Implied Volatility Skews and Stock Index Skewness and Kurtosis Implied by S&P 500 Index Option Prices

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The Black-Scholes option pricing model is used to value a wide range of option contracts. It often prices deep in-the-money and deep out-of-the-money options inconsistently, a phenomenon we refer to as a volatility "skew" or "smile."

This article applies an extension of the Black-Scholes model developed by Jarrow and Rudd to an investigation of S&P 500 index option prices. Non-normal skewness and kurtosis in option-implied distributions of index returns are found to contribute significantly to the phenomenon of volatility skews.

The Black-Scholes [1973] option pricing model provides the foundation of modern option pricing theory. In actual application, however, the model often inconsistently prices deep in-the-money and deep out-of-the-money options. Options professionals call this effect the volatility "skew" or "smile."

A volatility skew is the anomalous pattern that results from calculating implied volatilities across a range of strike prices. Typically, the skew pattern is systematically related to the degree to which the options are in or out of the money. This phenomenon is not predicted by the Black-Scholes model, since volatility is a property of the underlying instrument, and the same implied volatility value should be observed across all options on the same instrument.

The Black-Scholes model assumes that stock prices are lognormally distributed, which implies in turn that stock log-prices are normally distributed. Hull [1993] and Natenburg [1994] point out that volatility skews are a consequence of empirical violations of the normality assumption.

In this article, we investigate volatility skew patterns embedded in S&P 500 index option prices. We adapt a method developed by Jarrow and Rudd [1982] to extend the Black-Scholes formula to account for non-normal skewness and kurtosis in stock returns. This method fits the first four moments of a distribution to a pattern of empirically observed option prices. The mean of this distribution is determined by option pricing theory, but an estimation procedure is employed to yield implied values for the standard deviation, skewness, and kurtosis of the distribution of stock index prices.

I. NON-NORMAL SKEWNESS AND KURTOSIS IN STOCK RETURNS

It is widely known that stock returns do not always conform well to a normal distribution. As a simple examination, we separately compute the mean, standard deviation, and coefficients of skewness and kurtosis of monthly S&P 500 index returns in each of the seven decades from 1926 through 1995. In Exhibit 1, Panel A reports statistics based on arithmetic returns, and Panel B reports statistics based on log-relative returns. Arithmetic returns are calculated as $P_t/P_{t-1} - 1$, and log-relative returns are calculated as $\log(P_t/P_{t-1})$, where P_t denotes the index value observed at the end of month t.

The returns series used to compute statistics reported in Exhibit 1 do not include dividends. We choose returns without dividends because dividends paid out over the life of a European option do not accrue to the optionholder. Thus European-style S&P 500 index option prices are properly determined by index returns that exclude any dividend payments.

The Black-Scholes model assumes that arithmetic returns are lognormally distributed, or equivalently, that log-relative returns are normally distributed. All normal distributions have a skewness coefficient of zero and a kurtosis coefficient of 3. All lognormal distributions are positively skewed with kurtosis always greater than 3 (see Stuart and Ord [1987]).

We have two observations to make about Exhibit 1. First, reported coefficients of skewness and kurtosis show significant deviations from normality occurring in the first two decades (1926-1935 and 1936-1945) and the most recent decade (1986-1995) of this seventy-year period.

Statistical significance is assessed by noting that population skewness and kurtosis for a normal distribution are 0 and 3, respectively. Also, variances of sample coefficients of skewness and kurtosis from a normal population are $6/n$ and $24/n$, respectively. For each decade, n = 120 months, which is sufficiently large to invoke the central limit theorem and to assume that sample coefficients are normally distributed.

Thus, 95% confidence intervals for a test of index return normality are given by $\pm 1.96 \times \sqrt{6/120} = \pm 0.438$ for sample skewness and $3 \pm 1.96 \times \sqrt{24/120} = 3 \pm 0.877$ for sample kurtosis. For statistics computed from log-relative returns, sample skewness and kurtosis values outside these confidence intervals indi-

EXHIBIT 1
HISTORICAL S&P 500 INDEX RETURN STATISTICS 1926–1995

PANEL A. ARITHMETIC RETURNS

Decade	Mean (%)	Standard Deviation (%)	Skewness	Kurtosis
1926-1935	7.4	35.1	0.77	7.50
1936-1945	5.8	22.4	−0.55	7.15
1946-1955	10.1	13.5	−0.28	3.02
1956-1965	7.8	11.7	−0.52	3.37
1966-1975	2.1	15.9	0.23	4.08
1976-1985	9.5	14.1	0.30	3.67
1986-1995	11.8	14.9	−1.19	9.27
1986-1995*	14.0	13.0	−0.04	4.06

PANEL B. LOG-RELATIVE RETURNS

Decade	Mean (%)	Standard Deviation (%)	Skewness	Kurtosis
1926-1935	1.4	34.5	−0.02	6.23
1936-1945	3.2	22.9	−1.12	8.30
1946-1955	9.2	13.5	−0.39	3.22
1956-1965	7.1	11.7	−0.62	3.43
1966-1975	0.9	15.9	0.03	3.90
1976-1985	8.5	13.9	0.15	3.63
1986-1995	10.6	15.2	−1.67	11.92
1986-1995*	13.0	12.9	−0.20	4.05

Means and standard deviations are annualized. 95% confidence intervals for normal sample skewness and kurtosis coefficients are ±0.438 and 3 ± 0.877, respectively.

