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1. Math Skills Review

Introduction

In this slide deck, you will explore the following:

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Equalities & Inequalities

Equalities

- $a = b$ — *a* equals *b*
- $a \neq b$ — *a* does not equal *b*
- $a \equiv b$ — *a* equals *b* <u>by definition</u>

$$\left.\begin{array}{l} a \approx b \\ a \sim b \\ a \cong b \end{array}\right\}$$

These can have subtly different meanings, but we can use “approximately equal” for all.

Equalities & Inequalities

Inequalities

- $a < b$ "strict" inequality

  a is less than b, (also, b is greater than a)

- $a > b$ "strict" inequality

  a is greater than b, (also, b is less than a)

- $a \leq b$ a is less than or equal to b

  a is at most b

  a is not greater than b

- $a \geq b$ a is greater than or equal to b

  a is at least b

  a is not less than b

Equalities & Inequalities

Double Inequalities & Intervals

- $a < x < b \Longleftrightarrow x \in (a,b)$
- $a < x \leq b \Longleftrightarrow x \in (a,b]$
- Given that $a < b$
  - $a + c < b + c$ (addition preserves inequalities)
  - $a - c < b - c$ (subtraction preserves as well)
  - $a \bullet c < b \bullet c$ (multiplication preserves only if c is positive)
    - (if c is negative, must reverse the direction of inequality)
  - $a / c < b / c$ (division preserves only if c is positive)
    - (if c is negative, must reverse the direction of the inequality)
  - Must also reverse the direction of the inequality if we swap sides. I.e., given than a < b, we can write b > a but clearly not b < a.

Absolute Value

Absolute value

- $|a| \equiv a$ *if* $a \geq 0$, and $-a$ if $a < 0$
- $|a| \equiv$ perform all operations inside the lines, then make the results positive.
- Rules:
  - $|a| \;\geq\; 0$
  - $|-a| \;=\; |a|$
  - $|a - b| \qquad |b - a|$
  - $|a - b| \qquad |a| \; - |b|$
  - $|a \bullet b| \qquad |a| \; |b|$
  - $|a / b| \qquad |a| \; / \, |b|$
  - $|a| < b \Leftrightarrow -b < a < b \Leftrightarrow a \in (-b, b)$

Ratios and Fractions

a / b = "The ratio of ____ to ____

– Common ratios you will see in finance:
  - *EPS* earnings per share
  - *P-E* price-to-earnings
  - *ROE* return on equity (= net income / book value of equity)

    = (net income/sales) x (sales/total assets) x (total assets/BV equity)

    This decomposition is the "DuPont Formula"
  - *M/B* market-to-book
  - *D/E* debt-to-equity (note
  - *D/A* debt to assets

– Financial ratios enable "common-size" analysis to compare companies of different sizes.

Ratios and Fractions

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \qquad \frac{a}{c} + \frac{b}{d} \; ? \; \frac{a+b}{c+d}$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a/c}{b/d} = \frac{a}{b} \bullet \frac{d}{c} = \frac{a \bullet d}{b \bullet c}$$

Ceteris paribus, increasing (decreasing) the numerator increases (decreases) the ratio's value.

Ceteris paribus, increasing (decreasing) the denominator decreases (increases) the ratio's value.

If a ratio is positive, adding the same positive amount to both numerator and denominator moves the value toward 1, and subtracting the same positive number from both moves the value toward 0.

Multiplying both by the same number has no effect.

Some changes can have ambiguous effects…be careful!

Solving for unknowns in ratios

Cross-multiplication can help us solve for unknowns in ratios. To cross-multiply, set the product of the left-hand side's numerator and the right-hand side's denominator equal to the product of the left-hand side's denominator and the right-hand side's numerator.

Example 1:

$$\frac{5}{8} = \frac{A}{120}$$

$$(5)(120) = (8)(A)$$

$$600 = 8A \Rightarrow A = 75$$

Example 2:

$$0.60 = \frac{980}{\text{Total equity} + 980}$$

$$\frac{0.60}{1} = \frac{980}{\text{Total equity} + 980}$$

$$(0.60)(\text{Total equity} + 980) = (1)(980)$$

$$(0.60)(\text{Total equity}) + (0.60)(980) = 980$$

$$(0.60)(\text{Total equity}) = 392$$

$$\text{Total equity} = 653.33$$

Percentage Growth

Definition

- *Percentage growth* (or percent change) in *a* from time *t-1* to time *t* :

$$100\left(\frac{a_t - a_{t-1}}{a_{t-1}}\right)\%$$

- Growth from \$90 to \$100:
- Growth of \$50 from \$75 base:

Applications: Inflation (% growth in a price index), stock return

Easy one: \$80 stock. It increase 10% one day and falls 10% the next.

- What is the total percent change?
- \$80.00 → \$88.00 → \$79.20
- 100 [(79.20 – 80.00) / 80.00]% = -1.000%

Summation

Definition

$$\sum_{i=1}^{n} f(i)$$ Evaluate *f*(*i*) for each integer from *i* = 1 to *n* and sum the terms

Applications

- There are many applications. Some examples include: profit through time, statistics, weighted averages, portfolio math, recursive formulas, or any finite or infinite series

$$\text{Computation example 1}: \sum_{i=1}^{n} X_i = X_1 + X_2 + X_3 + \ldots . X_n$$

$$\text{Computation example 2}: \textit{Suppose we have } Z_1 = 1,\ Z_2 = 3,\ Z_3 = 2,\ \textit{and }\ Z_4 = 5$$

$$\begin{aligned}\sum_{i=1}^{4} (Z_i - 3)^2 &= (Z_1 - 3)^2 + (Z_2 - 3)^2 + (Z_3 - 3)^2 + (Z_4 - 3)^2 \\ &= (1-3)^2 + (3-3)^2 + (2-3)^2 + (5-3)^2 \\ &= (-2)^2 + (0)^2 + (-1)^2 + (2)^2 \\ &= 4 + 0 + 1 + 4 \\ &= 9\end{aligned}$$

Summation

Recursive formula example: we start with some initial cash amount at t=0, then at t = 1 and later periods there are receipts and disbursements

- $\text{Cash}_t = \text{Cash}_{t-1} + \text{Receipts}_t - \text{Disbursements}_t$
- So, cash position at time *n* is $C_n = C_0 + \sum_{t=1}^{n}(R_t - D_t)$

Finite series: $$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}=\sum_{j=1}^{4}\frac{1}{2j}$$

Infinite series: $$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\ldots=\sum_{k=0}^{\infty}\frac{1}{3^{k+1}}$$

Careful to always note starting and ending points of the summation!

Two handy shortcuts:

$$\sum_{j=1}^{n} c = nc \qquad \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

Weighted vs. Simple Average

In general terms, an *average* is a measure of typical value or central tendency.

A sloppy use of the term is to assume the average is the *arithmetic mean* or *simple, equally-weighted average*:

$$\sum_{j=1}^{n} \frac{x_j}{n} = \frac{1}{n}\sum_{j=1}^{n} x_j = \frac{\sum_{j=1}^{n} x_j}{n}$$

In finance, many situations call for a *weighted average.*

$$\sum_{j=1}^{n} w_j x_j \ \ where \sum_{j=1}^{n} w_j = 1 \ \ (sum\ of\ weights = 1)$$

– Note the arithmetic mean is a special case where every $w_j = (1/n)$

Weighted Average Example

Dividend yield for a portfolio of three stocks:

- 30% invested in Ford, which pays dividend yield of 4%
- 25% invested in Apple, which pays dividend yield of 2.4%
- 55% invested in MM Microtech, which pays dividend yield of 0%

$$DY_{port} = \sum_{j=1}^{3} w_j \, DY_j = (0.30)(4.0\%) + (0.25)(2.4\%) + (0.50)(0\%) = 1.8\%$$

Expected return over the next year of a portfolio consisting of:

- \$2,500 invested in ABD, which you expect to return 8%
- \$7,000 invested in MNO, which you expect to return 5%
- \$12,000 invested in XYZ, which you expect to return 10%
- Total invested is \$21,500

$$E(r_{port}) = \left(\frac{2.5}{21.5}\right)(8\%) + \left(\frac{7.0}{21.5}\right)(5\%) + \left(\frac{12.0}{21.5}\right)(10\%)$$

Or,

$$E(r_{port}) = \frac{(2.5)(8\%) + (7.0)(5\%) + (12.0)(10\%)}{21.5} = 8.14\%$$

Product

Definition

$\prod_{i=1}^{n} f(i)$ Evaluate *f*(*i*) for each integer from *i* = 1 to *n* and multiply the terms

Main application in finance is with compound interest

– $1000 invested at 5% per year will grow to how much after 20 years?

$$\$1000(1+0.05)(1+0.05)(1+0.05)\ldots(1+0.05) \quad (20\ terms\ total)$$

$$which\, is: \quad \$1000\prod_{i=1}^{20}(1.05)=\$2653.30$$

– Typically we would instead write $1000(1.05)^{20}, but what about some principal amount *P* invested a *various* yearly rates r_i for 20 years?

$$\$P\prod_{i=1}^{20}(1+r_i)=\$2653.30$$

Functions

Definition: $y = f(x)$

- A rule (f) that assigns each value of a variable (x) one and only one value [f(x)]
- x = independent variable, explanatory variable, exogenous variable
- y = dependent variable, endogenous variable

Univariate functions

- Economics: quantity demanded = f(price)
- Marketing: sales = f(advertising)
- Finance: expected return = f(risk)

