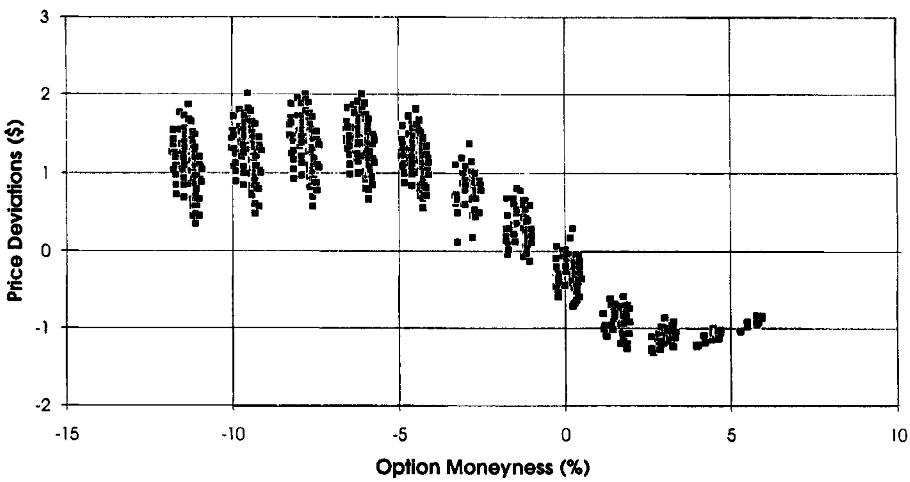

FIGURE 2
Black–Scholes formula.

actions typically have a fixed charge of $25 to $30 plus $2 to $4 per contract. Therefore, an investor believing the Black–Scholes model accurately measures option value might be motivated to trade on a contract mispricing of $34, or less for multiple contract trades. This should not be interpreted as indicating the availability of arbitrage opportunities. In an efficient market, arbitrage opportunities should not exist. But this does suggest that an investor believing the options market is inefficient because market prices deviate from Black–Scholes prices has sufficient economic incentive to trade on that belief. This is not to suggest, however, that any option pricing model can be used to identify genuine arbitrage opportunities in the very active S&P 500 index options market.

Figure 2 visually represents deviations of observed call prices from theoretical Black–Scholes call prices for the 773 price quotations recorded on March 12, 1990. The horizontal axis measures option moneyness and the vertical axis measures price deviations. As previously defined, option moneyness is the percentage difference between a discounted strike price and a dividend-adjusted stock index level. A negative (positive) percentage corresponds to in-the-money (out-of-the-money) call options with low (high) strike prices. Price deviations measured on the vertical axis are observed prices minus theoretical prices. So defined, the zero horizontal axis corresponds to theoretical Black–Scholes prices and the dots correspond to observed call prices relative to Black–Scholes prices. Because many price quotations are unchanged updates made throughout the day, the number of visually distinguishable dots is somewhat smaller than the actual number of price quotations.

Figure 2 graphically reveals that the Black–Scholes formula tends to overvalue out-of-the-money options and undervalue in-the-money options for this sample of S&P 500 index call options. For example, observed option prices deep in-the-money or out-of-the-money often deviate from theoretical prices by one dollar or more. Options dealers often rectify these deficiencies by assigning a different volatility value to each strike price. But this remedy is ad hoc, and even highlights model deficiency since volatility is a unique property of the underlying index.

The Jarrow–Rudd Option Pricing Formula

In the second set of estimation procedures, on a given day within a given option maturity class return standard deviation, skewness and kurtosis parameters are estimated simultaneously by minimizing the following sum of squares with respect to the arguments, ISD, L_1, and L_2, respectively.

$$\min_{ISD, L_1, L_2} \sum_{j=1}^{N} [C_{OBS,j} - (C_{BS,j}(ISD) + L_1\, Q_3 + L_2\, Q_4)]^2 \tag{6}$$