*Indicates exclusion of October 1987 crash-month return.

cate statistically significant departures from normality. For statistics obtained from arithmetic returns, a negative sample skewness value outside the appropriate confidence interval indicates a statistically significant departure from lognormality.

Second, statistics reported for the decade 1986-1995 are sensitive to the inclusion of the October 1987 return when the S&P 500 index fell by −21.76%. Including the October 1987 return yields log-relative skewness and kurtosis coefficients of −1.67 and 11.92, respectively, which deviate significantly from a normal specification. Excluding the October 1987 return yields skewness and kurtosis coefficients of −0.20 and 4.05, which are not significant deviations from normality.

The contrasting estimates of S&P 500 index return skewness and kurtosis in the decade 1986-1995 raise an interesting empirical issue regarding the pricing of S&P 500 index options. Specifically, do post-crash option prices embody an ongoing market perception of the possibility of another market crash similar to that of October 1987?

If post-crash option prices retain no memory of the crash, then the near-normal skewness and kurtosis obtained by omitting the October 1987 return suggest that the Black-Scholes model should be well-specified. If post-crash option prices "remember" the crash, however, we would expect to see non-normal skewness and kurtosis in the option-implied distribution of stock returns similar to the sample skewness and kurtosis obtained by including the October 1987 return.

II. JARROW-RUDD SKEWNESS- AND KURTOSIS-ADJUSTED MODEL

The Jarrow-Rudd [1982] option pricing model provides a useful analytic tool to examine the contrasting hypotheses. They propose a method to value European options when the underlying security price at option expiration follows a distribution F known only through its moments. They derive an option pricing formula from an Edgeworth series expansion of the security price distribution F about an approximating distribution A.

Their simplest option pricing formula is the expression for an option price:

$$C(F) = C(A) - e^{rt}\frac{\kappa_3(F) - \kappa_3(A)}{3!}\frac{da(K)}{dS_t} + e^{-rt}\frac{\kappa_4(F) - \kappa_4(A)}{4!}\frac{d^2a(K)}{dS_t^2} + \varepsilon(K) \quad (1)$$

The left-hand side term C(F) denotes a call option price based on the unknown price distribution F. The first right-hand side term C(A) is a call price based on a known distribution A, followed by adjustment terms based on the cumulants $\kappa_j(F)$ and $\kappa_j(A)$ of the distributions F and A, respectively, and derivatives of the density of A. The density of A is denoted by $a(S_t)$, where S_t is a random stock price at option expiration. These derivatives are evaluated at the strike price K. The remainder $\varepsilon(K)$ continues the Edgeworth series with terms based on higher-order cumulants and derivatives.

Cumulants are similar to moments. In fact, the first cumulant of a distribution is equal to the mean, and the second cumulant is equal to the variance. The Jarrow-Rudd model uses third and fourth cumulants. The relationships between third and fourth cumulants and moments for a distribution F are: $\kappa_3(F) = \mu_3(F)$ and $\kappa_4(F) = \mu_4(F) - 3\mu_2^2(F)$, where μ_2^2 is the squared variance, and μ_3, μ_4 denote third and fourth central moments (Stuart and Ord [1987, p. 87]). Thus the third cumulant is the same as the third central moment, and the fourth cumulant is equal to the fourth central moment less three times the squared variance.

Jarrow and Rudd [1982] suggest that with a good choice for the approximating distribution A, higher-order terms in the remainder $\varepsilon(K)$ are likely to be negligible. In essence, the Jarrow-Rudd model relaxes the strict distributional assumptions of the Black-Scholes model without requiring an exact knowledge of the true underlying distribution.

Because of its preeminence in option pricing theory and practice, Jarrow-Rudd suggest the lognormal distribution as a good approximating distribution. When the distribution A is lognormal, C(A) becomes the familiar Black-Scholes call price formula.