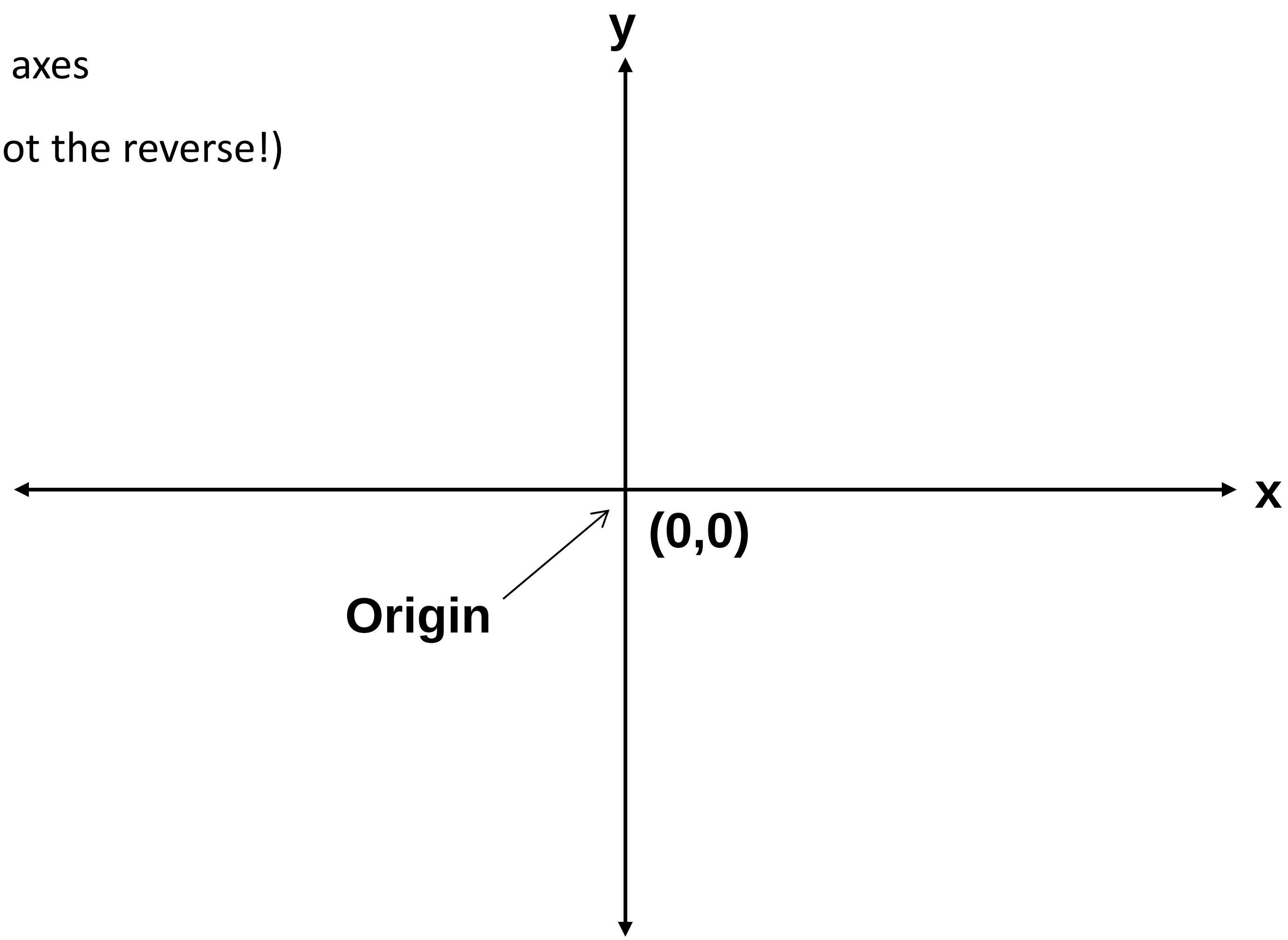
Multivariate functions

- Economics: $Q_X = f(P_X, P_Y, P_Z, \text{Income}, \text{Advertising})$
- Finance: Value of an option = f(stock price, exercise price, interest rate, time to maturity, expected volatility of stock price)

Cartesian coordinate system

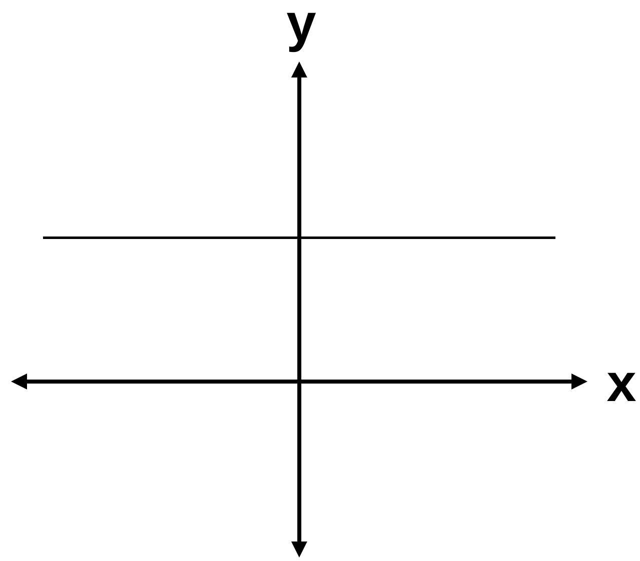
X and Y axes

(x,y) (not the reverse!)



Constant function

$y = c$

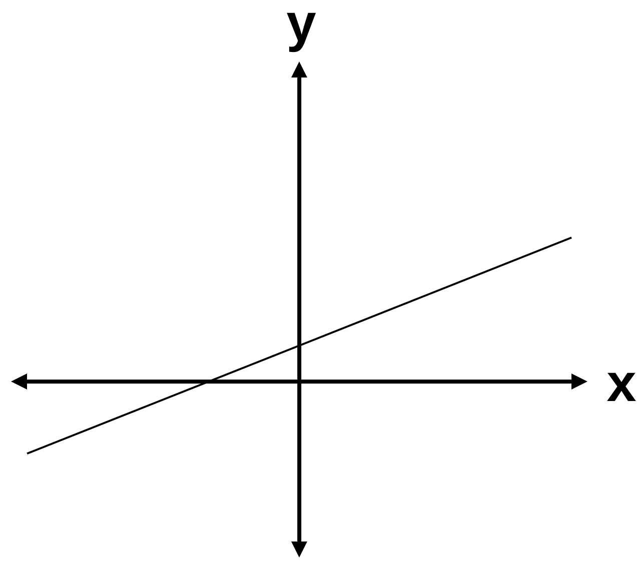


Applications

- License fee with state agency unaffected by sales, profits, etc.
- Cost of a course is unaffected by your performance

Linear function

$$y = mx + b$$

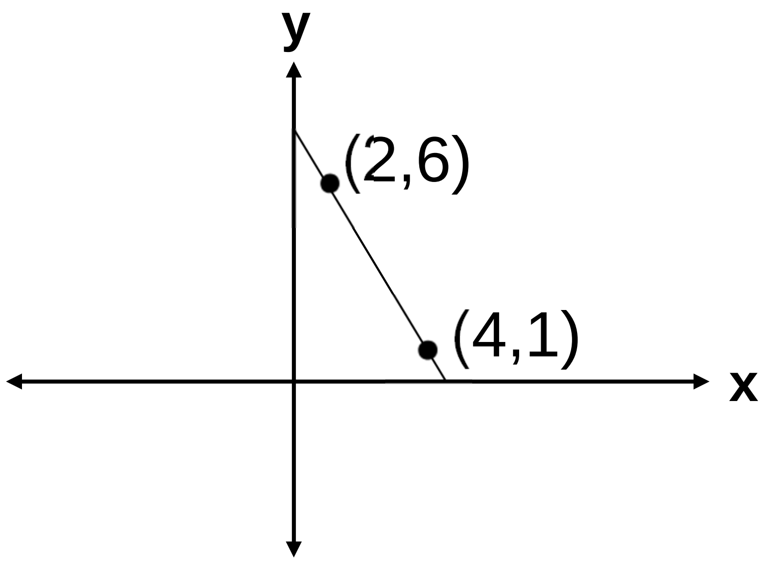


Application

- Suppose the quantity demanded is a linear function of the price.
- The cost of producing a product can be a fixed cost plus a variable cost directly proportional to the quantity produced.

Linear function

y = mx + b

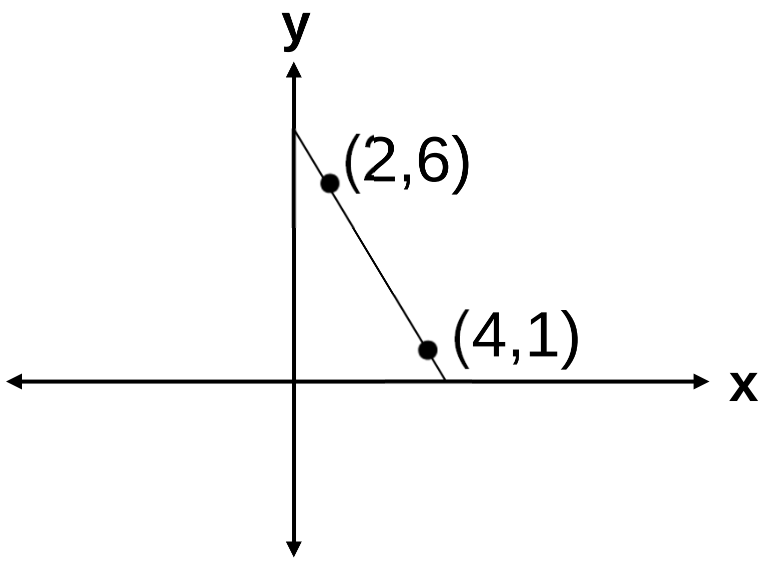


Slope = m $= \dfrac{\Delta y}{\Delta x} = \dfrac{rise}{run} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_1 - y_2}{x_1 - x_2}$

What is the slope of above line?

Linear function

y = mx + b



Find Q $= \dfrac{Percentage\ change\ in\ y}{Percentage\ change\ in\ x}$

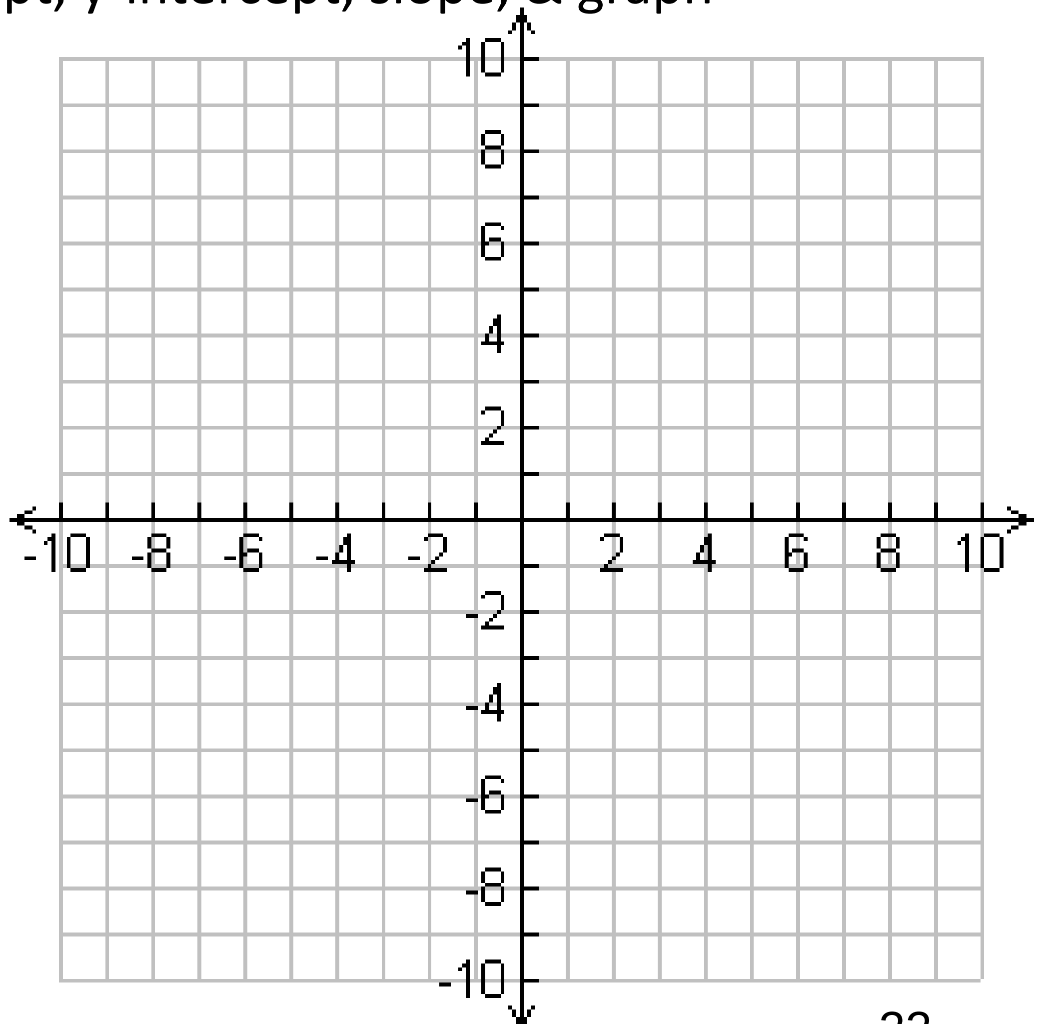
How does this relate to the slope?

