A Contingent-Claims Approach to the Inventory-Stocking Decision

John D. Stowe and Tie Su

Inventory management is a major area of study in the fields of engineering economics and production/operations management, and applying modern techniques of inventory management (along with improvements in telecommunications and transportation) has resulted in major gains in efficiencies and service levels. Inventory management is critical because the size of the investment in this asset is typically between 10% and 40% of the total assets of most large corporations. Economists believe that inventory levels play a pivotal role in the onset of economic recessions. Nevertheless, financial economists have given only cursory attention to inventories. This paper's underlying theme is that the tools of financial economics can address basic issues in inventory management.

The conventional approach to inventory-stocking decisions relies on a criterion that maximizes expected profits, given a distribution of demand for the inventory item. The contingent-claims approach presented here maps the demand and payoffs from an inventory policy onto the price of an underlying state variable, builds a portfolio of options that replicates the payoffs, and then derives the value of an inventory policy using option-pricing models in which the value of the policy is contingent on the financial characteristics of the underlying asset.

The conventional approach is particularly unsatisfying when we consider that financial economists approach other corporate finance decisions with models that account for risk in an economically meaningful way, but do not do so when modeling inventory decisions. This is puzzling since these critical decisions should be made according to the same sound financial principles that guide decision making in the rest of the organization.

Opportunity models have become a powerful technology in corporate finance, because they can be used to address problems for which the traditional discounted cash flow approaches are inappropriate. The literature that applies opportunity models to capital budgeting, often referred to as real options, is extensive. Applications include the option to defer an investment, to stage investments, to expand or contract, to abandon, to switch inputs, and to grow. (See Lee, 1997; Gilster, 1997; Jarrow, 1997; Gastineau, 1997; and Ross, 1995.) Trigeorgis (1993) gives an extensive summary of these applications. Opportunity models are also applied to financing decisions, such as the option to default, bond-refunding, and tax-timing options (see Ikenberry and Vermaelen, 1996; and Berkman and Bradbury, 1996). This paper positions the investment in inventory as another real option—an option on future sales.
The paper is organized as follows. Section I overviews the conventional expected profit-maximization approach to inventory decisions and summarizes the weaknesses of this approach. This section also reviews the relevant finance literature. Section II introduces the general logic of valuing an inventory policy by using a contingent-claims approach. An appendix to the paper provides more background on this process. Sections II and III illustrate the application of the contingent-claims approach to the two basic inventory problems that commonly appear in the inventory literature. Section II shows how the new approach can be applied when sales follow a discrete distribution, and Section III applies the new approach when sales have a continuous distribution. Both Sections II and III give a straightforward view of how the contingent-claims approach can be seen as a logical extension of the familiar profit-maximization (or cost-minimizing) approach. Section IV provides a conceptual and operational comparison of the conventional and contingent-claims approaches. Section V summarizes and concludes the paper.

I. Background and Relevant Literature

This section presents two examples of the conventional approach to the inventory-stocking decision when sales are discretely or continuously distributed. Later, in Sections II and III, we reevaluate the examples using the contingent-claims approach. This section also summarizes the relevant financial literature.

A. Conventional Approach to Inventory-Stocking Levels

The conventional approach to the inventory-stocking decision relies on a distribution of demand that is either discrete or continuous. We illustrate the conventional profit-maximization approach for each type of distribution with an example.

Taking the discrete distribution first, we assume given probabilities of each sales level. Also, we assume that there are four possible levels of demand, with the probabilities and the quantity demanded of each state shown in Table I. The example makes several assumptions:

- There are four states of nature, and the quantity demanded will be four, five, six, or seven units.
- The probability of each state is 0.25.
- The selling price is $100 per unit, the investment is $60 per unit, and excess (overstock) units are sold for $10 per unit.

If \( Q \) is the number of units stocked in inventory, and \( D \) is the quantity demanded, the profit is

\[
\text{Profit} = 100 \min(Q, D) + 10 \max(Q - D, 0) - 60Q \tag{1}
\]

Total revenue is $100 times the number of units sold, which is the lesser of the units stocked (\( Q \)) or the number demanded (\( D \)). Excess stock, if any, can be sold for $10 per unit, and the product cost is $60 per unit.

Table I shows the profit associated with each stocking level and each state of nature. For example, if the firm stocks five units and demand is four units (state 1), its profit is $110. Total revenue is $410, which is $400 for the four units sold at full price and $10 for one excess unit sold for salvage. Total costs are $300 for five units, which gives the $110 net profit. We use a similar calculation to determine the state-contingent profit for each stocking/demand level pair.

Table I also shows the expected profit for each stocking level. Given the probability of each state and the state-contingent profits, the profit-maximizing stocking level is five units, with an expected profit of $177.50.

The other general application of the conventional approach deals with continuously distributed demand. We assume that demand is normally distributed, with a mean of 200 units and a standard deviation of 20 units. We also assume the investment in inventory is $6 per unit, that the price is $10 per unit sold, and that unsold units can be salvaged for $4. The expected profit for stocking \( Q \) units where demand is \( x \) is

\[
\text{Profit} = \int_0^Q (10x + 4(Q-x))f(x)dx + \int_Q^\infty 10Qf(x)dx - 6Q \tag{2}
\]

We find the highest profit level by taking the derivative with respect to \( Q \), which results in

\[
6(1 \cdot f(x)dx) - 2 = 0, \text{ or } f(x)dx = 2/3 \tag{3}
\]

The optimal order quantity is the quantity for which the cumulative density function of sales is two thirds, and the probability of demand exceeding \( Q \) (the stockout probability) is one third. As a result, stocking

\[Q = \frac{1}{2}(100 + \sqrt{100^2 + 4 \cdot 16 \cdot 120}) = 179.09\]

The inventory problem is often presented in a framework in which the decision maker's objective is to minimize the opportunity cost of alternative stocking levels. The optimal decision minimizes the expected opportunity cost. The profit-maximizing and the opportunity-cost frameworks are logically consistent. We prefer the simplicity of the former.