The coefficients, L_1 and L_2, estimate the parameters, λ_1 and λ_2, respectively, defined in eq. (3), where the terms, Q_3 and Q_4, are also defined. These estimates yield implied coefficients of skewness (ISK) and kurtosis (IKT) calculated as follows, where $\gamma_1(A)$ and $\gamma_2(A)$ are defined in eq. (4).

$$ISK = L_1 + \gamma_1(A(ISD)) \qquad IKT = 3 + L_2 + \gamma_2(A(ISD))$$

ISK estimates the skewness parameter, $\gamma_1(F)$, and IKT estimates the kurtosis parameter, $3 + \gamma_2(F)$. The simultaneously estimated values for ISD, ISK, and IKT represent maximum likelihood estimates of implied standard deviation, implied skewness, and implied kurtosis parameters based on N price observations. Substituting ISD, L_1, and L_2 estimates into eq. (2), a skewness- and kurtosis-adjusted Jarrow–Rudd option price, i.e., C_{JR}, is then expressed as the following sum of a Black–Scholes option price plus adjustments for skewness and kurtosis deviations from lognormality.

$$C_{JR} = C_{BS}(ISD) + L_1\, Q_3 + L_2\, Q_4 \tag{7}$$

Equation (7) yields theoretical skewness- and kurtosis-adjusted Black–

Scholes option prices. Based on this equation, deviations of theoretical prices are calculated from market-observed prices.

Table II summarizes calculations for the same S&P 500 index call option prices used to compile Table I. As a result, column 1 in Table II correctly lists the same even-numbered dates and column 2 lists the same number of price quotations as Table I. To assess out-of-sample forecasting power of skewness- and kurtosis-adjustments, implied standard deviation (*ISD*), implied skewness (*ISK*), and implied kurtosis (*IKT*) parameters are estimated from prices observed on trading days immediately prior to dates listed in column 1. For example, for the date March 2, 1990, columns 3, 4, and 5 report standard deviation, skewness, and kurtosis estimates obtained from March 1 prices. Thus, out-of-sample parameters, *ISD, ISK,* and *IKT,* reported in columns 3, 4, and 5, respectively, are prior-day estimates. These prior-day estimates of *ISD, ISK,* and *IKT* provide inputs used to calculate theoretical skewness- and kurtosis-adjusted Black–Scholes option prices according to eq. (7) for all price observations on even-numbered dates listed in column 1. These theoretical prices based on out-of-sample *ISD, ISK,* and *IKT* values are then used to calculate daily proportions of theoretical prices lying outside bid–ask spreads (column 6) and daily averages of deviations from spread boundaries (column 7). Like Table I, the bottom row of Table II reports column averages.

In Table I, skewness coefficients (column 4) on all days are negative, with a column average of -1.16. All kurtosis coefficients (column 5) are greater than 3, with a column average of 4.33. Proportions of skewness- and kurtosis-adjusted prices lying outside their respective bid–ask spread boundaries (column 6) have a column average of 61%. Average absolute deviations of theoretical prices from bid–ask spread boundaries for only those prices lying outside bid–ask spreads (column 7) have a column average deviation of \$0.29. This is less than half the size of the average \$0.64 deviation reported in Table I. An average deviation of \$0.29 for a quoted price is a deviation of \$29 for a single contract. A \$29 contract mispricing may not be trivial to CBOE members, but it does eliminate the incentive to trade for investors who must pay brokerage commissions in the range of \$20 to \$30 per contract.

The statistical significance of the performance improvement from out-of-sample adjustments for skewness and kurtosis can be assessed using the following Z-statistic for the difference between two proportions [Hoel (1984)].

$$Z = \frac{p_1 - p_2}{\sqrt{p_1(1 - p_1)/N_1 + p_2(1 - p_2)/N_2}}$$

In this statistic, p_1 and p_2 are sample proportions, and N_1 and N_2 are sample sizes corresponding to these proportions. For example, from Table I the volume-weighted average proportion is $p_1 = .88$ and from Table II the volume-weighted average proportion is $p_2 = .62$. These proportions are almost identical to simple averages reported in Tables I and II. In this test, N_1 and N_2 are both equal to the sample size of $N = 9300$ observations for all even-numbered days in the month. A quick computation yields a Z-statistic value of 42.9, which is statistically significant at more than a 99.99% level of confidence.