- Slope = -0.83333 • (y_1/x_1) = -0.83333 • (6/2) = -2.50
- Note Q $= \left[\dfrac{(1-6)}{(4-2)}\right]\left[\dfrac{2}{6}\right] = [slope]\left[\dfrac{x_1}{y_1}\right]$

Linear function

$10x - 20y - 40 = 0$

Find the y=mx+b form, x-intercept, y-intercept, slope, & graph



Linear function

$10x - 20y - 40 = 0$

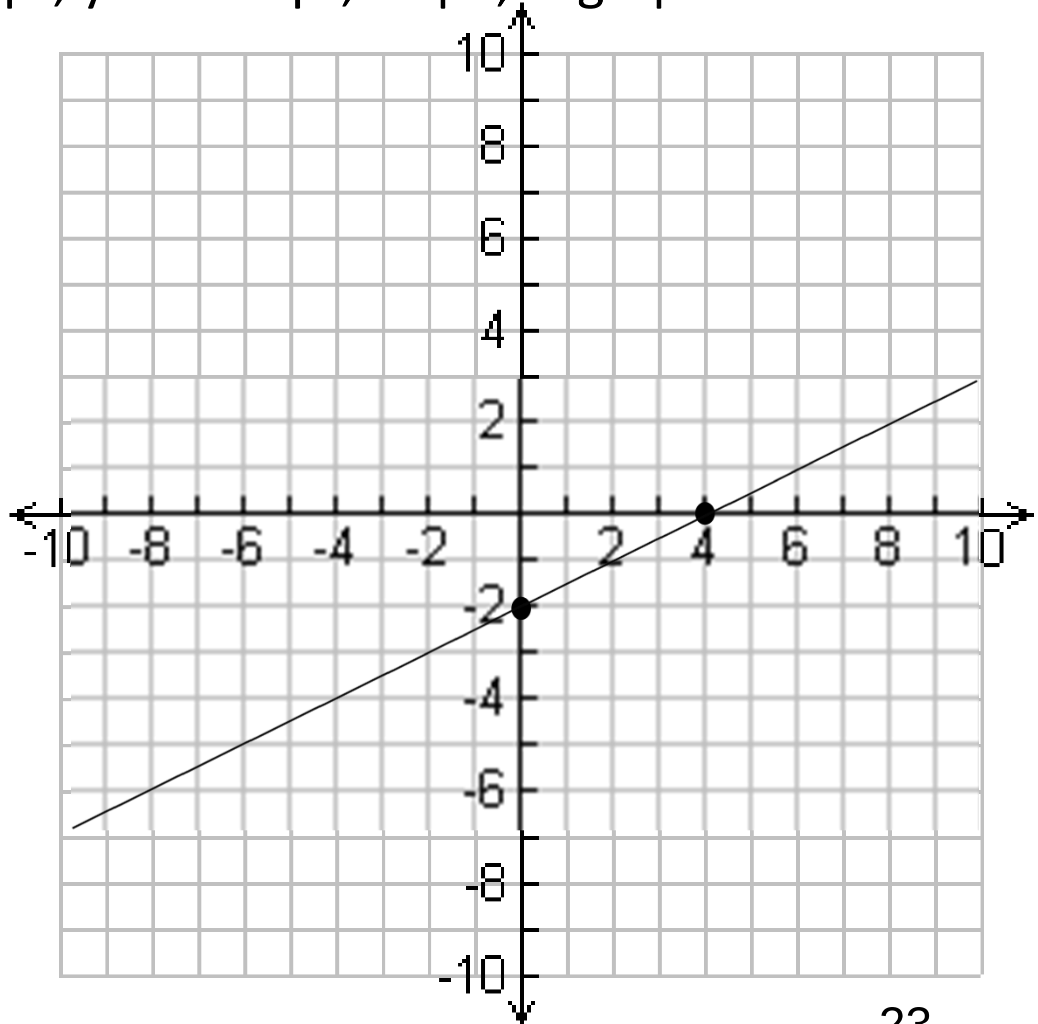
Find the y=mx+b form, x-intercept, y-intercept, slope, & graph

Rearrange/simplify into $y = 0.5x - 2$

x-int: let $y = 0$ and solve for $x = 4$

y-int: let $x = 0$ and solve for $y = -2$

Slope = 0.5



Linear function

Find the y=mx+b form of a line given

- Slope = 5, point = (10, 40)
  - Answer: y =
- Slope = 3, y-intercept = -5
  - Answer: y =
- Slope = 6, x-intercept = -2
  - Answer: y =
- Points (3,10) and (13, 5)
  - Answer: y =

Linear function

Sales of TZPPB (Tim's Zany Ping Pong Balls) increased from \$780m in 2011 to \$900m in 2013. If sales are approximated by a linear function, express sales (S) as a function of elapsed time (t) since it went public in 2000

Find slope (m): m = rise/run =

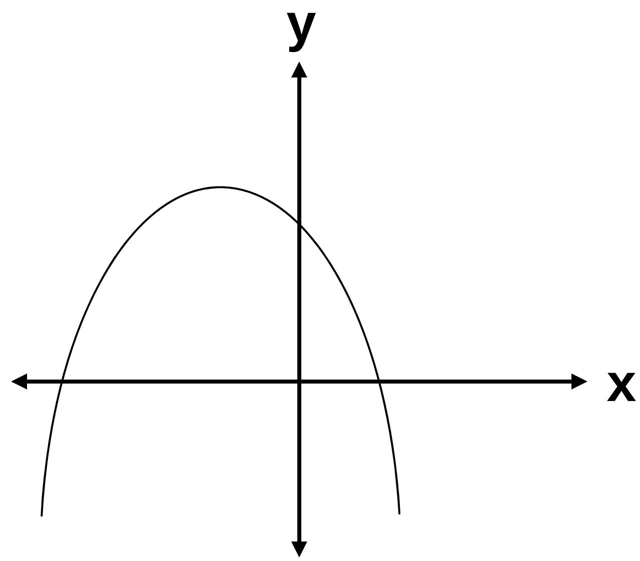
So, thus far we have: S =

Find b:

Final equation: S =

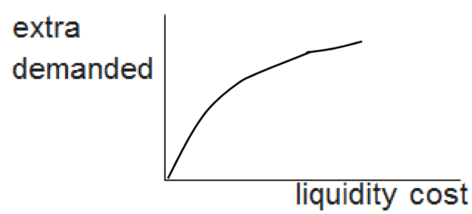
Quadratic function

$y = ax^2 + bx + c$



Applications

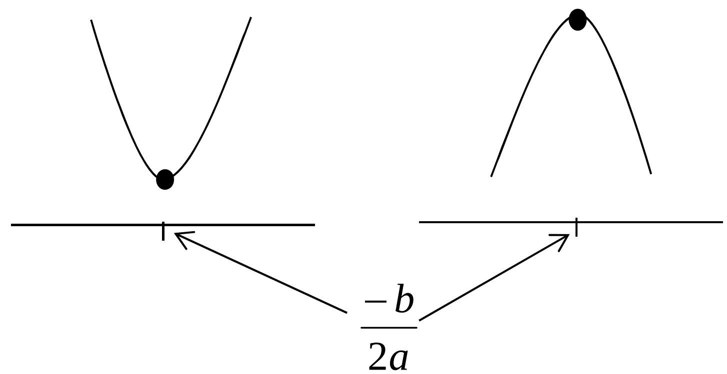
- Advertising increases profits up to a point, but eventually the increased cost is no longer sufficiently recovered through increased sales.
- Investors demand a higher expected return when the liquidity costs (costs of trading) are higher, but the extra expected return demanded increases at a decreasing rate.



Quadratic function

$y = ax^2 + bx + c$

The value of X that minimizes/maximizes (depending on the function) is $\frac{-b}{2a}$



Plugging this in, the min (or max) value of Y is: $c - \frac{b^2}{4a}$

This point is called the vertex, & the values of (x,y) are: $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$

The roots, or values of X that make Y = 0: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Quadratic function: example

Suppose that the weekly quantity demanded for your new invention is a linear function of price, P is as follows:

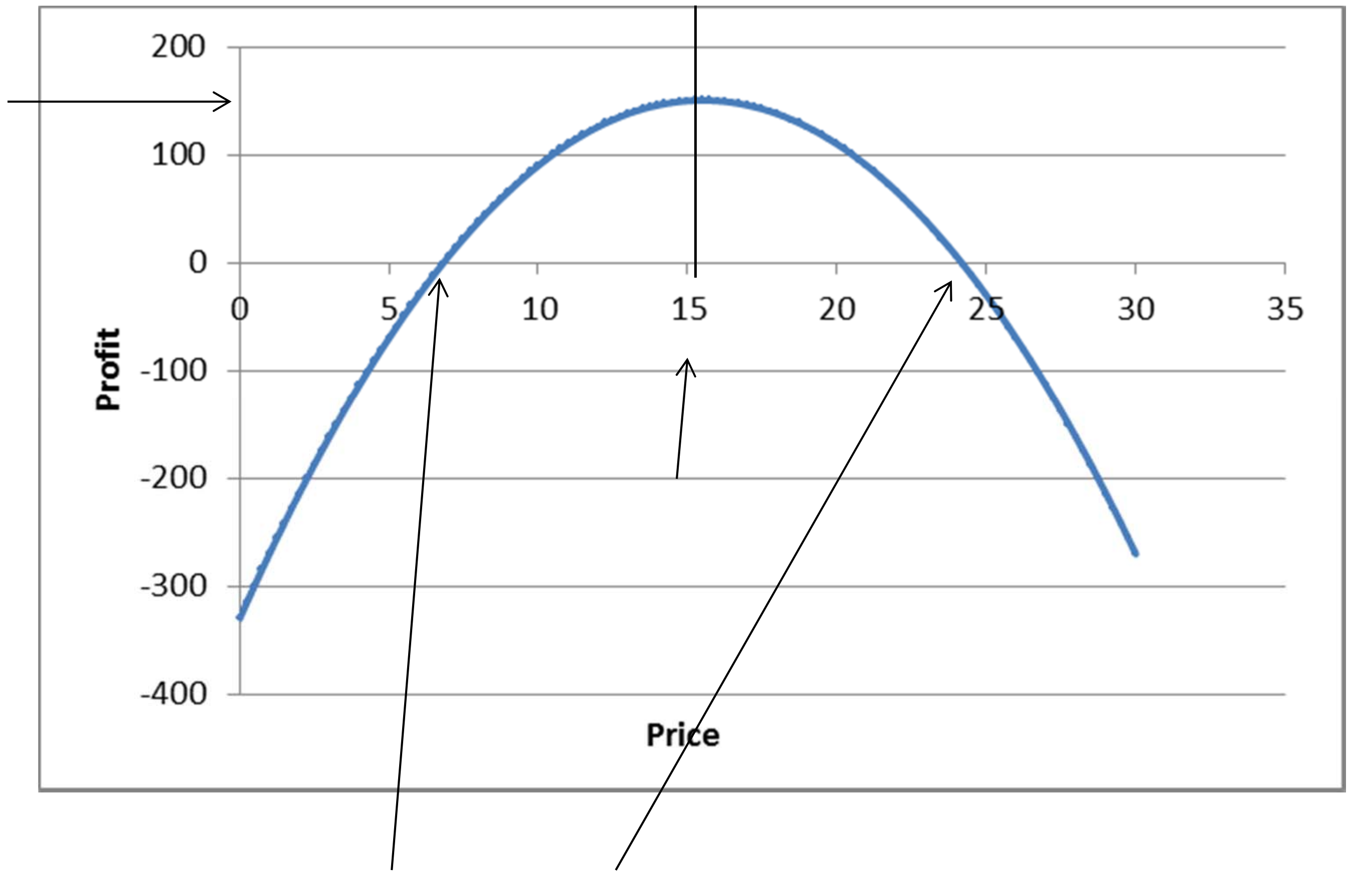
- Demand = 50 – 2p

Each unit cost you $6 to make (materials and labor), plus there is a weekly overhead expense (building rental, etc.) of $30

What is the profit function?