While it is common to assume normally distributed demand, an obvious flaw is that demand can assume negative values. Allowing for negative demand and assuming that excess stocks are sold for salvage value would result in a profit of

\[
\text{Profit} = \int_0^Q f(x)dx + \int_Q^{\infty} (10x + 4(Q-x))f(x)dx + \int_Q^{\infty} 10Qf(x)dx - 6Q
\]

A realistic way to avoid the negative demand problem is to assume another distribution, such as the lognormal distribution or the exponential distribution.
Table 1. Contingent and Expected Profits for Discrete, Conventional Approach

The conventional approach calculates an expected profit based on the contingent profit for a given stocking level. The expected profit is based on the contingent profits resulting in each state of demand. The contingent profit in each state for demand (D) and stocking level (Q) is

\[ \text{Profit} = 100 \min(Q, D) + 10 \max(Q-D, 0) - 60Q \]

where the price per unit sold is $100, the investment per unit is $60, and excess units (where stocks exceed demand) are salvaged for $10 per unit. The optimal stocking level is the one that results in the highest expected profit.

<table>
<thead>
<tr>
<th>State</th>
<th>Demand (D)</th>
<th>Probability</th>
<th>Order Level (Q)</th>
<th>State-Contingent Profit ($)</th>
<th>Expected Profit ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>4</td>
<td>0.25</td>
<td>160</td>
<td>160</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>5</td>
<td>0.25</td>
<td>160</td>
<td>160</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>6</td>
<td>0.25</td>
<td>200</td>
<td>177.5</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>7</td>
<td>0.25</td>
<td>240</td>
<td>172.5</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>190</td>
<td>145</td>
</tr>
</tbody>
</table>

Q = 208.62 units is the optimal policy, and the maximum expected profit is $356.36.

While the conventional approach is widely used, especially by engineering economists, it has some obvious deficiencies. Specifically, it does not carefully handle the size and timing of the cash flows associated with alternative inventory policies. More importantly, it does not address risk in an economically meaningful way.

B. Related Literature

Studies on the conventional approach to inventory management represent a well-developed area of engineering economics and production/operations management. Two of the classic texts in this area are Starr and Miller (1962) and Buffa and Miller (1979). Our approach uses valuation methods that postdate conventional inventory theory and can be viewed as a logical extension of that theory. Some traditional problems in inventory management can be updated and recast into a framework that permits solutions that are consistent with the principles of financial economics.

The literature that applies financial theory to inventory decisions is sparse. Kim and Chung (1989) and Morris and Chang (1991) have used the capital-asset pricing model (CAPM) as an alternative to the conventional profit-maximization approach. A key insight of this approach is that a demand beta (sensitivity of demand to the market index) is inversely related to the stocking level.

Three studies have applied option-pricing approaches to the inventory problem. Stowe and Gehr (1991) provide two financial approaches to handling the inventory problem above. The first is a replicating portfolio approach comparing the value of a portfolio of securities, with payoffs that replicate those of the inventory policy, to the inventory investment. This helps establish a net present value. The second approach is an option-pricing solution to the discrete sales problem. Each state of nature is mapped onto a range of prices of an underlying asset (such as a stock index), state prices are computed for each state of nature, and the state prices are used to find the present values of the inventory payoffs. (We use this approach in Section II.) Chung (1990) also shows another option-pricing solution for inventory payoffs that are a linear function of demand when demand and the investors' discount rate are jointly log-normally distributed. Becker (1994) solves for the inventory reorder point using a binomial option-pricing approach.

Our approach is an extension of these earlier studies. Our intuition is that if any discrete or continuous inventory payoff function can be mapped onto an underlying state variable, the payoff can be replicated by a portfolio of options. Then, we can use modern option-pricing models to make value-maximizing inventory decisions. In this paper, we give an application of this technology to the two special cases of payoff functions.\(^3\)

II. A Contingent-Claims Solution for the Discrete Demand Case

In this section, we reconsider the inventory problem in which the demand has a discrete distribution. For the basic problem presented in Section I, instead of

\(^3\)See the Appendix for three simple properties that can be used to value these payoff functions.
four demand levels with given probabilities assigned to each, there are now four demand levels that can be associated with, or mapped onto, price ranges for an underlying state variable. The underlying state variable can be any priced asset, such as stock or fixed-income indexes, an individual security, or the value of a foreign currency.

In the example here, we map the four levels of demand onto the values for an underlying stock index. The first state, which has a demand of four units, occurs if the index has a value below 575. The second state, which has a demand of five units, occurs if the index has a value between 575 and 650. The third state, six units of demand, is mapped onto a range of 650 to 725. The fourth state, seven units of demand, is mapped onto an index range of 725 and above.

The payoffs for each state of nature depend on the level of demand in that state ($D_i$) and the choice of inventory-stocking level ($Q_k$) chosen. The payoffs for each inventory policy and each state of nature are shown in Figure 1 and Table 2. For the numerical example, the end-of-period payoff for state i and inventory-stocking level k is

$$\text{Payoff}_{ik} = 100 \min(Q_k, D_i) + 10 \max(Q_k - D_i, 0)$$  (4)

and the investment is $60Q_k$.