Figure 3 graphically displays deviations of market-observed option prices from theoretical skewness- and kurtosis-adjusted Black–Scholes option prices for the 773 price quotations observed on March 12, 1990. Like Figure 2, Figure 3 measures option moneyness on the horizontal axis, where negative (positive) moneyness corresponds to in-the-money (out-of-the-money) call options with low (high) strike prices. Dollar deviations measured on the vertical axis are calculated as market-observed prices less theoretical prices. To aid visual comparison, Figures 2 and 3 share the same scale. The comparison of Figure 3 with Figure 2 is dramatic. Figure 3 reveals that out-of-sample skewness and kurtosis adjustments account for almost all strike price biases of the Black–Scholes model for this sample of S&P 500 index options.

Joint and Separate Skewness and Kurtosis Adjustments

Results presented above illustrate the effectiveness of simultaneously estimated adjustments for skewness and kurtosis. To assess the effectiveness of separately estimated adjustments for skewness and kurtosis, Table III reports results from separate skewness estimates used to adjust the Black–Scholes formula. Similarly, Table IV reports results from separate kurtosis estimates used to adjust the Black–Scholes formula. By format construction, Tables III and IV are directly comparable with each other and with Table II.

Comparing Tables II and III, note that skewness adjustments estimated separately provide almost the same benefit as skewness and kurtosis adjustments estimated simultaneously. For example, average proportions (column 6) of theoretical prices lying outside of bid–ask spreads in both tables are almost the same, 0.61 (Table II) versus 0.64 (Table III). Similarly, average deviations (column 7) of theoretical prices from spread boundaries are almost the same, \$0.29 (Table II) versus \$0.31 (Table III). By contrast, comparing Tables II and IV, note that separate

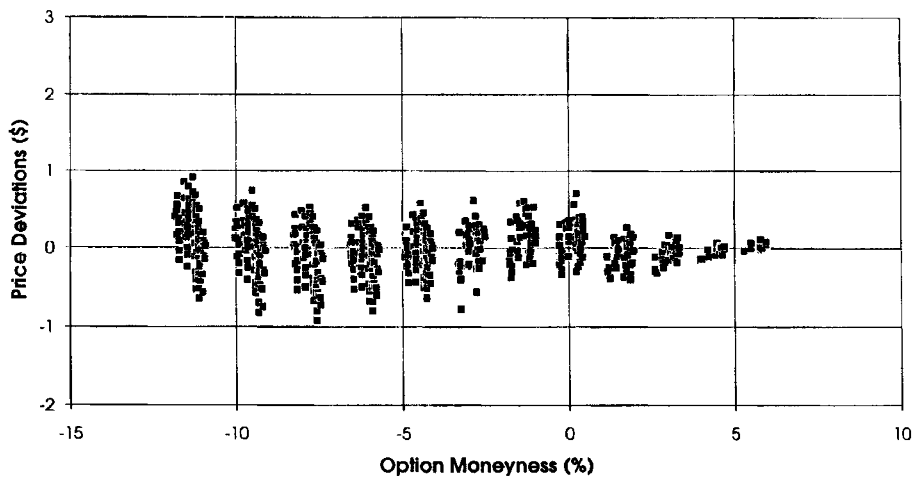

FIGURE 3
Jarrow–Rudd formula.