In notation followed throughout, the Black-Scholes call price formula is stated in Equation (2), where S_0 is current stock price, K is strike price, r is interest rate, t is time until option expiration, and the parameter σ^2 is the instantaneous variance of the security log-price:

$$C(A) = S_0N(d_1) - Ke^{-rt}N(d_2) \tag{2}$$

$$d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)t}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

Evaluating the lognormal density $a(S_t)$ and its first two derivatives at the strike price K yields the expressions:

$$a(K) = (K\sigma\sqrt{t2\pi})^{-1}\exp(-d_2^2/2)$$

$$\frac{da(K)}{dS_t} = \frac{a(K)(d_2 - \sigma\sqrt{t})}{K\sigma\sqrt{t}}$$

$$\frac{d^2a(K)}{dS_t^2} = \frac{a(K)}{K^2\sigma^2 t} \times$$

$$[(d_2 - \sigma\sqrt{t})^2 - \sigma\sqrt{t}(d_2 - \sigma\sqrt{t}) - 1] \tag{3}$$

The risk-neutral valuation approach adopted by Jarrow and Rudd [1982] implies equality of the first cumulants of F and A, i.e., $\kappa_1(F) = \kappa_1(A) = S_0e^{rt}$. This is equivalent to the equality of the means of F and A, as the first cumulant of a distribution is its mean.

Also, the call price in Equation (1) corresponds to Jarrow and Rudd's first option price approximation method. This method selects an approximating distribution that equates second cumulants of F and A, i.e., $\kappa_2(F) = \kappa_2(A)$. This is equivalent to the equality of the variances of F and A, as the second cumulant of a distribution is equal to its variance. Consequently, Jarrow and Rudd show that when the distribution A is lognormal, the volatility parameter σ^2 is specified as a solution to the equality $\kappa_2(F) = \kappa_1^2(A)(e^{\sigma^2 t} - 1)$.

Dropping the remainder term $\varepsilon(K)$, the Jarrow-Rudd option price in Equation (1) is conveniently restated as

$$C(F) = C(A) + \lambda_1 Q_3 + \lambda_2 Q_4 \tag{4}$$

where the terms λ_j and Q_j, j = 1, 2, are defined as follows:

$$\lambda_1 = \gamma_1(F) - \gamma_1(A)$$

$$Q_3 = -(S_0e^{rt})^3(e^{\sigma^2 t} - 1)^{3/2}\frac{e^{-rt}}{3!}\frac{da(K)}{dS_t} \tag{5A}$$

$$\lambda_2 = \gamma_2(F) - \gamma_2(A)$$

$$Q_4 = (S_0e^{rt})^4(e^{\sigma^2 t} - 1)^2\frac{e^{-rt}}{4!}\frac{d^2a(K)}{dS_t^2} \tag{5B}$$

In Equation (5), $\gamma_1(F)$ and $\gamma_1(A)$ are skewness coefficients for the distributions F and A. Similarly, $\gamma_2(F)$ and $\gamma_2(A)$ are excess kurtosis coefficients. Skewness and excess kurtosis coefficients are defined in terms of cumulants as follows (Stuart and Ord [1987, p. 107]):

$$\gamma_1(F) = \frac{\kappa_3(F)}{\kappa_2^{3/2}(F)} \qquad \gamma_2(F) = \frac{\kappa_4(F)}{\kappa_2^2(F)} \tag{6}$$

When the substitution $q^2 = e^{\sigma^2 t} - 1$ is used to simplify the algebraic expression, coefficients of skewness and excess kurtosis for the lognormal distribution A are defined as:

$$\gamma_1(A) = 3q + q^3$$

$$\gamma_2(A) = 16q^2 + 15q^4 + 6q^6 + q^8 \tag{7}$$

For example, when σ = 15% and t = 0.25, skewness is $\gamma_1(A) = 0.226$, and excess kurtosis is $\gamma_2(A) = 0.091$. Notice that skewness is always positive for the lognormal distribution.

Non-lognormal skewness and kurtosis for $\gamma_1(F)$ and $\gamma_2(F)$ as defined in Equation (6) give rise to implied volatility skews. To illustrate this effect, we generate option prices according to the Jarrow-Rudd option price in Equation (4) using parameter values $\lambda_1 = -0.5$, $\lambda_2 = 5$, $S_0 = 450$, $\sigma = 20\%$, $t = 3$ months, and $r = 4\%$, and strike prices ranging from 400 to 500. Implied volatilities are then calculated for each skewness- and kurtosis-impacted option price using the Black-Scholes formula.

The resulting volatility skew is plotted in Exhibit 2, where the horizontal axis measures strike price, and the vertical axis measures implied standard deviation value. While the true volatility value is $\sigma = 20\%$, Exhibit 2 reveals that implied volatility is greater than true volatility for deep out-of-the-money options, but less than true volatility for deep in-the-money options.

Exhibit 3 shows an empirical volatility skew obtained from S&P 500 index call option price quotes recorded on December 2, 1993, for options expiring in February 1994. The horizontal axis measures option moneyness as the percentage difference between a dividend-adjusted stock index level and a discounted strike price. Positive (negative) moneyness corresponds to in-the-money (out-of-the-money) options with low (high) strike prices.