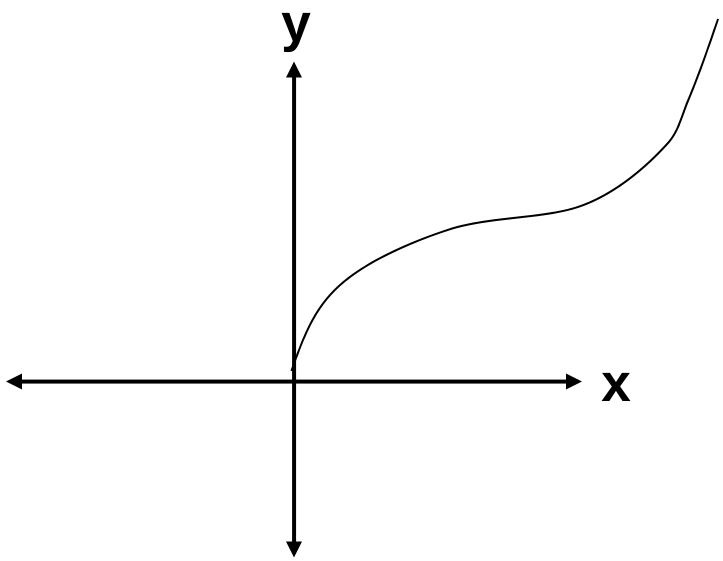
Quadratic function: example

Graph the profit function, and find optimal price to charge and the resulting profit *without using calculus.*



Cubic function

$y = ax^3 + bx^2 + cx + d$

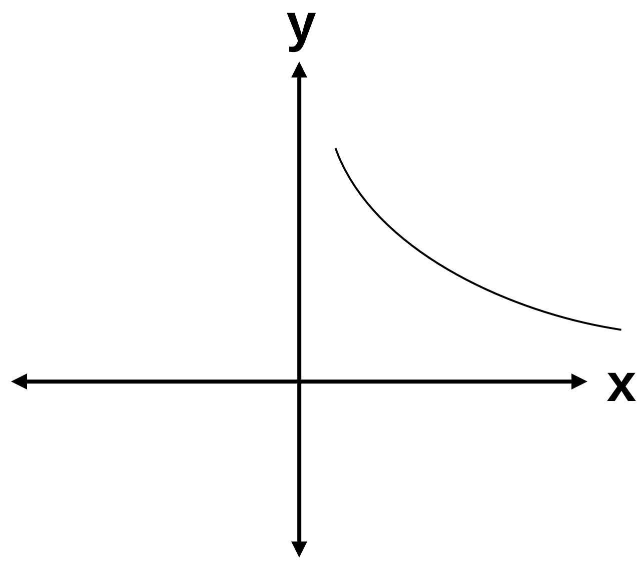


Application

- Total costs rise as output increases, first at a decreasing rate, then at an increasing rate

Hyperbolic function

$y = a / x$

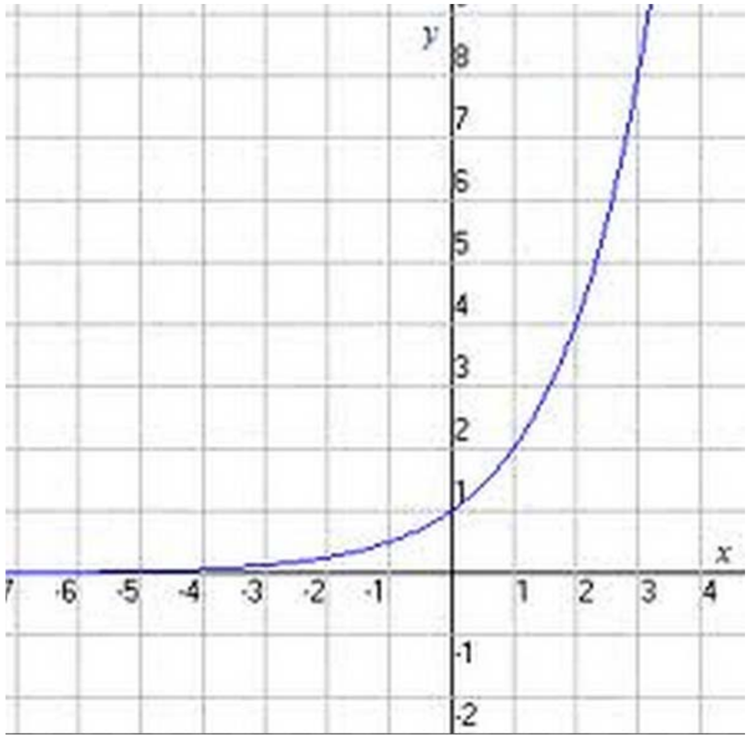


Application

– The present value of an annual perpetuity of $P each year, as a function the discount rate used to value it, looks like the above graph (y = present value, and x = discount rate)

Special case: Exponential function

y = e^x

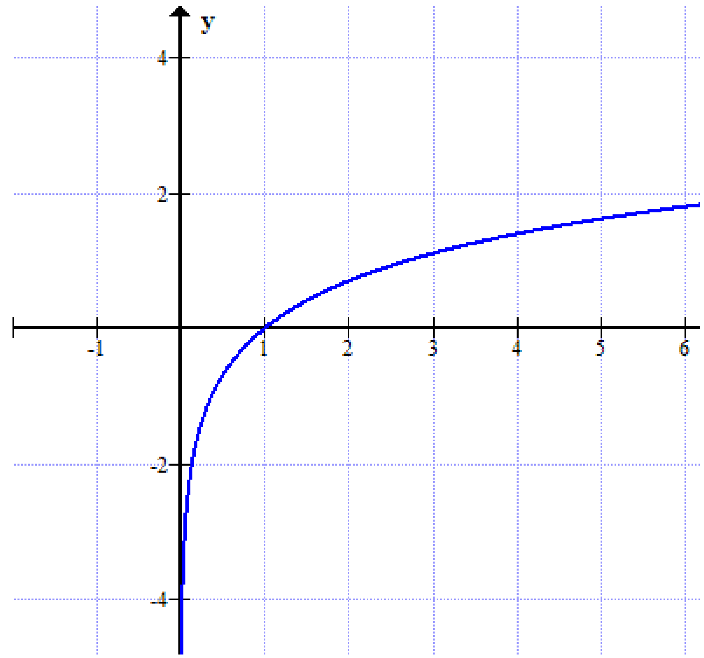


Applications

- In certain cases, an interest rate is “continuously compounded”…and in this case, calculating how money grows uses the exponential function – we’ll see this later.
- Some financial models also use exponentials (e.g., Nelson-Siegel term structure model)

Special case: Natural log

y = ln(x)



Application

– This is the inverse of the exponential, and is extremely helpful in using algebra to solve certain sorts of problems

  • E.g., In how many years will $100 grow to $862.71 if the annual return is 8%? This asks us to solve:

– Statistics of stock returns: stock returns are not "normally distributed," but are approximately log-normally distributed

  • That is, the log return, which is $\ln(P_1/P_0)$, is approximately normal

Inverse function

If $y = f(x)$, then *by definition*

- $x = f^{-1}(y)$

Example

- If $Q = f(P)$ is $Q = 20 – 4P$,

  then $P = f\text{-}1(Q)$ would be $P = 5 – 0.25Q$

Applications

- This is simply manipulating a given equation to reverse the input and output
- $y = \exp(x)$ and $y = \ln(x)$ are inverse functions of each other
  - $e^{\ln(x)} = x$ and $\ln(e^x) = x$

Working with natural logs

$y = \ln(x)$

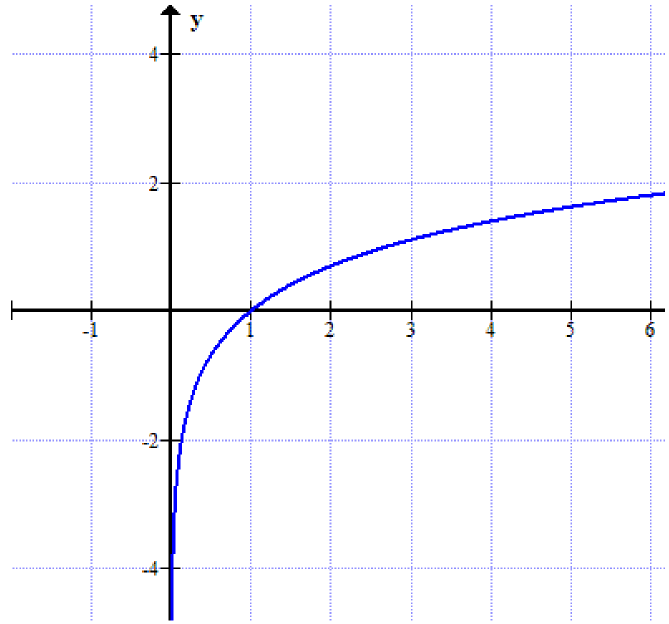
$\ln(1) = 0$, $\ln(0) = -\infty$

ln(non-positive number): undefined

Product rule: $\ln(x \bullet y) = \ln(x) + \ln(y)$

Quotient rule: $\ln(x / y) = \ln(x) - \ln(y)$

Power rule: $\ln(x^y) = y \bullet \ln(x)$



Let's solve: $862.71 = 100\,(1.08)^n$

Example application: we are solving for the number of years it takes \$100 to grow into \$862.71 if the interest rate is 8% per year

Exponents

$a^m a^n = a^{m+n}$ (please note: $a^m b^n \neq ab^{m+n}$)

$a^m / a^n = a^{m-n}$

$(a^m)^n = a^{mn}$

$a^{m/n} = \sqrt[n]{a^m}$

$a^0 = 1$

$a^{-3} = 1 / a^3$ (because $= a^{0-3} = a^0 / a^3 = 1/a^3$)

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Exponents

Practice:

$10^4 \times 10^3 =$

-

$10^3 \times 10^5 \times 10^{-2}$

-

$[\ 10^2 \times 10^4\ ] / [\ 10^3 \times 10^5]$

-

$$\sqrt{10^4} =$$

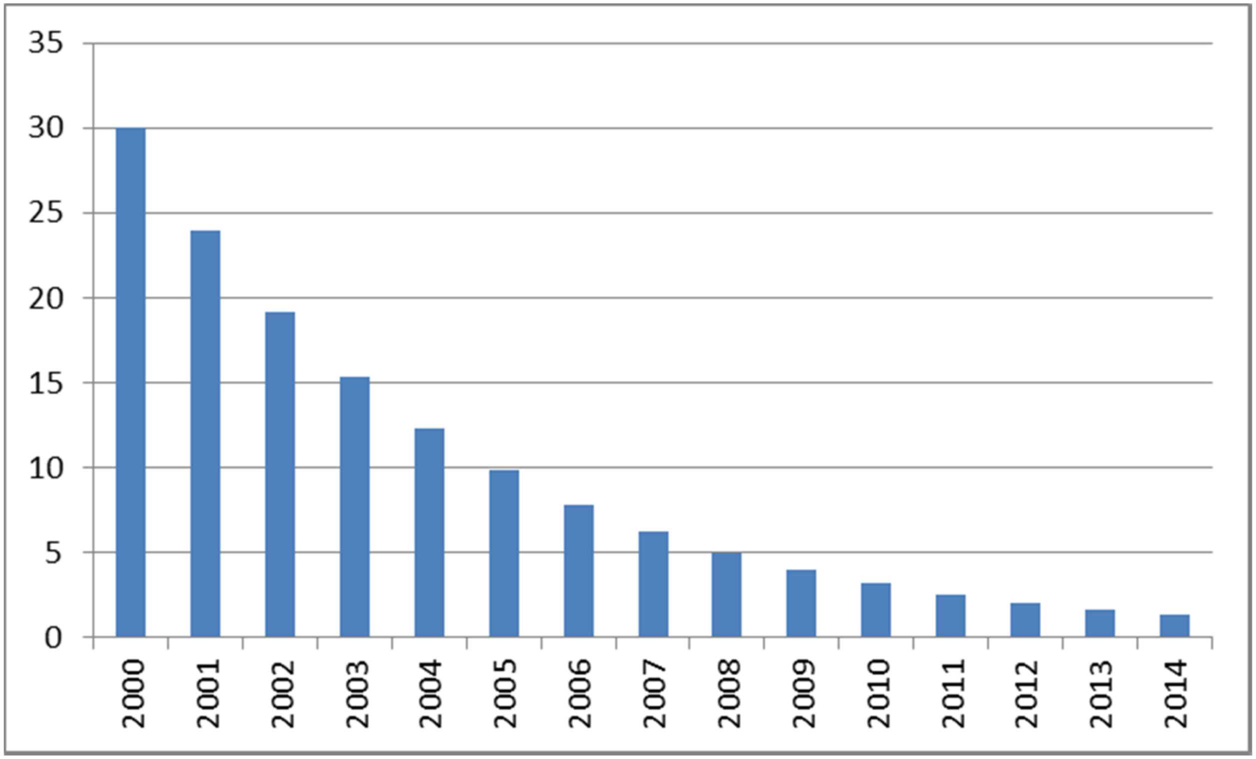
-

$$\frac{10^5 / 10^2}{\sqrt[3]{10^9}} =$$

Exponents

Suppose Blueberry is a maker of cell phones that had 30m customers in 2000, but has lost 20% of its customers each year ever since

Customers as of 2010?



Exponents

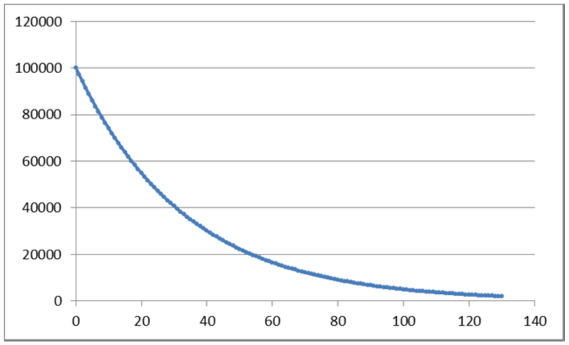
You may encounter exponential notation ("scientific notation") either in a problem or, more likely, in a solution your calculator provides.

$10000 = 1 \times 10^{4}$	$12345 = 1.2345 \times 10^{4}$
$1000 = 1 \times 10^{3}$	$1234 = 1.123 \times 10^{3}$
$100 = 1 \times 10^{2}$	$123 = 1.23 \times 10^{2}$
$10 = 1 \times 10^{1}$	$12 = 1.2 \times 10^{1}$
$1 = 1 \times 10^{0}$	
$1/10 = 1 \times 10^{-1}$	$0.123 = 1.23 \times 10^{-1}$
$1/100 = 1 \times 10^{-2}$	$0.0123 = 1.23 \times 10^{-2}$
$1/1000 = 1 \times 10^{-3}$	$0.00123 = 1.23 \times 10^{-3}$
$1/10000 = 1 \times 10^{-4}$	$0.000123 = 1.23 \times 10^{-4}$

On a calculator, 9.12E8 means $9.12 \times 10^{8} = 912{,}000{,}000$

Exponential Decay

An exponential decay process is one characterized by a constant percentage decrease in value over time. For example, the resale value V of a certain type of industrial equipment has been found to behave according to $V(t) = 100{,}000e^{-0.03t}$ where t = years since original purchase.



a. What was the original value?

c. What is the expected resale value after 5 years? After 6 years?

c. What is the annual rate of decay? In other words, by what percent does V decrease each year?

a. $V(0) = 100{,}000e^{0} = \$100{,}000$

b. $V(5) = 100{,}000e^{-0.15} = \$86{,}070.80$, and $V(6) = 100{,}000e^{-0.18} = \$83{,}527.02$

c. 83,527.02 / 86,070.80 = 0.97045, so the value declines by approximately 3% each year. (Note the decline each year is 2.9554466%, which is $1 - e^{-0.03}$)

Linear Algebra

Two of the most popular ways to solve a system of equations: Substitution and Elimination. Choose whichever seems most convenient.

Substitution: solve one equation for y (or x), then substitute in the other to solve for the other

$$6x - 4y = 20$$
$$5x + 10y = -10$$

From first equation: $y = (6x-20)/4 = 1.5x - 5$

Using second: $5x + 10(1.5x-5) = -10 \rightarrow x = 2$

And using either equation, now find $y = -2$

Linear Algebra

Elimination…look to add or subtract equations so that one variable is eliminated. May need to multiply one or both equations first.

$$6x - 4y = 20$$
$$5x + 10y = -10$$

E.g., multiplying first by 5 and second by 2 yields

$$30x - 20y = 100$$
$$10x + 20y = -20$$

In this case we can add equations ➔ 40x = 80, x = 2

And use either equation to solve for y = -2

Tips:

– Do *not* count on being able to use a graphing calculator

– Always plug back in to check your answer!

Linear Algebra

Let's try this one:

$$\begin{aligned} 2x - 3y &= 12 \\ -6x + 9y &= -36 \end{aligned}$$

Linear Algebra

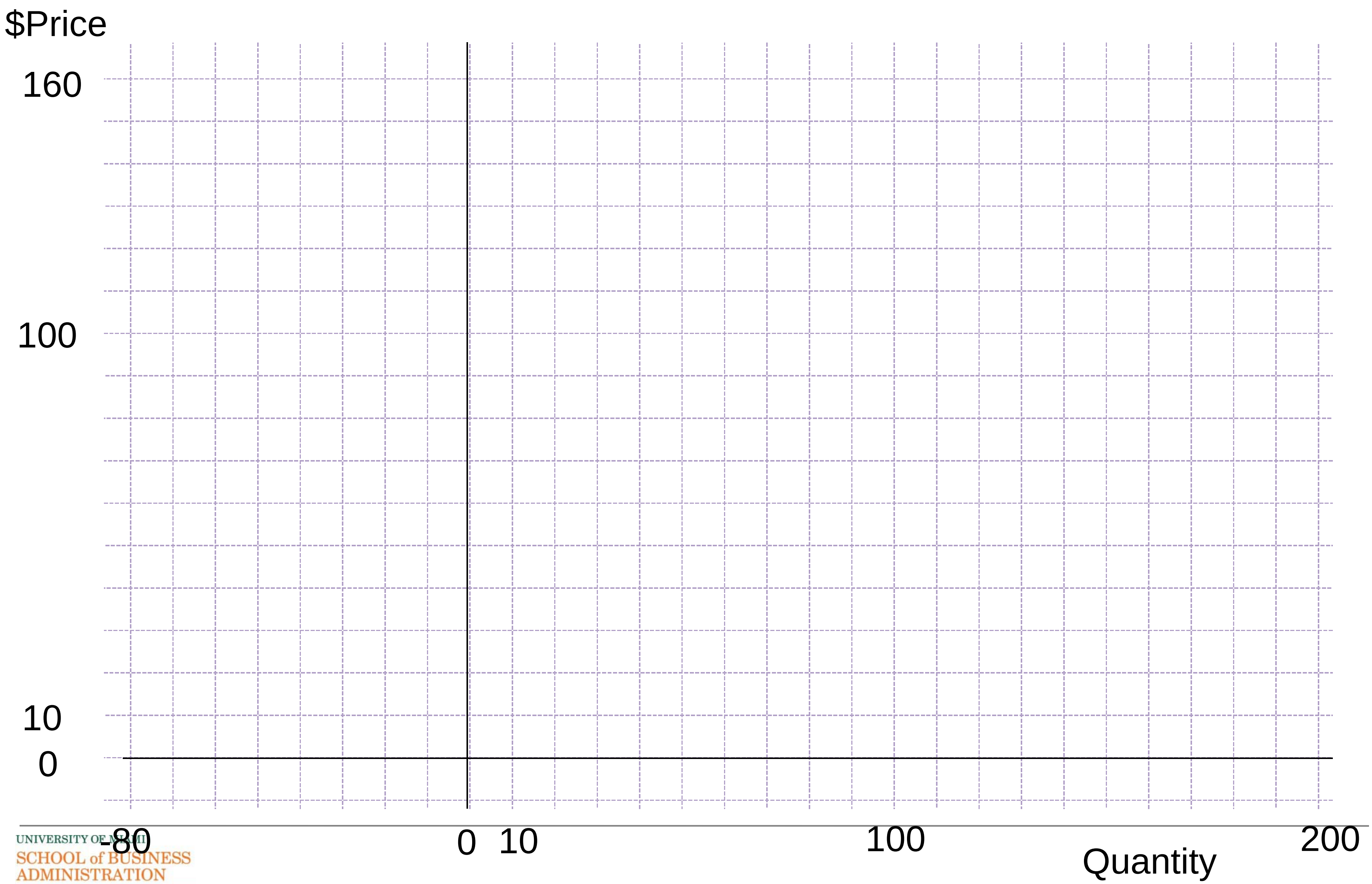
You own a yo-yo shop and you sell your high-end yo-yos for $50 each and have two potential markets: teenagers and university professors. Monthly demand curves are as follows:

– $Q_{TA} = -2P + 200$, and $Q_{UP} = -4P + 150$

Graph the demand curves on the page that follows

Will your pricing strategy attract both customers?