TABLE III

Comparison of Skewness- and Kurtosis-Adjusted Black–Scholes Prices and Observed Prices of S&P 500 Index Options (SPX)

Date	*Number of Price Observations*	*Implied Standard Deviation (%)*	*Implied Skewness (ISK)*	*Implied Kurtosis (IKT)*	*Proportion of Theoretical Prices Outside the Bid–Ask Spread*	*Average Deviation of Theoretical Prices From Spread Boundaries ($)*
90/3/2	706	17.71	−1.33	n/a	0.51	0.27
90/3/6	560	17.23	−1.47	n/a	0.45	0.21
90/3/8	813	16.06	−1.48	n/a	0.59	0.25
90/3/12	773	16.16	−1.46	n/a	0.71	0.31
90/3/14	673	16.39	−1.41	n/a	0.43	0.21
90/3/16	756	15.37	−1.56	n/a	0.80	0.49
90/3/20	1108	14.80	−1.35	n/a	0.62	0.27
90/3/22	1224	15.63	−1.18	n/a	0.80	0.54
90/3/26	856	16.78	−1.15	n/a	0.72	0.30
90/3/28	861	17.32	−1.68	n/a	0.80	0.33
90/3/30	970	16.20	1.19	n/a	0.66	0.22
Average	845	16.33	−1.39	n/a	0.64	0.31

On each day indicated, implied standard deviation (*ISD*), skewness (*ISK*), and kurtosis (*IKT*) parameters are estimated from one-day lagged price observations. Theoretical option prices are then calculated using these out-of-sample implied parameters. All observations correspond to call options traded in March, 1990 and expiring in April, 1990.

TABLE IV

Comparison of Skewness- and Kurtosis-Adjusted Black–Scholes Prices and Observed Prices of S&P 500 Index Options (SPX)

Date	*Number of Price Observations*	*Implied Standard Deviation (%)*	*Implied Skewness (ISK)*	*Implied Kurtosis (IKT)*	*Proportion of Theoretical Prices Outside the Bid–Ask Spread*	*Average Deviation of Theoretical Prices From Spread Boundaries ($)*
90/3/2	706	20.78	n/a	3.09	0.51	0.86
90/3/6	560	20.16	n/a	3.08	0.81	0.62
90/3/8	813	18.56	n/a	3.06	0.85	0.56
90/3/12	773	18.29	n/a	3.06	0.91	0.70
90/3/14	673	19.11	n/a	3.06	0.71	0.47
90/3/16	756	18.17	n/a	3.05	0.80	0.67
90/3/20	1108	17.04	n/a	3.04	0.62	0.36
90/3/22	1224	17.80	n/a	3.04	0.84	0.68
90/3/26	856	19.47	n/a	3.04	0.76	0.36
90/3/28	861	20.97	n/a	3.04	0.70	0.57
90/3/30	970	18.66	n/a	3.03	0.67	0.26
Average	845	19.00	n/a	3.05	0.74	0.56

On each day indicated, implied standard deviation (*ISD*), skewness (*ISK*), and kurtosis (*IKT*) parameters are estimated from one-day lagged price observations. Theoretical option prices are then calculated using these out-of-sample implied parameters. All observations correspond to call options traded in March, 1990 and expiring in April, 1990.

kurtosis estimates and adjustments yield noticeably inferior results. However, the separate skewness estimates in Table III (*ISK*, column 4) are slightly larger than the simultaneous skewness estimates in Table II (*ISK*, column 4) on every day listed, suggesting that simultaneous skewness and kurtosis estimation may be important in avoiding an omitted variable bias.

Higher Order Moment Estimates

At first inspection, it might appear that price adjustments beyond skewness and kurtosis could add further improvements to the procedures specified above. Seemingly, this is no more problematic than deriving higher order analogues for the terms Q_3 and Q_4, say Q_5 and Q_6, to augment the estimation procedure specified in eq. (6). These would then yield estimates of fifth and sixth moment deviations from lognormality. Unfortunately, including additional terms creates severe collinearity problems since all even-numbered subscripted terms, e.g., Q_4 and Q_6, are highly correlated with each other. Similarly, all odd-numbered subscripted terms, e.g., Q_3 and Q_5, are also highly correlated. Consequently, adding the terms, Q_5 and/or Q_6, to skewness and kurtosis estimation procedures leads to highly unstable parameter estimates.