The vertical axis measures implied standard deviation values. Solid marks represent implied volatilities calculated from observed option prices

EXHIBIT 2
IMPLIED VOLATILITY SKEW

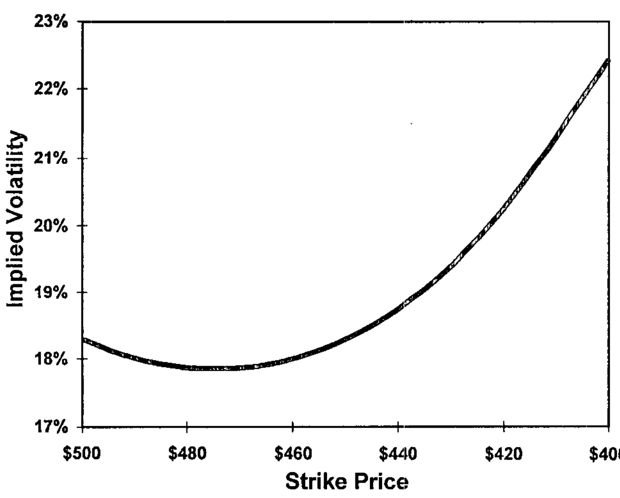
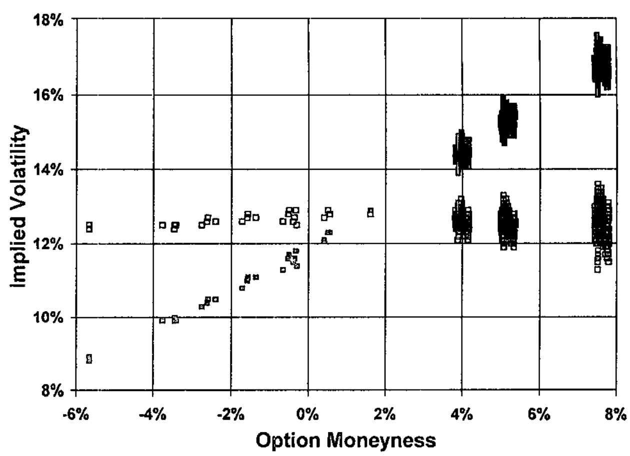


EXHIBIT 3
IMPLIED VOLATILITIES (SPX: 12/02/93)



using the Black-Scholes formula. Hollow marks represent implied volatilities calculated from observed option prices using the Jarrow-Rudd formula.

The Jarrow-Rudd formula uses a single skewness parameter and a single kurtosis parameter across all price observations. The skewness parameter and the kurtosis parameter are estimated by a procedure described in the empirical results section below. There are actually 1,354 price quotes used to form this graph, but the number of visually distinguishable dots is smaller.

Exhibit 3 reveals that Black-Scholes implied volatilities range from about 17% for the deepest in-the-money options (positive moneyness) to about 9% for the deepest out-of-the-money options (negative moneyness). By contrast, Jarrow-Rudd implied volatilities are all close to 12% or 13%, regardless of option moneyness. Comparing Exhibit 3 with Exhibit 2 reveals that the Black-Scholes implied volatility skew for these S&P 500 index options is consistent with negative skewness in the distribution of S&P 500 index prices.

III. DATA SOURCES

We base this study on the market for S&P 500 index options at the Chicago Board Options Exchange (CBOE), the SPX contracts. Rubinstein [1994] argues that this market best approximates conditions required for the Black-Scholes model, although Jarrow and Rudd [1982] point out that a stock index distribution is a convolution of its component distributions. Therefore, when the Black-Scholes model is the correct model for individual stocks, it is only an approximation for stock index options.

Intraday price data come from the Berkeley Options Data Base of CBOE options trading. S&P 500 index levels, strike prices, and option maturities also come from the Berkeley data base. To avoid bid-ask bounce problems in transaction prices, we take option prices as midpoints of CBOE dealers' bid–ask price quotations. The risk-free interest rate is taken as the U.S. Treasury bill rate for a bill maturing closest to option contract expiration. Interest rate information comes from the *Wall Street Journal*.

Since S&P 500 index options are European-style, we use Black's [1975] method to adjust index levels by subtracting present values of dividend payments made before each option's expiration date. Daily S&P 500 index dividends are collected from the "S&P 500 Information Bulletin."

Following data screening procedures in Barone-Adesi and Whaley [1986], we delete all option prices under \$0.125 and all transactions listed as occurring before 9:00am. Obvious outliers are also purged from the sample, including recorded option prices lying outside well-known no-arbitrage option price boundaries (Merton [1973]).

IV. EMPIRICAL RESULTS

We first assess the out-of-sample performance of the Black-Scholes option pricing model without adjusting for skewness and kurtosis. Specifically, using option prices for all contracts within a given maturity series observed on a given day, we estimate a single implied standard deviation using Whaley's [1982] simultaneous equations procedure.

We then use this implied volatility as an input to the Black-Scholes formula to calculate theoretical option prices corresponding to all actual option prices within the same maturity series observed on the following day. Thus, theoretical option prices for a given day are based on a prior-day, out-of-sample implied standard deviation estimate. We then compare these theoretical prices with the actual market prices observed that day.