The net present value (NPV) of a particular inventory policy is the present value of the payoffs from that policy minus the outlay.

$$\text{NPV}_k = \sum_{i=1}^{4} \text{Payoff}_{ik} \cdot e^{-r_i} - \text{Outlay}_k$$  (5)

The present value of the future payoffs is the sum of the products of the cash flow in each state of nature (Payoff$_{ik}$) and its respective state price ($p_i$). The state price is the value of a claim that pays off $1.00 in that state of nature, and zero otherwise. These state prices are the basic element of the contingent-claims approach to the inventory problem, just as were the probabilities of the various demand levels for the conventional approach to this problem.

We can establish state prices, such as those for the four states shown in Figure 1 and Table 2, using option-pricing models. Banz and Miller (1978) and Breeden and Litzenberger (1978) give the state price for a claim that pays off $1.00 if the underlying asset price is between two values $Y_1$ and $Y_2$. For the Black-Scholes option-pricing model, the state price is

$$p(Y_1, Y_2) = e^{-\sigma^2/2} \left\{ \frac{1}{\sigma \sqrt{t}} \left[ \left( \frac{S}{X} \right)^{\frac{1}{2}} N(d_1) + \left( \frac{Y_1}{X} \right)^{\frac{1}{2}} N(d_2) \right] - \frac{d_2}{d_1} \right\}$$  (6)

where

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)t}{\sigma \sqrt{t}}$$
$$d_2 = \frac{\ln(S/X) + (r - \sigma^2/2)t}{\sigma \sqrt{t}}$$
$$N(x) = \text{cumulative normal for } x$$
$$S = \text{current price of underlying state variable}$$
$$X = \text{exercise price of the option}$$
$$t = \text{time until maturity of the option}$$
$$r = \text{riskless rate of return}$$
$$\sigma = \text{volatility of the underlying state variable}$$

A convenient way to view the state prices discussed above is to recognize that they are combinations of binary options (see Hull, 1995; or Chance, 1995). They also resemble a cash-or-nothing (digital) option that pays a fixed dollar amount ($1.00 in this case) if the option finishes in-the-money.

A state price, $p(Y_1, Y_2)$, is the difference of two digital options:

$$p(Y_1, Y_2) = dc(Y_1) - dc(Y_2)$$

where $dc(\cdot)$ is the price of a digital call option.

State prices account for the likelihood that each state of nature will occur and also for the marginal utilities of money in each state of nature. This method of finding state prices is intuitive when we recall that $N(d_1)$ represents the risk-neutral probability of an option expiring in the money. The difference between two such risk-neutral probabilities, when multiplied by a discount factor for a riskless rate, forms the state price. In the option-pricing approach, state prices perform a role similar to that of probabilities in the conventional approach. Critically, these state prices reflect both the probabilities of each state and the effects of the time value of money and risk.

To solve for the state prices in the example above, we assume that the current price of the index is 660, the option term is six months (0.5 years), the standard deviation of the index return is 0.20, and the risk-free rate is 0.10. Using Equation (6) and these parameters results in the state prices shown in Table 2. Using these state prices, we can value the state-contingent payoff for each state. The total value of the state-contingent payoffs minus the outlay gives us the NPV for an inventory policy. As Table 2 shows, the optimal stocking level is five units. With an initial outlay of $300 and a total present value of $476.24 (the sum of the present values of the four state-contingent payoffs shown in Table 2), the NPV is $176.24.

Of course, the optimal inventory policy depends on the shape of the payoff function, market conditions, and the characteristics of the underlying asset. Changes in these factors will change the NPV of an inventory policy in predictable ways.
### Table 2. The Contingent-Claims Approach to the Discrete Case

The contingent-claims approach maps states of demand onto the price of an underlying asset. It uses option-pricing theory to establish state prices for each outcome. The present value of a state-contingent payoff is the state price times the state-contingent payoff. The sum of the present values of the state-contingent payoffs minus the investment is the NPV for a particular stocking level. The optimal policy is the one that results in the highest NPV.

<table>
<thead>
<tr>
<th>State (i)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demand (D)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Price Range</td>
<td>0-575</td>
<td>575-650</td>
<td>650-725</td>
<td>725-∞</td>
</tr>
<tr>
<td>State Prices (p)</td>
<td>0.09916</td>
<td>0.23184</td>
<td>0.28590</td>
<td>0.33433</td>
</tr>
<tr>
<td>Order Level (Q_s) / Outlay ($)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>400</td>
<td>400</td>
<td>400</td>
<td>400</td>
</tr>
<tr>
<td>240</td>
<td>39.66</td>
<td>92.74</td>
<td>114.36</td>
<td>133.73</td>
</tr>
<tr>
<td>5</td>
<td>410</td>
<td>500</td>
<td>510</td>
<td>520</td>
</tr>
<tr>
<td>300</td>
<td>40.66</td>
<td>115.92</td>
<td>145.81</td>
<td>173.85</td>
</tr>
<tr>
<td>6</td>
<td>420</td>
<td>510</td>
<td>600</td>
<td>610</td>
</tr>
<tr>
<td>360</td>
<td>41.65</td>
<td>118.24</td>
<td>171.54</td>
<td>203.94</td>
</tr>
<tr>
<td>7</td>
<td>430</td>
<td>520</td>
<td>610</td>
<td>700</td>
</tr>
<tr>
<td>420</td>
<td>42.64</td>
<td>120.56</td>
<td>174.40</td>
<td>234.03</td>
</tr>
</tbody>
</table>