Variance Correction Terms

Jarrow and Rudd's (1982, p. 352) general formula contains a price-variance correction, which is a part of their second approximation method. A price-variance correction term is defined here as Q_2 analogously to skewness and kurtosis correction terms defined in eq. (3). Adding Q_2 to the skewness and kurtosis estimation procedure defined in eq. (6) yields unstable parameter estimates, because Q_2 and Q_4 are highly correlated. Furthermore, squared implied standard deviations estimated via eq. (6) are estimates of log-price variances, i.e., σ^2. Consequently, simultaneous estimates of price variances and log-price variances suffer from overidentification. These problems are avoided by using Jarrow and Rudd's (1982, p. 354) first approximation method based on equating second cumulants of the distributions, F and A, i.e., $\kappa_2(F) = \kappa_2(A)$. In this case, when the approximating distribution, A, is lognormal the log-price variance, σ^2, is a solution to the equality, $\kappa_2(F) = \kappa_1^2(A)(e^{\sigma^2 t} - 1)$ [Jarrow and Rudd (1982, p. 354)].

Further Empirical Results from Put Options

All procedures leading to Table I and II are applied also to S&P 500 index put options traded in March, 1990 and maturing in April, 1990. Put options present no new problems since S&P 500 index options are European style and a skewness- and kurtosis-adjusted Black–Scholes formula is simply derived through put–call parity. Tables V and VI present empirical results obtained from this sample of S&P 500 index put options. Table V reports an average number of daily price observations of 562 (column 2), with an average option price of \$8.25 (column 6) and an average bid–ask spread of \$0.39 (column 7). The average proportion of theoretical prices lying outside their corresponding bid–ask spreads is 83% (column 4) with an average deviation of \$0.55 for observations lying outside a spread boundary.

Table VI summarizes calculations for the same S&P 500 put option prices used to compile Table V, where out-of-sample parameters, *ISD*, *ISK*, and *IKT*, represent one-day lagged estimates used to calculate theoretical skewness- and kurtosis-adjusted Black–Scholes put option prices. Again, all skewness coefficients (column 4) are negative, with a column average of -1.13; and all kurtosis coefficients (column 5) are greater

TABLE V

Comparison of Black–Scholes Prices and Observed Prices of S&P 500 Index Options (SPX)

Date	*Number of Price Observations*	*Implied Standard Deviation (%)*	*Proportion of Theoretical Prices Outside Bid–Ask Spreads*	*Average Deviation of Theoretical Price From Spread Boundaries ($)*	*Average Put Price ($)*	*Average Bid–Ask Spread ($)*
90/3/2	468	16.92	0.86	0.67	10.40	0.43
90/3/6	550	16.12	0.89	0.74	9.11	0.43
90/3/8	507	15.52	0.85	0.64	8.43	0.42
90/3/12	401	15.87	0.87	0.63	8.49	0.40
90/3/14	463	15.71	0.94	0.56	7.73	0.39
90/3/16	455	14.35	0.80	0.42	9.09	0.34
90/3/20	745	15.12	0.70	0.42	7.23	0.40
90/3/22	988	18.09	0.81	0.76	7.89	0.40
90/3/26	325	15.47	0.89	0.46	6.99	0.35
90/3/28	515	15.41	0.82	0.35	9.50	0.29
90/3/30	772	16.78	0.71	0.43	5.85	0.38
Average	562	15.98	0.83	0.55	8.25	0.39

On each day indicated, a Black-Scholes implied standard deviation (*BSISD*), is estimated from current price observations. Theoretical Black-Scholes option prices are then calculated using *BSISD*. All observations correspond to put options traded in March, 1990 and expiring in April, 1990.

than 3, with a column average of 4.77. Column 6 lists proportions of skewness- and kurtosis-adjusted prices lying outside bid–ask spread boundaries, where the column average proportion is 66%. Column 7 lists average absolute deviations of theoretical prices from bid–ask spread boundaries. The column average deviation is $0.26, which is about one-half the average deviation of $0.55 reported in Table V. Overall, results obtained from put options parallel those obtained from call options.

In addition to the empirical results reported in this article, all procedures are applied to data for S&P 500 index option contracts traded in December, 1990; October, 1993; and December, 1993 with essentially identical results. These results are contained in an appendix available from the authors upon request.