Next, we assess the skewness- and kurtosis-adjusted Black-Scholes option pricing formula developed by Jarrow and Rudd [1982] using an analogous procedure. Specifically, on a given day, we estimate a single implied standard deviation, a single skewness coefficient, and a single excess kurtosis coefficient using an expanded version of Whaley's [1982] simultaneous equations procedure.

We then use these three parameter estimates as inputs to the Jarrow-Rudd formula to calculate theoretical option prices corresponding to all option prices within the same maturity series observed on the following day. Thus, these theoretical option prices for a given day are based on prior-day, out-of-sample implied standard deviation, skewness, and excess kurtosis estimates. We then compare these theoretical prices with the actual market prices.

Black-Scholes Option Pricing Model

The Black-Scholes formula specifies five inputs: a stock price, a strike price, a risk-free interest rate, an option maturity, and a return standard deviation. The first four inputs are directly observable market data. Since the return standard deviation is not directly observable, we estimate a return standard deviation implied by option prices using Whaley's [1982] simultaneous equations procedure.

This procedure yields a Black-Scholes implied standard deviation (BSISD) that minimizes the sum of squares:

$$\min_{BSISD} \sum_{j=1}^{N} [C_{OBS,j} - C_{BS,j}(BSISD)]^2 \qquad (8)$$

where N denotes the number of price quotations available on a given day for a given maturity series, C_{OBS} represents a market-observed call price, and $C_{BS}(BSISD)$ specifies a theoretical Black-Scholes call price based on the parameter BSISD.

Using a prior-day BSISD estimate, we calculate theoretical Black-Scholes option prices for all contracts in a current-day sample within the same maturity series. We then compare these theoretical Black-Scholes option prices to their corresponding market-observed prices.

Exhibit 4 summarizes results for S&P 500 index call option prices observed in December 1993 for options expiring in February 1994. To save space, we list in the stub only even-numbered dates within the month. Column (1) lists the number of price quotations available on each date. The Black-Scholes implied standard deviation (BSISD) used to calculate theoretical prices for each date is in column (2).

To assess differences between theoretical and observed prices, the next-to-last column gives the proportion of theoretical Black-Scholes option prices lying outside their corresponding bid-ask spreads, either below the bid price or above the asked price. The last column shows the average absolute deviation of theoretical prices from spread boundaries for those prices lying outside their bid-ask spreads.

Specifically, for each theoretical option price lying outside its corresponding bid-ask spread, we compute an absolute deviation according to:

$$\max[C_{BS}(BSISD) - Ask, Bid - C_{BS}(BSISD)]$$

This absolute deviation statistic measures deviations of theoretical option prices from observed bid-ask spreads.

Finally, the two middle columns list day-by-day averages of observed call prices and day-by-day averages of observed bid-ask spreads. Since option contracts are indivisible, all prices are stated on a per contract basis, which for SPX options is 100 times a quoted price.

The last row in Exhibit 4 lists column averages for all variables. For example, the average number of daily price observations is 1,218, with an average contract price of $2,231.35, and an average bid-ask spread of $56.75. The average implied standard deviation is 12.88%. The average proportion of theoretical Black-Scholes prices lying outside their corresponding bid-ask spreads is 75.21%, with an average absolute deviation of $69.77 for those observations lying outside a spread boundary.

The average absolute price deviation of $69.77 per contract for observations lying outside a spread boundary is slightly larger than the average bid-ask spread of $56.75. Price deviations are larger for deep in-the-money and deep out-of-the-money options, however.

EXHIBIT 4

Comparison of Black-Scholes Prices and Observed Prices of S&P 500 Options

Date	Number of Price Observations	Implied Standard Deviation (%)	Average Observed Call Price ($)	Average Observed Bid-Ask Spread ($)	Proportion of Theoretical Prices Outside Bid-Ask Spreads (%)	Average Absolute Deviation of Theoretical Price from Spread Boundaries ($)
12/02/93	1,354	15.29	2,862.74	67.87	59.68	48.98
12/06/93	1,667	14.94	3,113.35	67.26	56.63	54.14
12/08/93	956	14.77	3,012.24	59.95	62.03	40.53
12/10/93	2,445	14.56	2,962.24	61.60	60.00	32.87
12/14/93	3,100	15.14	3,003.08	68.75	66.58	37.46
12/16/93	1,944	14.55	2,754.71	61.79	78.65	59.65
12/20/93	115	10.66	1,453.04	55.76	91.30	194.51
12/22/93	199	9.93	1,203.78	45.63	99.50	117.83
12/28/93	166	9.07	608.45	27.52	83.13	46.56
12/30/93	242	9.86	1,339.86	51.32	94.63	65.18
Average	1,218	12.88	2,231.35	56.75	75.21	69.77

Black-Scholes implied standard deviations (BSISD) estimated from prior-day option price observations. Current-day theoretical Black-Scholes option prices calculated using prior-day volatility parameter estimate. Prices stated on a per contract basis, i.e., 100 times quote price.