NPV ($) = Total Present Value - Investment

Assumptions: \[ P_o = 660, \quad t = 0.5, \quad r = 0.10, \quad \sigma = 0.20 \]

Payoff_{s,t} = 100 \min (Q_o, D_s) + 10 \max (Q_o - D_s, 0)

Present Value_{s,t} = Payoff_{s,t} \cdot p_s

NPV_s = \sum_{t=1}^{T} \text{Payoff}_{s,t} \cdot p_s \cdot \text{Outlay}_{s,t}

### Figure 1. Payoffs from Inventory Policies with Discrete Demand

This figure shows the payoffs for each inventory policy (stocking level) as a function of the price of the underlying asset. The present values of these payoff functions minus the original outlays yield the NPVs of each inventory policy.
III. The Contingent-Claims Solution for the Continuous Demand Case

Evaluating inventory policies when demand is continuously distributed is also straightforward. The first step is to map the payoffs from an inventory policy onto the price of the underlying asset. The second step is to construct a portfolio of options that implements or replicates those payoffs. The final step is to value that portfolio and then subtract the investment to obtain the NPV. In this section, we demonstrate this process for a simple inventory problem, followed by more complex problems.

A. Case 1: The Basic Case

We illustrate the contingent-claims decision-making process with three cases.

The first case has no salvage value for excess inventory and no penalties for shortages. Assume that sales are tied to an underlying asset with a current price of $50, standard deviation of 0.35, an interest rate of 0.10, and a payoff maturity of 0.5 years. Sales will be zero if the asset price is below $30, but will increase by ten units per $1.00 increase above this price. Assume that the investment per unit is $6, that the price per unit sold is $10, and that the salvage value for overstocks is zero.

Panel A of Figure 2 shows the payoffs from a particular inventory policy under which the firm stocks 300 units. The payoff is zero if the asset price is below $30, increases with a slope of 100 for S > $30, and reaches its maximum of $3,000 when the asset price is $60. The option strategy in Panel A is thus a vertical (money) spread. The portfolio of options that replicates this payoff is a long position of 100 call options with an exercise price of $30, and a short position of 100 call options with an exercise price of $60. The NPV of this inventory policy is the value of the call options (the call spread) minus the initial outlay:

\[ NPV = 100C(X=30) - 100C(X=60) - 6(300) \]
\[ NPV = 100(21.50) - 100(2.47) - 1,800 = $103.00 \]

Of course, this is not necessarily the optimal inventory policy. At the margin, adding an additional unit to the inventory stock increases the investment by $6 and raises the potential payoff by $10. The additional $10 payoff is achieved by increasing the exercise price of the short position in 100 call options by $0.10. If Q is the number of units stocked, the NPV is

\[ NPV = 100C(X=30) - 100C(X=30+0.10Q) - 6Q \]

We find the optimal inventory policy by taking the derivative of this function with respect to Q, and setting it to zero, which results in

\[ \frac{\partial NPV}{\partial Q} = -100 \frac{\partial C(X = 30 + 0.10Q)}{\partial Q} - 6 = 0 \]
\[ \frac{\partial NPV}{\partial Q} = -100 \frac{\partial C(X = 30 + 0.10Q)}{\partial X} \frac{\partial X}{\partial Q} - 6 = 0 \]

Since \( \partial X/\partial Q = 0.10 \) in this example, the optimal stocking level occurs where

\[ \frac{\partial C(X = 30 + 0.10Q)}{\partial X} = 0.6 \]

Since \( \partial C/\partial X = e^{-\mu t}N(d_1) \), we may find Q and X by iterating until \( \partial C/\partial X \) has the desired value. This occurs when \( Q = 169.35 \) units and \( X = 46.94 \). The NPV for the optimal inventory solution is

\[ NPV = 100C(X=30) - 100C(X=46.94) - 6(169.35) \]
\[ NPV = 100(21.50) - 100(7.81) - 6(169.35) = $352.90 \]

B. Case 2: Salvage Value for Excess Inventories

The example above assumed a salvage value of zero. Now, we include a positive salvage value for leftover units of $4 per unit. The positive salvage value can come from several sources. One is to return the product to the firm's suppliers. In this case, the return price of $4 (as opposed to the original cost of $6) reflects the costs of making the return. Another source of a $4 salvage value could be that if the items cannot be sold for their full price of $10, they can be sold at the greatly discounted price of $4. A final possible source of salvage value is that the firm holds the items for sale at a future time, and $4 is the present value of this future sale.

Panel B of Figure 2 shows how the payoffs are altered by the existence of a salvage value. Assume initially that the firm uses an inventory order quantity of Q = 300 units. In this case, the minimum payoff is $4Q = $1,200, which occurs if the underlying asset price is below $30. The maximum payoff is $10Q = $3,000, which occurs if the underlying asset price is above $60. The

---

4Inventory carrying costs can also be built into the option-pricing model by treating them as a dividend yield, which is used to adjust the stock price before it is input into the Black-Scholes option-pricing model:

\[ C = Se^{\mu t}N(d_1) - Xe^{\mu t}N(d_2) \]

where

\[ d_1 = \frac{\ln(S/X) + rT + \sigma^2/2}{\sigma\sqrt{T}} \]
\[ d_2 = d_1 - \sigma\sqrt{T} \]
\[ \delta = \text{annualized dividend yield} \]
payoff function between these two prices has a slope of 60, which is composed of ten units of sales per $1.00 increase in price times the price differential between the full sales price ($10) and the salvage value ($4).