Related Empirical Studies

Jarrow and Rudd (1983) test their extended formula on an options sample for individual stocks. Unlike methods employed in this article, they rely on historical estimates of variance, skewness, and kurtosis to construct price adjustments. They find that their extended formula correctly pre-

TABLE VI

Comparison of Skewness- and Kurtosis-Adjusted Black–Scholes Prices and Observed Prices of S&P 500 Index Options (SPX)

Date	*Number of Price Observations*	*Implied Standard Deviation (%)*	*Implied Skewness (ISK)*	*Implied Kurtosis (IKT)*	*Proportion of Theoretical Prices Outside the Bid–Ask Spread*	*Average Deviation of Theoretical Prices From Spread Boundaries ($)*
90/3/2	468	18.15	−1.33	4.24	0.51	0.12
90/3/6	550	17.64	−1.43	4.53	0.61	0.34
90/3/8	507	17.19	−1.51	4.60	0.55	0.20
90/3/12	401	17.36	−1.50	4.49	0.52	0.16
90/3/14	463	17.20	−1.31	4.85	0.70	0.28
90/3/16	455	15.63	−0.96	5.67	0.68	0.20
90/3/20	745	16.54	−0.78	5.41	0.68	0.26
90/3/22	988	19.62	−0.92	5.01	0.91	0.57
90/3/26	325	16.63	−1.13	4.21	0.81	0.29
90/3/28	515	16.07	−0.82	5.04	0.67	0.14
90/3/30	772	18.15	−0.69	5.85	0.61	0.31
Average	562	17.34	−1.13	4.77	0.66	0.26

On each day indicated, implied standard deviation (*ISD*), skewness (*ISK*), and kurtosis (*IKT*) parameters are estimated from one-day lagged price observations. Theoretical option prices are then calculated using these out-of-sample implied parameters. All observations correspond to put options traded in March, 1990 and expiring in April 1990.

dicts the sign of Black–Scholes price deviations from dealers' bid–ask spread boundaries 10¢ or more in magnitude almost 70% of the time. Jarrow and Rudd conclude that significant differences between market prices and Black–Scholes prices can be partially attributed to departures from lognormality in the underlying security prices.

Empirical procedures employed in this article can be viewed as extensions to specification tests of the Black–Scholes model used by Whaley (1982). For example, Whaley separately regresses percentage deviations of observed stock option prices from theoretical Black–Scholes prices on a battery of variables, of which the most relevant to this study is option moneyness. Whaley finds that percentage deviations of observed stock option prices from Black–Scholes prices are negatively correlated with option moneyness. In further tests, Whaley (1982) uses the Black–Scholes model to identify potential arbitrage opportunities among traded stock options, but finds that options so selected are efficiently priced since no abnormal profits are realizable after accounting for reasonable trading costs. Whaley's findings convincingly argue that the options market is efficient, as evidenced by the Black–Scholes model's inability to identify genuine arbitrage opportunities. Indeed, in an efficient market it is expected that no model is able to identify genuine arbitrage opportunities. This article assumes that the options market is efficient. Consequently, the inquiry is confined to tests assessing the ability of an option pricing model to accurately mimic market prices.

SUMMARY AND CONCLUSION

This study empirically tests an expanded version of the Black–Scholes (1973) option pricing model originally proposed by Jarrow and Rudd (1982) that accounts for skewness and kurtosis deviations from lognormality in security price distributions. The Jarrow and Rudd extended formula is applied to estimate coefficients of skewness and kurtosis implied by S&P 500 index option prices. Relative to a lognormal distribution, significant negative skewness and positive excess kurtosis are found in the distribution of S&P 500 index prices. The methodology employed in this article extends the widely used procedure of obtaining implied standard deviations to also include procedures to simultaneously obtain implied coefficients of skewness and kurtosis. It is found that out-of-sample price adjustments for skewness and kurtosis effectively accounts for most strike price biases associated with the Black–Scholes formula for S&P 500 index options.

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