For example, Exhibit 4 shows that the Black-Scholes implied standard deviation (BSISD) estimated using Whaley's simultaneous equations procedure on December 2 option prices is 15.29%, while Exhibit 3 reveals that contract-specific Black-Scholes implied volatilities range from about 18% for deep in-the-money options to about 8% for deep out-of-the-money options.

Given December 2 SPX input values (S = \$459.65, r = 3.15%, t = 78 days), a deep in-the-money option with a strike price of 430 yields call contract prices of \$3,635.76 and \$3,495.68, respectively, from volatility values of 18% and 15.29%. Similarly, a deep out-of-the-money option with a strike price of 490 yields call contract prices of \$46.02 and \$395.13, respectively, from volatility values of 8% and 15.29%. These prices correspond to contract price deviations of \$140.08 for deep in-the-money options and \$349.11 for deep out-of-the-money options, significantly larger than the average deviation of \$56.75 per contract.

Price deviations of this magnitude indicate that CBOE market makers quote deep in-the-money (out-of-the-money) call option prices at a premium (discount) compared to Black-Scholes prices, although the Black-Scholes formula is a useful first approximation to these option prices.

Skewness- and Kurtosis-Adjusted Jarrow-Rudd Model

To examine the improvement in pricing accuracy obtained by adding skewness- and kurtosis-adjustment terms, in the second set of estimation procedures, on a given day within a given option maturity series, we simultaneously estimate a single return standard deviation, a single skewness parameter, and a single kurtosis parameter by minimizing the sum of squares with respect to the arguments ISD, L_1, and L_2, respectively:

$$\min_{ISD, L_1, L_2} \sum_{j=1}^{N} \times$$

$$[C_{OBS,j} - (C_{BS,j}(ISD) + L_1 Q_3 + L_2 Q_4)]^2 \quad (9)$$

The coefficients L_1 and L_2 estimate the parameters λ_1 and λ_2 defined in Equation (5), where the terms Q_3 and Q_4 are also defined. These daily estimates yield implied coefficients of skewness (ISK) and kurtosis (IKT) calculated as follows, where $\gamma_1(A)$ and $\gamma_2(A)$ are as defined in Equation (7):

$$ISK = L_1 + \gamma_1[A(ISD)]$$

$$IKT = 3 + L_2 + \gamma_2[A(ISD)]$$

Thus ISK estimates the skewness parameter $\gamma_1(F)$, and IKT estimates the kurtosis parameter $3 + \gamma_2(F)$.

Substituting estimates of ISD, L_1, and L_2 into Equation (4) yields skewness- and kurtosis-adjusted Jarrow-Rudd option prices (C_{JR}) expressed as the sum of a Black-Scholes option price plus adjustments for skewness and kurtosis deviations from lognormality:

$$C_{JR} = C_{BS}(ISD) + L_1 Q_3 + L_2 Q_4 \quad (10)$$

Equation (10) yields theoretical skewness- and kurtosis-adjusted Black-Scholes option prices from which we compute deviations of theoretical prices from market-observed prices.

Exhibit 5 summarizes results for the same S&P 500 index call option prices used to compile Exhibit 4. Consequently, the stub lists the same even-numbered dates and column (1) the same number of price quotations given in Exhibit 4.

To assess the out-of-sample forecasting power of skewness and kurtosis adjustments, the implied standard deviation (ISD), implied skewness coefficient (ISK), and implied kurtosis coefficient (IKT) for each date are estimated from prices observed on the trading day immediately prior to each date listed. For example, the first row of Exhibit 5 lists the date December 2, 1993, but that day's standard deviation, skewness, and kurtosis estimates provided are obtained from December 1 prices. Thus, out-of-sample parameters ISD, ISK, and IKT reported correspond to one-day lagged estimates.

We use these one-day lagged values of ISD, ISK, and IKT to calculate theoretical skewness- and kurtosis-adjusted Black-Scholes option prices according to Equation (10) for all price observations on the even-numbered dates listed. In turn, these theoretical prices based on out-of-sample ISD, ISK, and IKT val-

EXHIBIT 5

COMPARISON OF SKEWNESS- AND KURTOSIS-ADJUSTED BLACK-SCHOLES PRICES AND OBSERVED PRICES OF S&P 500 OPTIONS

Date	Number of Price Observations	Implied Standard Deviation (%)	Implied Skewness (ISK)	Implied Kurtosis (IKT)	Proportion of Theoretical Prices Outside Bid-Ask Spread (%)	Average Absolute Deviation of Theoretical Price from Spread Boundaries ($)
12/02/93	1,354	12.70	−1.57	4.33	17.73	8.60
12/06/93	1,667	11.90	−1.54	5.19	23.10	15.15
12/08/93	956	12.12	−1.36	5.68	6.59	17.22
12/10/93	2,445	11.58	−1.44	4.72	8.06	10.37
12/14/93	3,100	12.06	−1.46	5.73	11.03	8.15
12/16/93	1,944	12.04	−1.58	5.32	12.65	14.18
12/20/93	115	11.42	−2.13	5.86	62.61	28.27
12/22/93	199	11.06	−2.22	6.12	72.86	30.82
12/28/93	166	10.55	−1.91	5.21	50.60	12.36
12/30/93	242	10.75	−1.62	5.77	53.31	13.42
Average	1,218	11.62	−1.68	5.39	31.85	15.85