The portfolio of options that replicates this payoff is a vertical discontinuity of $1,200 at X = 0, a long position of 60 call options with X = $30, and a short position of 60 call options with X = $60. (As described in the Appendix, the value of a $1 vertical discontinuity at X = 0 is $1\,\partial C/\partial X (X=0) = e^{-\alpha}.) The NPV of this inventory policy is the value of this portfolio minus the investment of $6 each in 300 units of inventory:

\[
\begin{align*}
NPV &= 1,200 \, \frac{\partial C}{\partial X}(X=0) + 60 \, C(X=30) - 60 \, C(X=60) - 6(300) \\
NPV &= 1,200(0.9512294) + 60(21.50) - 60(2.47) - 1,800 \\
NPV &= 1,141.48 + 1,290.00 - 148.20 - 1,800 = $483.28
\end{align*}
\]

Of course, this may not be the optimal policy. The NPV for inventory-stocking level Q is

\[
\begin{align*}
NPV &= 4 \, Q \, \frac{\partial C}{\partial X}(X=0) + 60 \, C(X=30) - 60 \, C(X=30 + 0.10Q) - 6Q
\end{align*}
\]

Taking the derivative with respect to Q, and setting it to zero gives us

\[
\begin{align*}
\frac{\partial NPV}{\partial Q} &= 4 \, \frac{\partial C}{\partial X}(X=0) - 60 \, \frac{\partial C}{\partial X}(X=30 + 0.10Q) - 6 = 0 \\
\frac{\partial NPV}{\partial Q} &= -60 \, \frac{\partial C}{\partial X}(X=30 + 0.10Q) \frac{\partial X}{\partial Q} (6 - 4e^{-\alpha}) = 0
\end{align*}
\]

Since $\partial X/\partial Q = 0.10$ in this example, the optimal stocking level occurs where

\[
\begin{align*}
\frac{\partial C}{\partial X}(X = 30 + 0.10Q) = 6 - 4e^{-0.05} = 0.36585
\end{align*}
\]

The existence of a salvage value increases Q and X. Now Q = 248.18 units and X = $54.82. The NPV for the optimal inventory solution is

\[
\begin{align*}
NPV &= 4(248.18)e^{-0.05} + 60 \, C(X=30) - 60 \, C(X=54.82) - 6(248.18) \\
NPV &= 944.30 + 1,290.00 - 241.80 - 1,489.08 = $503.42
\end{align*}
\]

C. Case 3: Salvage Value for Excess Inventories and Penalties for Shortages

If demand exceeds the quantity stocked, the firm may incur stockout costs on the inventory shortage. Stockout costs are conventionally viewed as a loss of profit on the lost sale as well as resulting in additional penalties.

The first of these, the profit on the lost sale, is already captured in the payoffs shown for Cases 1 and 2. By not stocking an additional unit of inventory, the firm avoids the investment in this unit, but it also forgoes the additional payoff that would occur if demand is sufficiently high that the unit is sold.

However, there can be additional costs caused by stockouts. To the extent that a customer is injured by a stockout, the customer will try to reduce this cost by switching future sales (and the current sale, if the stockout is anticipated) to other suppliers. Thus, the firm loses the profit on these sales. The customer can also impose a stockout cost on a supplier by paying a lower price for goods from the supplier.

Panel C of Figure 2 incorporates the effect of a salvage value and stockout costs on the payoff function.\(^3\) The figure assumes that a stockout (the failure to stock one more unit) reduces the payoff by the price per unit (flattening the payoff function above $60, as in Panels A and B of Figure 2) and imposes an additional penalty of $15 per unit stockout.

Assume again that the firm uses an order quantity of Q = 300 units. The payoff function, shown in Panel C, has a minimum payoff of $1,200. At X = $30, the payoff function has a slope of 60, and at X = $60, the slope changes to -150. (At X = $60, without the penalty of $15 per unit shortage, the slope would change to zero. The slope changes to -150 because the demand is ten units per $1 price of the underlying asset and the penalty is $15 per unit times ten, or $150.)

The portfolio of options that replicates this payoff is a vertical discontinuity of $1,200 at X = $0, long 60 calls with an X = $30, and short 210 calls with an X = $60. (The slope changes from +60 to -150, representing a change of 210.) The NPV is the value of this portfolio minus the investment:

\[
\begin{align*}
NPV &= 1,200 \, \frac{\partial C}{\partial X}(X=0) + 60 \, C(X=30) - 210 \, C(X=60) - 6(300) \\
NPV &= 1,200(0.9512294) + 60(21.50) - 210(2.47) - 1,800 \\
NPV &= 1,141.48 + 1,290.00 - 518.70 - 1,800 = $112.78
\end{align*}
\]

Again, Q = 300 units is not the optimal policy. The net present value for inventory purchase size Q is

\[
\begin{align*}
NPV &= 4Q \, \frac{\partial C}{\partial X}(X=0) + 60 \, C(X=30) - 210 \, C(X=30 + 0.10Q) - 6Q
\end{align*}
\]

\(^3\)Another case is that of no salvage value and a penalty for stockouts. This is a straightforward adaption that we do not present to save space. It would have the payoff shown in Panel A in Figure 2 for an underlying asset price below $60 and the payoff shown in Panel C above $60.
Figure 2. Payoffs from Inventory Policies with Continuously Distributed Demand

Demand is a function of the price of the underlying asset. This example assumes that demand is 10 units per $1 that the asset price exceeds $30. Since the price per unit sold is $10, sales revenues are 100 times the excess of the asset price over $30.