$Q_{TA} = -2P + 200$, and $Q_{UP} = -4P + 150$



Linear Algebra

Another application is break-even analysis

Profit = Units sold x (Price – Variable cost) – Fixed costs

We break even when profit = 0, or

- Units sold x (Price – Variable cost) = Fixed costs
- Units = Fixed costs / (Price – Variable cost)

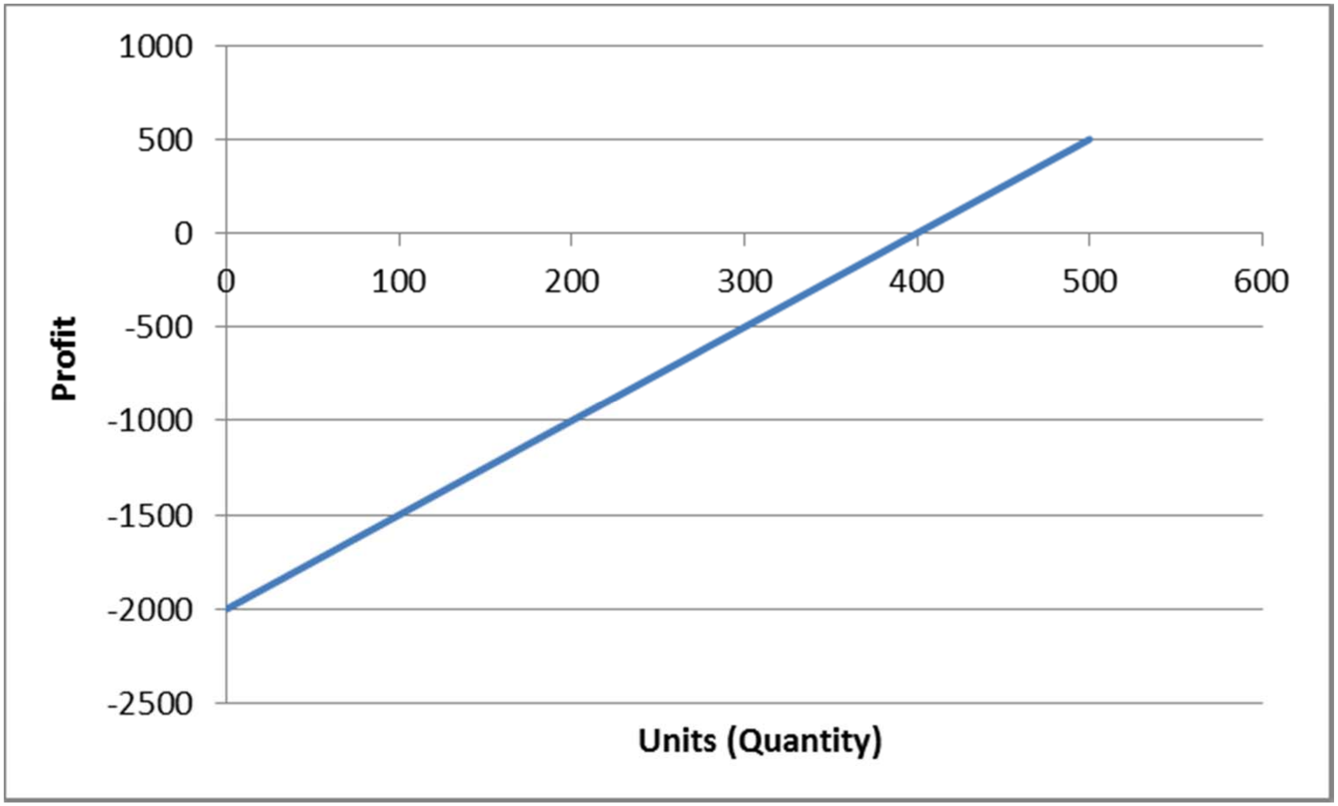
Example: You are thinking of opening a stand in Dadeland Mall to sell ear muffs.

- Retail price = $20 per unit
- Variable cost (materials, labor, etc.) = $15 per unit
- Fixed cost per month (renting space) = $2,000

Linear Algebra

Units to break even: 2000 / (20-15) = 400 per month

Profit function looks as follows: $\pi = (20-15)Q - 2000$



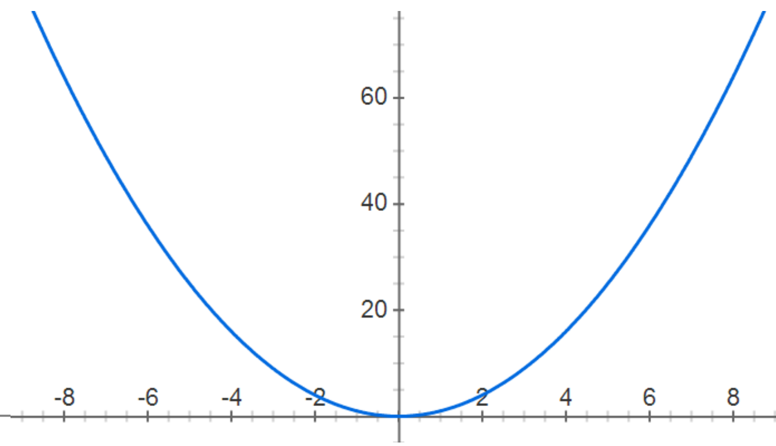
Derivatives (from basic Calculus)

Derivative = slope of a curve at a particular point

Slope of $y = 4 + 8x$

- Slope = 8
  - As x increases 1 unit, y increases 8 units
  - The change in y is 8 times greater than the change in x
  - In math speak: $\frac{\Delta y}{\Delta x} = 8$

What about $y = x^2$



Derivatives

To "take the derivative" we calculate the slope over a very small interval (from x = 1 to x = 1+k)

$$\frac{\Delta y}{\Delta x} = \frac{f(x+k) - f(x)}{k}$$

Then, take the limit as k moves to 0

$$\lim_{k \to 0} \frac{\Delta y}{\Delta x} = \lim_{k \to 0} \frac{f(x+k) - f(x)}{k}$$

For y = x^2 , the derivative is

$$\lim_{k \to 0} \frac{+k)^2 - (x)^2}{k} = \quad \lim_{k \to 0} \frac{x^2 + 2kx + k^2 - x^2}{k} = \lim_{k \to 0} \; 2x + k = \underline{\qquad\qquad}$$

Derivatives

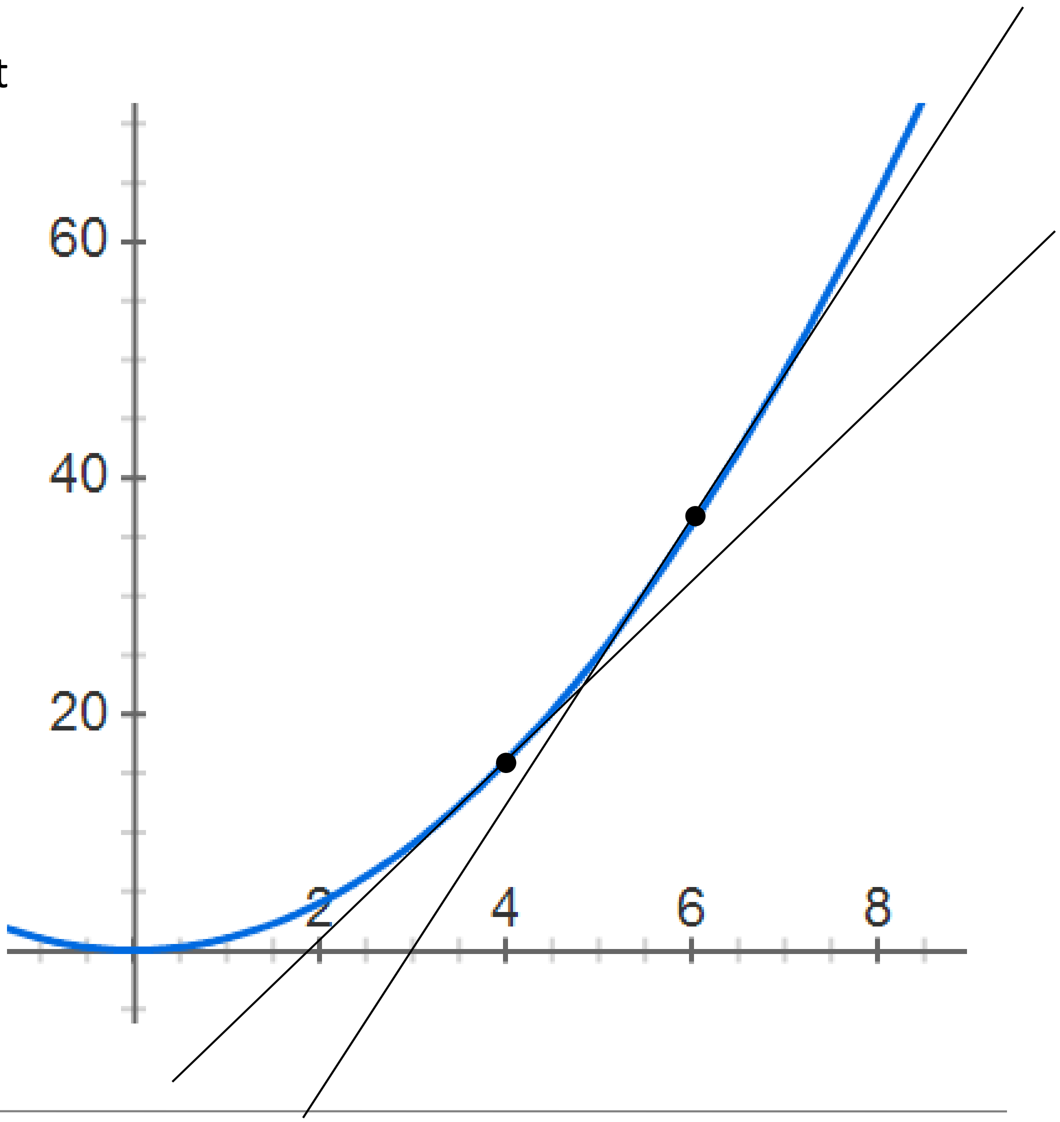
For $y = x^2$, the slope is not constant -- it is a function of the value of x: $f'(x) = 2x$

The slope at any point x on the graph is the slope of a tangent line at that point

At x = 4, slope is 8

- The tangent line at x = 4 is the equation y= 8x - 16 because two points are (2,0) and (4,16)
- Slope of y=8x – 16 is 8

At x = 6, the tangent line is much steeper, has the equation y = 12x – 36, two points are (3,0) and (6,36), and slope is 6



Derivatives

So the derivative is

- (1) a limit
- (2) the slope of the function at a particular value of x
- (3) a description of how y and x relate…if we change x, how much does y change?

Notations for the derivative

- "arc elasticity" $\frac{\Delta y}{\Delta x}$
- "total derivative" $\frac{dy}{dx}$
- "partial derivative" $\frac{\partial y}{\partial x}$
- generic notation $f'(x)$

Derivatives

Many business applications, including

- marginal revenue
- marginal cost
- minimizing the error to fit a curve to data
- measuring the sensitivity of bond values to changes in interest rates
- finding the optimal weights on assets in a portfolio

In the coursework you will see that understanding derivatives will make it much easier to understand certain formulas and their intuition.