Implied standard deviation (ISD), skewness (ISK), and kurtosis (IKT) parameters estimated from prior-day price observations. Current-day theoretical option prices calculated using out-of-sample parameter estimates.

ues are then used to compute daily proportions of theoretical prices outside bid-ask spreads and daily averages of deviations from spread boundaries. Column averages are reported in the last row of Exhibit 5.

All daily skewness coefficients are negative, with a column average of −1.68. Daily kurtosis coefficients average 5.39. These option-implied coefficients may be compared with sample coefficients reported in Exhibit 1 for the decade 1986-1995.

For example, option-implied skewness of −1.68 compares to log-relative return skewness of −1.67 and arithmetic return skewness of −1.19 calculated including the October 1987 return, but option-implied kurtosis of 5.39 is less extreme than arithmetic return kurtosis of 9.27 and log-relative return kurtosis of 11.92 calculated including the October 1987 return. This appears to suggest that any memory of the October 1987 crash embodied in S&P 500 option prices is more strongly manifested by negative option-implied skewness than option-implied excess kurtosis.

The next-to-last column in Exhibit 5 lists the proportion of skewness- and kurtosis-adjusted prices lying outside their corresponding bid-ask spread boundaries. The average proportion is 31.85%. The last column lists average absolute deviations of theoretical prices from bid-ask spread boundaries for only those prices lying outside their bid-ask spreads. The column average is $15.85, which is about one-fourth the size of the average bid-ask spread of $69.77 reported in Exhibit 4.

Moreover, Exhibit 3 reveals that implied volatilities from skewness- and kurtosis-adjusted option prices (hollow markers) are unrelated to option moneyness. In turn, this implies that the corresponding price deviations are also unrelated to option moneyness.

Comparison of implied volatility values in Exhibits 4 and 5 suggests that the implied volatility series obtained using the Jarrow-Rudd model is smoother than the series obtained using the Black-Scholes model. Indeed, the average absolute value of daily changes in implied volatility is 0.42% for the Jarrow-Rudd model, less than half the 0.91% of the Black-Scholes model. Using a matched-pairs t-test on absolute values of daily changes in implied volatilities for both models, we obtain a t-value of 4.0, indicating a significantly smoother time series of implied volatilities from the Jarrow-Rudd model. Thus the Jarrow-Rudd model not only flattens the implied volatility skew, but it also produces more stable volatility estimates.

The empirical results will vary slightly depending on the interest rate assumed. If the assumed rate is too low, implied standard deviation estimates will be biased upward. Likewise, if the assumed rate is too high, implied volatility estimates will be biased downward.

We follow the standard research practice, and use Treasury bill rates, which may understate the true cost of funds to option market participants. Treasury bill repurchase (repo) rates likely better represent the true cost of borrowed funds to securities firms. For individual investors, the broker call money rate would better represent the true cost of funds.

To assess the robustness of our results to the interest rate assumed, we repeat all empirical analyses leading to Exhibits 4 and 5 using Treasury bill repurchase rates and broker call money rates. On average, repurchase rates were 7 basis points higher than Treasury bill rates in December 1993. Call money rates were on average 196 basis points higher.

Average daily Black-Scholes implied standard deviations are 12.81% using repurchase rates and 10.72% using call money rates. These are both lower than the 12.88% average implied volatility reported in Exhibit 4. Using repurchase rates for the Jarrow-Rudd model yields an average daily implied volatility of 11.55%, and using call money rates yields an average volatility of 9.69%. Both are lower than the 11.62% average implied volatility reported in Exhibit 5.

Using repurchase rates for the Jarrow-Rudd model yields an average daily skewness coefficient of −1.66 and an average daily kurtosis coefficient of 5.34, while using call money rates yields an average daily skewness coefficient of −1.11 and an average daily excess kurtosis coefficient of 3.46. These are all smaller than the average skewness of −1.68 and average kurtosis of 5.39 reported in Exhibit 5. Yet whichever interest rate is used to measure the cost of funds to S&P 500 options market participants, the option-implied distributions of S&P 500 returns are still noticeably non-normal.

Overall, we conclude that skewness and kurtosis adjustment terms added to the Black-Scholes formula significantly improve pricing accuracy for deep in-the-money or deep out-of-the-money S&P 500 index options. Furthermore, these improvements are obtainable from out-of-sample estimates of skewness and kurtosis.

Of course, there is an added cost, in that two additional parameters must be estimated. But this cost is slight, because once the computer code is in place, the additional computation time is trivial on modern computers.