Panel A. Payoff with Zero Salvage Value

This example assumes a stocking level of 300 units, that any excess stocks (where demand is less than stocks) have no salvage or residual value, and that there is no penalty for understocks (where demand exceeds stocks). The maximum revenue of $3,000 occurs when demand is 300 units, which occurs if the underlying asset price is $60 (X = 30 + 0.10Q where Q = 300).

Panel B. Payoff with a Positive Salvage Value

This payoff is the same as in Panel A, except that excess stocks have a positive salvage value of $4 per unit. The minimum payoff (for S < 30) is 4(300) = $1,200 and the maximum payoff (for S > 60) is 10(300) = $3,000. For 30 < S < 60, the slope is 60, which is a net revenue of $6 per unit ($10 - $4) times 10 units of sales per $1 increase in the underlying asset price.

Panel C. Payoffs with a Positive Salvage Value and a Penalty for Stockouts

This payoff is the same as Panel B, except that stockouts (where demand exceeds stocks) cause the firm to incur an additional penalty for stockouts. The penalty per unit stockout is $15, so the slope is -150 (= 15x10) for S > $60. The total slope change at point S = $60 is -210, which is a change in slope from 60 to -150.
Taking the derivative with respect to $Q$ and setting it to zero gives

$$\frac{\partial NPV}{\partial Q} = 4\frac{\partial C(X = 0)}{\partial X} - 210\frac{\partial C(X = 30 + 0.10Q)}{\partial Q} - 6 = 0$$

$$\frac{\partial NPV}{\partial Q} = -210\frac{\partial C(X = 30 + 0.10Q)}{\partial Q} = 6 - 4e^{-0.05} = 0$$

Since $\frac{\partial X}{\partial Q} = 0.10$, the optimal-stocking level occurs where

$$\frac{\partial C(X = 30 + 0.10Q)}{\partial X} = \frac{6 - 4e^{-0.05}}{21} = 0.10453$$

The presence of a penalty for stockouts increases $Q$ and $X$. Now, $Q = 390.68$ units and $X = 69.07$. The NPV for the optimal inventory solution is

$$NPV = 4(390.68)e^{-0.05} + 60C(X=30) - 210C(X=69.07) - 6(390.68)$$

$$NPV = 1,486.51 + 1,290.00 - 201.60 - 2,344.08 = 5230.83$$

Compared to Case 2, the NPV is lower at any given $Q$ because of the additional stockout penalties. The optimal $Q$ is now higher to accommodate these stockout costs, and the NPV is $5230.83$.

D. Summary of the Three Cases

These three cases illustrate how the payoff function depends on the relation between demand and the underlying asset price, the amount of inventory stocked, the price and cost of the item, the salvage value, and any stockout penalties. The value of the portfolio that replicates the payoff function depends on market conditions and the characteristics of the underlying asset. The value of the portfolio (minus the initial outlay) is the NPV of an inventory policy.

Figure 3 shows the NPV for each of the three cases for various stocking levels. As the figure shows, Case 2, which introduces a salvage value for excess stocks, increases the NPV and increases the optimal stocking level over Case 1. The inclusion of a stockout penalty causes the NPV to be lower and the optimal stocking level to be larger for Case 3 than for Case 2.

As mentioned previously for the discrete case, state prices are used instead of probabilities in an inventory model. The advantage of state prices is that they incorporate risk (the marginal utility of money) and the time value of money as well as probabilities into their values. Hence, they are economically richer than probabilities alone.

IV. Comparison of Conventional and Contingent-Claims Solutions

There are important conceptual and operational contrasts between the conventional and contingent-claims solutions to the inventory-stocking decision.

A. Conceptual Difference

For continuously distributed demand, it is important to see the essential difference between the contingent-claims and conventional approaches. We establish the optimal policy for the conventional approach by finding the inventory level that maximizes profits and for the contingent-claims approach by finding the inventory level that maximizes the NPV. This difference, as mentioned previously, is important. In this section, we look at the marginal conditions for both approaches to further emphasize this distinction.

In the conventional approach, analysts choose the optimal stockout probability by equating the expected benefit of stocking an additional unit with the expected cost of stocking that additional unit:

$$p\pi + (1 - p)s + n\pi = c$$

(7)

The expected benefit of stocking an additional unit of inventory is the left-hand-side of the above equation, where:

$$\pi$$ is the stockout probability
$$p$$ is the selling price per unit
$$c$$ is the cost per unit
$$s$$ is the salvage value per unit
$$n$$ is the penalty for a stockout

An incremental unit costs $c$. The expected revenue from selling the unit is $p\pi$, the expected salvage value is $(1 - \pi)s$, and the expected stockout penalty avoided is $n\pi$. For the continuous case, the optimal stockout probability is

$$\pi = \frac{c - s}{p - s + n}$$

(8)

We can use Equation (8) to illustrate the conventional approach to choosing stocking levels for the three cases that we discussed in the previous section. We can consider the numerator of the equation as the carrying cost on an additional unit of inventory. The
denominator is the benefit of selling an additional unit of inventory, which is the price of one more unit sold less the salvage value of one less unit left over plus the benefit of avoiding the stockout penalty on one unit.