Derivatives

Simple derivative rules

- Power rule
  - $y = x^n$ ➔ $\frac{\partial y}{\partial x} = nx^{n-1}$
  - For $y = x^2$, $\frac{\partial y}{\partial x} = 2x^{2-1} = 2x$
  - For $y = 4x^6$, $\frac{\partial y}{\partial x} = (6 \bullet 4)x^{6-1} = 24x^5$
  - For y = 5, $y = 5x^0$, so $\frac{\partial y}{\partial x} = (0 \bullet 5)x^{0-1} = \frac{0}{x} = 0$
    - So the slope = derivative of a constant is 0 (and as we know, y = horizontal line ➔ has zero slope)

Derivatives

Simple derivative rules

– Power rule

- y = x^n ➔ $\frac{\partial y}{\partial x} = nx^{n-1}$
- For $y = \frac{3}{x^4}$, $y = 3x^{-4}$, so $\frac{\partial y}{\partial x} = (-4 \bullet 3)x^{-4-1} = -12x^{-5}$
- For $y = \sqrt{x}$,

  $y = x^{1/2}$, so $\frac{\partial y}{\partial x} = (1/2)x^{(1/2)-1} = (1/2)x^{-1/2} = \frac{1}{2\sqrt{x}}$
- What about derivative of $y = \sqrt[8]{x^3}$? $\frac{dy}{dx} =$

Derivatives

Sums and differences

Let u(x) and v(x) be functions of x

The derivative of y = u(x) + v(x) is

$$-\frac{\partial y}{\partial x} = u'(x) + v'(x) \qquad \text{or} \quad \frac{\partial y}{\partial x} = \frac{\partial u(x)}{\partial x} + \frac{\partial v(x)}{\partial x}$$

Derivative of y = 5x^8 + 3x + 4

$$-\frac{\partial y}{\partial x} = (8 \bullet 5)x^{8-1} + 3 + 0 = 40x^7 + 3$$

Derivatives

Products

Let u(x) and v(x) be functions of x

The derivative of y = u(x) • v(x) is $\frac{\partial y}{\partial x} = u'(x)v + v'(x)u$

– Or $\frac{\partial y}{\partial x} = \frac{\partial u(x)}{\partial x} v(x) + \frac{\partial v(x)}{\partial x} u(x)$

– This is derivative of first term times the second term, plus derivative of second term times the first term

y = (3 + 4x)(5 + x^3)

$$\frac{\partial y}{\partial x} = [4][5 + x^3] + [3x^2][3 + 4x] = 16x^3 + 9x^2 + 20$$

Derivatives

For y = ln(x), $\frac{\partial y}{\partial x} = \frac{1}{x}$

So what about $y = \ln(x^3)$

1. y = 3ln(x) (recall)

2. By product rule for derivatives,

$$\frac{\partial y}{\partial x} = \frac{\partial 3}{\partial x} \bullet \ln(x) + \frac{\partial \ln(x)}{\partial x} \bullet 3$$

$$\frac{\partial y}{\partial x} = 0 \bullet \ln(x) + \frac{1}{x} \bullet 3$$

$$\frac{\partial y}{\partial x} = \frac{3}{x}$$

Let's solve another way.

1. $y = \ln(x^3) = \ln(x \bullet x \bullet x)$

2. Recall y = ln(x)+ln(x)+ln(x)

3. by sum rule for derivatives,

$$\frac{\partial y}{\partial x} = \frac{\partial \ln(x)}{\partial x} + \frac{\partial \ln(x)}{\partial x} + \frac{\partial \ln(x)}{\partial x}$$

$$\frac{\partial y}{\partial x} = \frac{1}{x} + \frac{1}{x} + \frac{1}{x} = \frac{3}{x}$$

Derivatives

Chain rule – for composite functions

$$y = u(v(x)) \; then \; \frac{\partial y}{\partial x} = u'(v(x))v'(x)$$

Example: y = $(4+6x^2)^3$

$$\frac{\partial y}{\partial x} = 3(4+6x^2)^2(12x) = 36x(4+6x^2)^2$$

Example: y = $\ln(x^3)$

$$\frac{\partial y}{\partial x} = \frac{1}{x^3}(3x^2) = \frac{3}{x}$$

Derivatives

For y = e^x, $\frac{\partial y}{\partial x} = e^x$

$$y = e^x \ \textit{implies} \ \ln(y) = x \ (\textit{because} \ \ln(x) \ \textit{and} \ e^x \ \textit{are inverse functions})$$

$$\ln(y) = x$$

$$\frac{1}{y} y' = 1 \ (\textit{taking derivatives of both sides})$$

$$y' = y$$

- So, when y = e^x, the derivative of y (which is y') = y. Thus e^x is its own derivative!
- For the derivative of y = $e^{f(x)}$, the chain rule implies $\frac{\partial y}{\partial x} = e^{f(x)} \cdot f'(x)$

$$\textit{If} \ y = e^{3x^2} \ \textit{then} \ \frac{\partial y}{\partial x} = \left(e^{3x^2}\right) 6x = 6xe^{3x^2}$$

Derivatives

Knowing the *Quotient rule* is actually optional because you can always use the product rule!

<u>Quotient rule</u>

$$y = \frac{u(x)}{v(x)} \qquad \frac{\partial y}{\partial x} = \frac{u'(x)v - v'(x)u}{[v(x)]^2}$$

Let's take derivative of $y = \frac{(6+3x)}{(2+x^2)}$ but using the product rule and chain rule.

$$y = (6+3x)(2+x^2)^{-1}$$

$$\frac{\partial y}{\partial x} = (3)(2+x^2)^{-1} + (-1)(2+x^2)^{-2}(2x)(6+3x)$$

$$= \frac{3(2+x^2) - 2x(6+3x)}{(2+x^2)^2}$$

Derivatives

Second derivative: the derivative of the derivative

Suppose y = f(x) = $6x^2 + 3x - 4$

The first and second derivatives are:

$$-\frac{\partial y}{\partial x} = f'(x) = 12x + 3$$

and

$$\frac{\partial^2 y}{\partial x^2} = f''(x) = 12$$

The first derivative tells us the slope; the second derivative tells us concavity/convexity

Graphing Using First and Second Derivatives

As we have seen, the first derivative is the slope

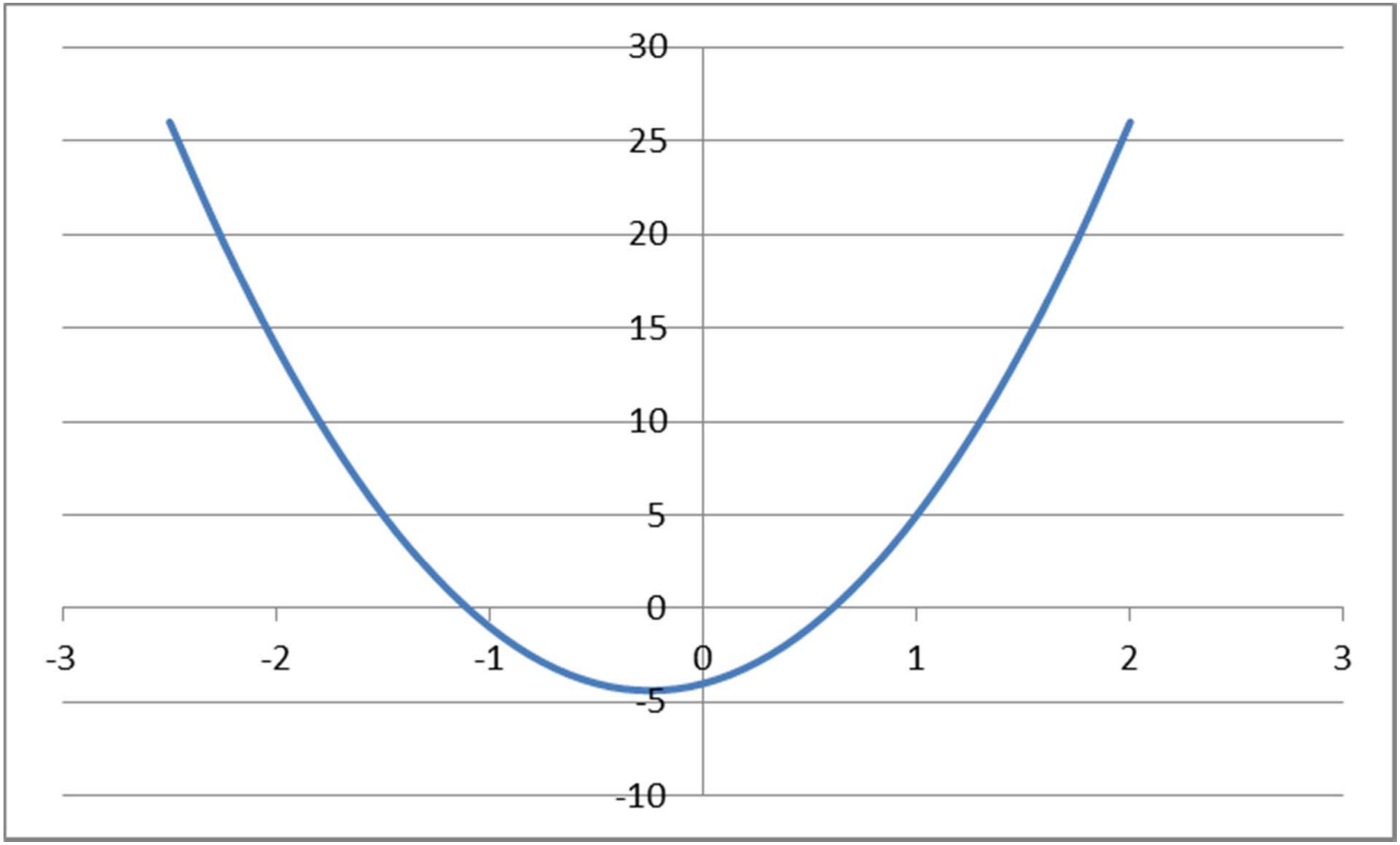
Second derivative > 0 ➔ convex

Second derivative < 0 ➔ concave

Second derivative = 0 ➔ inflection point

- $f(x) = 6x^2 + 3x - 4$, $f'(x) = 12x+3$, and $f''(x) = 12$
- At $x = -1/4$ the slope is 0 because $x = -1/4$ makes $f' = 0$
- $f'' > 0$, so this is a convex function
- Now find two points, such as (1,5) and (-1,-1)

Graphing Using First and Second Derivatives



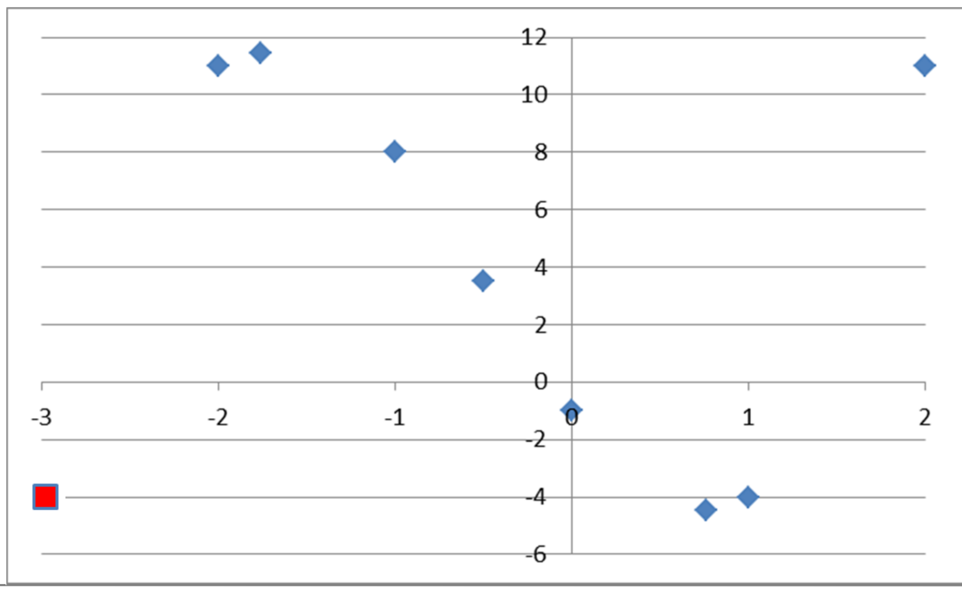
Graphing Using First and Second Derivatives