V. HEDGING IMPLICATIONS OF THE JARROW-RUDD MODEL

To explore the Jarrow-Rudd model's implications for hedging strategies using options, we derive formulas for an option's delta and gamma based on the model. Delta is used to calculate the number of contracts needed to form an effective hedge based on options. Gamma states the sensitivity of a delta-hedged position to stock price changes. By definition, delta is the first partial derivative of an option price with respect to the underlying stock price. Similarly, gamma is the second partial derivative.

Taking first and second derivatives of the Jarrow-Rudd call option price formula yields delta and gamma formulas, where the variables λ_j and Q_j are defined as in Equation (5):

$$\text{Delta}: \frac{\partial C}{\partial S_0} = N(d_1) + \lambda_1\left(\frac{3}{S_0}\right)Q_3 + \lambda_2\left(\frac{4}{S_0}\right)Q_4 \tag{11A}$$

$$\text{Gamma}: \frac{\partial^2 C}{\partial S_0^2} = n(d_1)(S_0\sigma\sqrt{t})^{-1} + \lambda_1\left(\frac{6}{S_0^2}\right)Q_3 + \lambda_2\left(\frac{12}{S_0^2}\right)Q_4 \tag{11B}$$

The first terms on the right-hand sides of Equations (11A) and (11B) are the delta and gamma for the Black-Scholes model. Adding the second and third terms yields the delta and gamma for the Jarrow-Rudd model.

Exhibit 6 illustrates how a hedging strategy based on the Jarrow-Rudd model might differ from a hedging strategy based on the Black-Scholes model. In this example, S&P 500 index options are used to delta-hedge a hypothetical $10 million stock portfolio with a beta of one. The example assumes an index level of S_0 = $700, an interest rate of r = 5%, a dividend yield of y = 2%, and a time until option expiration of t = 0.25. For the volatility parameter in the Black-Scholes model, we use the average implied volatility of 12.88% reported in Exhibit 4. For the Jarrow-Rudd model, we use the average implied volatility of 11.62% and the average skewness and kurtosis values of $\lambda_1 = -1.68$ and $\lambda_2 = 5.39$ reported in Exhibit 5.

Strike prices range from 660 to 750 in increments of 10. For each strike price, the BS and JR columns list the number of S&P 500 index option contracts needed to delta-hedge the assumed $10 million stock portfolio, according to the Black-Scholes and the Jarrow-Rudd models.

In both cases, numbers of contracts required are computed as follows, where the option contract size is 100 times the index level (Hull [1993]):

EXHIBIT 6
NUMBER OF OPTION CONTRACTS NEEDED TO DELTA-HEDGE $10 MILLION STOCK PORTFOLIO WITH BETA OF ONE

In-the-Money Options			Out-of-the-Money Options		
Strike	**(BS)**	**(JR)**	**Strike**	**(BS)**	**(JR)**
660	167	161	710	303	302
670	179	173	720	370	377
680	197	190	730	465	486
690	221	215	740	601	651
700	256	250	750	802	900

Index level S_0 = $700, interest rate r = 5%, dividend yield y = 2%, time until option expiration t = 0.25. Black-Scholes (BS) model assumes volatility of σ = 12.88%. Jarrow-Rudd (JR) model assumes σ = 11.62%, and skewness and kurtosis parameters of $\lambda_1 = -1.68$ and $\lambda_2 = 5.39$.

$$\text{Number of Contracts} = \frac{\text{Portfolio Value/Contract Size}}{\text{Option Delta}}$$

Exhibit 6 reveals that for in-the-money options a delta-hedge based on the Black-Scholes model specifies more contracts than a delta-hedge based on the Jarrow-Rudd model. But for out-of-the-money options, a delta-hedge based on the Jarrow-Rudd model requires more contracts (except in one case).

Differences in the number of contracts specified by each model are greatest for out-of-the-money options. For example, in the case of a delta-hedge based on options with a strike price of 740, the Black-Scholes model specifies 601 contracts, while the Jarrow-Rudd model specifies 651 contracts.

VI. SUMMARY AND CONCLUSION

We have empirically tested an expanded version of the Black-Scholes [1973] option pricing model developed by Jarrow and Rudd [1982] that accounts for skewness and kurtosis deviations from lognormality in stock price distributions. The Jarrow-Rudd model is applied to estimate coefficients of skewness and kurtosis implied by S&P 500 index option prices.

We find significant negative skewness and positive excess kurtosis in the option-implied distribution of S&P 500 index prices. This observed negative skewness and positive excess kurtosis induces a volatility smile when the Black-Scholes formula is used to calculate option-implied volatilities across a range of strike prices.

By adding skewness and kurtosis adjustment terms developed in the Jarrow-Rudd model, the volatility smile is effectively flattened. We conclude that skewness and kurtosis adjustment terms added to the Black-Scholes formula significantly improve accuracy and consistency for pricing deep in-the-money and deep out-of-the-money options.

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