The first case involves an investment of $6 per unit of inventory and no salvage value. The benefit is $10 if it is sold, so the optimal stockout probability is $\pi = (6 - 0) / (10 - 0 + 0) = 0.60$. The second case included a salvage value of $4 for unsold units, so the optimal stockout probability is $\pi = (6 - 4) / (10 - 4 + 0) = 2 / 6 = 0.3333$. The third case includes a $4 salvage value as well as a $15 per unit penalty for stockouts. The optimal-stockout probability is now $\pi = (6 - 4) / (10 - 4 + 15) = 2 / 21 = 0.0952$.

We can alter Equation (7) so that it is consistent with the contingent-claims approach by replacing objective probabilities with risk-neutral probabilities, and by discounting the expected benefits on the left-hand side of the equation at the riskless rate. Using $N(d_1)$ as the risk-neutral probability of a stockout (and of the option being in-the-money) results in:

$$e^{n[pN(d_1) + s(1 - N(d_1)) + nN(d_1)]} = c$$

Equation (9) sets the present value of the expected benefit of stock ing an additional unit of inventory equal to the investment in one unit of inventory.

We can rewrite Equation (9) in terms of the state price of the contingent claim:

$$\frac{\partial C}{\partial X} = e^{n}[pN(d_2) + s[1 - N(d_2)] + nN(d_2)]$$

In this new equation, $\frac{\partial C}{\partial X}$ is the state price, the value of a $1.00 payoff if the underlying asset price is above the exercise price. This state price is the product of a present value factor and $N(d_2)$. The economic interpretation of $N(d_2)$ is that it is the risk-neutral probability of the underlying asset price exceeding the exercise price, and that the option is in the money.

In Equation (10), the state price replaces the objective probability in Equation (8), and the right-hand side of equation (10) has an additional adjustment for the time value of money involved in carrying costs. The state price is economically richer than the objective probability alone because it also impounds the effects of risk and the time value of money. This economically sound risk adjustment is a fundamental advantage of the contingent-claims approach over the conventional approach.

Notice that the critical state prices calculated using Equation (10) are identical to the critical values for $\frac{\partial C}{\partial X}$ for the contingent-claims approach that we presented in the previous section. For Case 1, the optimal-state price is $\frac{\partial C}{\partial X} = 6 / 10 = 0.60$. For Case 2 (with a salvage value), the optimal-state price is $\frac{\partial C}{\partial X} = (6 - 4e^{-0.1005}) / (10 - 4) = 2.195 / 6 = 0.3658$. The third case included an additional penalty of $15 per unit short, so the optimal-state price is $\frac{\partial C}{\partial X} = (6 - 4e^{-0.1005}) / (10 - 4 + 15) = 2.195 / 21 = 0.1045$. These state prices are consistent with the value-maximizing inventory order levels shown in the previous section.

\[\text{The difference between objective probabilities and risk-neutral probabilities is important. For example, if the objective probabilities of a high and low payoff were 0.55 and 0.45, respectively, the risk-neutral probabilities could be 0.50 and 0.50, reflecting the differing marginal utilities of money in the two states of nature.}\]
In the contingent-claims approach, inventory is treated as an option on future sales. At the margin, the price of this call option is the right-hand side of Equation (9). The value of the call option, which depends on p, s, n, r, t, and the determinants of N(d), is the left-hand side of the equation. As long as the value of additional inventory exceeds its cost, the total NPV of the inventory policy should be increasing.

B. Operational Difference

An important operational difference between the conventional and contingent-claims approaches is the economic use of the variable employed in forming the sales forecasts. The conventional approach relies on an estimate of the distribution of future demand. This distribution is then used in an inventory model to make an optimal-stocking decision. If we use a forecasting model to predict the distribution of sales, it is ironic that the conventional approach then discards the underlying variables once the forecast is made.

In the contingent-claims approach, the underlying variable (or variables) used to forecast sales is the same variable that is used to value the portfolio of options. For example, if sales are related to short-term interest rates, mortgage rates, stock indices, foreign currency exchange rates, or other variables that are priced in the financial marketplace, we can use this relation to create the payoff functions described in this paper. Once the payoff functions are specified, then we use option-pricing models to develop the optimal inventory policy. Whenever the underlying economic variables used to forecast sales are actionable, the contingent-claims approach benefits from incorporating the economic relevance of these variables in the inventory valuation model.

Of course, the sales forecast can be based on more than one underlying state variable. This paper describes how we can use option-pricing models when inventory payoffs are a function of a single such asset. If sales were related to two underlying state variables, we could develop a more sophisticated option-pricing approach under which the valuation process would depend on both underlying assets. Boyle (1988) and Boyle, Evnine, and Gibbs (1989) describe the valuation of options on two or more variables.

Here, we discuss only the one-shot inventory problem. Repeat orders are needed if the inventory is to be replenished over time. The conventional solutions to this problem often rely on fixed-order intervals or fixed-order sizes and determine what safety stock the firm should maintain. Just as the repeat-order solution is an extension of the single-order solution for the conventional approach, the contingent-claims approach can be adapted to the repeat-order problem.

V. Conclusion

This paper presents a contingent-claims approach to the inventory-stocking decision. This approach incorporates the economic principles of asset-pricing models, such as the Black-Scholes option-pricing model, to replace the expected profit maximization logic of the conventional approach. The conventional approach has no mechanism for addressing the financial riskiness of an inventory investment, and thus can over- or underinvest in inventories. Asset-pricing models have been incorporated into many of the other investing and financing pricing decisions of firms, and this paper advocates using financially sound pricing models for evaluating inventory investments as well.

The usefulness of the contingent-claims approach rests on the identification of financially traded assets that can be used to forecast product demand, allowing the specification of payoff functions for the inventoried item. Another requirement for application of the approach is the willingness of financial economists to adapt basic or more complicated option-pricing models to specific inventory situations. If these requirements can be met, there should be opportunities for successfully applying the contingent-claims approach to inventory decisions.

References


Appendix: Valuing Linear Payoff Functions

This appendix reviews some of the principles used to implement and value payoff functions. Consult texts like Chance (1995) or Hull (1995) for reference material.

We can implement payoff functions, such as those shown in Figure 4, with a portfolio of options, and the value of this portfolio is the value of the payoff function. We can implement a payoff function by using three rules:

1. The slope of a payoff function at any point is equal to the number of call options with an exercise price at or below that price.
2. A change in the slope of a payoff function at a point is equal to the number of call options written with an exercise price equal to that price.
3. We can implement a discontinuity in a payoff function with the derivative of the call option with respect to the exercise price, ∂C/∂X.

Once we have a portfolio of options that replicates the payoff function, then we establish the value of the portfolio with an option-pricing model. For example, with the Black-Scholes option-pricing model, the value of a call option is:

\[ C = S_0 N(d_1) - X e^{-rX} N(d_2) \]

where

- \( S_0 \) = current level of state variable
- \( N(d) \) = value of cumulative normal for \( d \)
- \( d_1 = \frac{\ln(S_0/X) + (r + \sigma^2/2)t}{\sigma\sqrt{t}} \)
- \( d_2 = d_1 - \sigma\sqrt{t} \)
- \( X \) = exercise price of the option
- \( t \) = time until maturity of the option
- \( r \) = riskless rate of return
- \( \sigma \) = volatility of the underlying state variable

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
The portfolio can include several options, some with differing exercise prices. We handle a discontinuity in the payoff function with a pure security that pays off $1 if the underlying stock is above the exercise price, and zero otherwise. With the Black-Scholes model, the value of a discontinuity of $1.00 at $X = Y$, is

$$
-\frac{\partial C}{\partial X} = e^{-rT}N(d_2(X=Y))
$$

These rules can be used to value a variety of payoff functions. The first case is that of continuous piecewise payoff functions, such as those shown in Panels A and B of Figure 4. The value of the portfolio in Panel A is
equal to the number of calls with exercise price \( Y_1 \). The number of calls is equal to the slope of the payoff function above \( Y_1 \). If this slope is \( a \), the value of the portfolio is

\[
P = aC(X=Y_1)
\]

In Panel B, we assume the slope is between \( Y_1 \) and \( Y_2 \). The slope increases by \( b \) between \( Y_2 \) and \( Y_1 \), and decreases to zero after \( Y_2 \). The portfolio would consist of a long position of a calls with an exercise price of \( Y_1 \), a long position of \( b \) calls with an exercise price of \( Y_2 \), and a short position of \( a+b \) calls with an exercise price of \( Y_2 \). The value of this portfolio is

\[
P = aC(X=Y_1) + bC(X=Y_2) - (a+b)C(X=Y_2)
\]

Step functions and discontinuities are portfolios of pure securities. In Panel C, we assume a discontinuity of height \( h \). The value of this payoff is

\[
P = -h \frac{\partial C}{\partial X} = he^{-n}N(d_1(X=Y_1))
\]

In Panel D, we assume a payoff of \( \$1 \) if \( Y_1 < S < Y_2 \). This value is the difference in value between two pure securities:

\[
P = -\frac{\partial C}{\partial X}(X=Y_1) \cdot \left( -\frac{\partial C}{\partial X}(X=Y_2) \right)
\]

\[
P = e^{-n}N(d_1(X=Y_1)) - e^{-n}N(d_1(X=Y_2)) = e^{-n} \left[ N(d_1(X=Y_1)) - N(d_1(X=Y_2)) \right]
\]

A final special case of a pure security is one in which the exercise price \( Y_1 = 0 \) and \( Y_2 = \infty \). In this case,

\[
P = -\frac{\partial C}{\partial X}(X=0) \cdot \left( -\frac{\partial C}{\partial X}(X=\infty) \right) = e^{-n}(1 - 0) = e^{-n}
\]

Since \( N(d_1) = 1 \) for \( X = 0 \) and \( N(d_2) = 0 \) for \( X = \infty \), this is the case of a certain payoff of \( \$1.00 \), which has a present value of \( e^{-n} \). This payoff is shown in Panel E. For \( X > 0, -\frac{\partial C}{\partial X} \) is a decreasing function of \( X \), approaching zero for high values of \( X \).

Finally, we can also value twice differentiable (or curvilinear) payoff functions, such as the one shown in Panel F. This is a portfolio consisting of a large number of options at various exercise prices that reflect the slope changes along the payoff function. The value of this portfolio is

\[
P = \int_{b}^{s} \frac{\partial^2 C}{\partial S^2}(X=S) dS
\]

\( \frac{\partial^2 C}{\partial S^2} \) is the second derivative of the payoff function at point \( S \). \( H \) is the height of the payoff function, \( H' \) is the slope, and \( H'' \) is the change in the slope. The paper does not include examples of twice differentiable payoff functions.

An actual payoff function can be a combination of piecewise linear payoffs, discontinuities, and curvilinear payoffs. A few such cases are included in the paper.