$f(x) = 2x^3 + 3x^2 – 8x – 1$, $f'(x) = 6x^2 + 6x – 8$, $f''(x) = 12x+6$

Helpful in graphing: Slope = 0 at two points: x = -1.7583 and 0.7583 (roots of a quadratic function!), concave when x < -1/2, and convex when x > -1/2 (so, x = -1/2 is an inflection point)

A few easy points might help: (-1,8), (0,-1), and (1, -4) are easy ones

Slightly harder but also potentially helpful: (-2, 11) and (2,11)



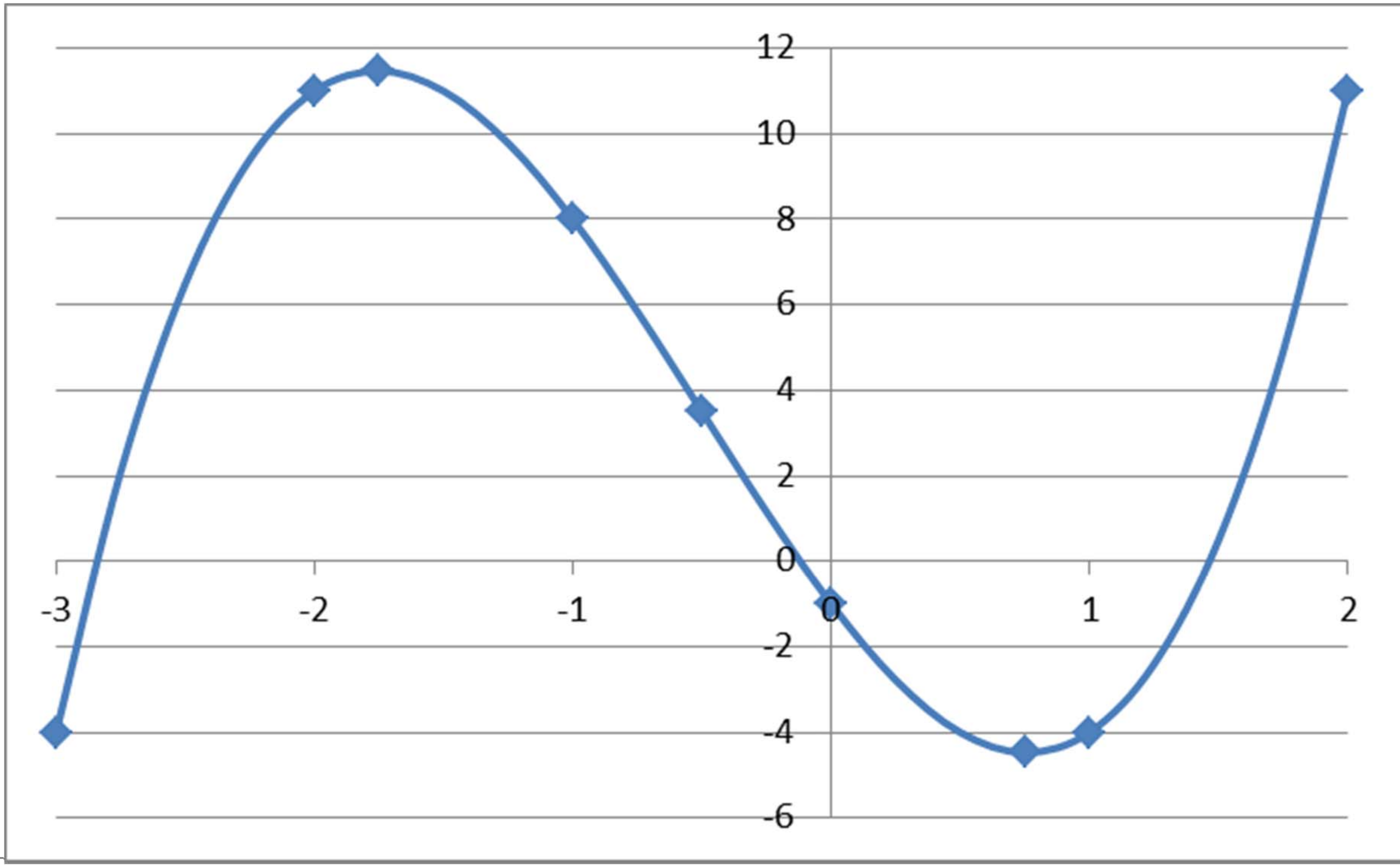
Ok, pretty good idea now…looks like it would be helpful to know about where y=0 when x < 0.

So try x = -3 for help:

When x=-3, y = -4

Graphing Using First and Second Derivatives

$f(x) = 2x^3 + 3x^2 - 8x - 1$ is graphed as follows with the points we found:



Optimization

We may want to:

- Minimize costs
- Maximize profits
- Maximize an investor's "utility" as he or she trades off risk versus expected return

So given some function, to find the

- minimum: we want first derivative=0, second deriv. > 0
- maximum: we want first derivative=0, second deriv. < 0

Optimization

Consider a situation in which the average of daily unit production costs, C, depend on the quantity produced, Q:

$$C = (Q - 10)^3 + 2(Q-12)^2 + 800$$

Find the production quantity Q* that minimizes per unit costs.

First derivative: $3(Q-10)^2 + 4(Q-12)$

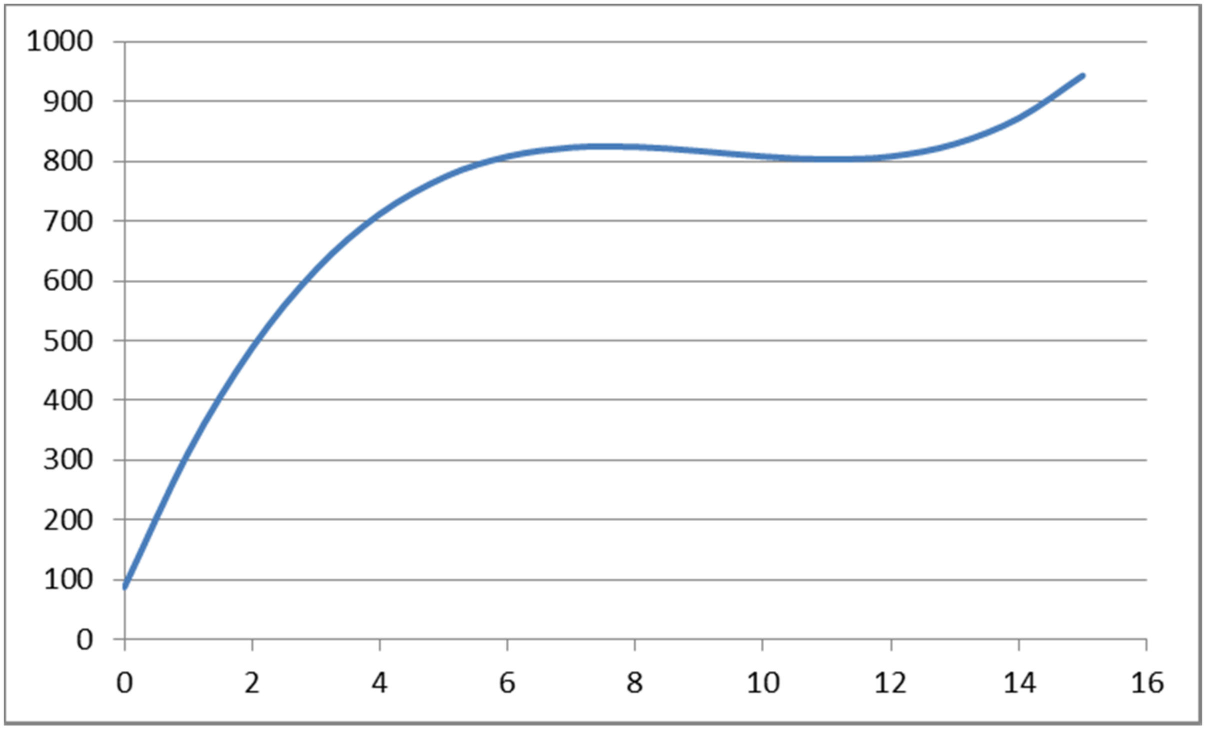
This = $3Q^2 - 56Q + 252$, & roots are Q = 7.569 and Q=11.097 (the places where slope = 0)

Second derivative = $6(Q-10)+4 = 6Q-56$

- Q = 7.569 is a "local maximum" because f"(7.569)<0
- Q = 11.097 is a "local minimum" because f"(11.097)>0

Optimization

The plot looks as follows



What could explain this?

- Costs rise while employees are low on the learning curve, then fall as efficiencies kick in, then rise again as capacity is strained and errors are made.

Accuracy

Obviously \$100.10 + \$200.20 ≠ \$300.00, yet many people make similar rounding errors, particularly when intermediate work is required.

Consider the following cash flow pattern, where PV = present value as of today (i.e., the value in today's dollars). What is the total present value rounded to the nearest penny?

	\$100.00	\$100.00	\$100.00
PV today:	92.5926	85.7339	79.3832

Correct: 92.5926 + 85.7339 + 79.3832 = \$257.71 (accurately rounded to the penny)

Incorrect: 92.59 + 85.73 + 79.38 = \$257.70

The more intermediate steps and the more numbers involved, the worse the error can become.

➔ For additive math (includes any subtractions) carry intermediate steps out a few more places than the accuracy desired in the final answer!

Accuracy

The cumulative rounding error is even more serious when using multiplicative math (includes division)

For example, consider $10,000,000 growing at 5.259% for 10 years followed by 3.238% for 10 years...what is the ending value?

Correct: $\$10,000,000 \times (1.05256)^{10} \times (1.03238)^{10} = \$22,954,321$

Incorrect: $\$10,000,000 \times (1.0526)^{10} \times (1.0324)^{10} = \$22,967,494$

Advice: If the interest rates you are using result from intermediate steps, carry these intermediate steps out to 12.345678% (or 0.12345678).

- Better yet, LEARN TO USE YOUR CALCULATOR'S MEMORY KEYS AND PARENTHESES TO AVOID ROUNDING!

Selected Math Symbols You May Occasionally See

Symbol	Meaning
$\therefore$	therefore
$\because$	because
$\Leftrightarrow$	"iff" if and only if;
	$A \Leftrightarrow B$ means if B is true, A is also true,
	and if B if false, A is also false
$\forall$	for all; for any; for each
$\in (or \notin)$	is a member or is an element of (or is not...)
$\bigcup$	the union of
$\bigcap$	the intersection of
$\forall$	for all; for any; for each
$\propto$	is proportional to; varies as
$\Rightarrow$	implies $(A \Rightarrow B$ means A implies B$)$

Logic and proofs

Be careful with critical writing, how you reach and support your conclusions, etc. This is often a weakness.

What can we conclude, if anything, if we observe that red cars on the highway tend to travel at higher rates of speed than green cars? We can speculate that:

- Red car paint causes engines to be faster
- Red car paint makes car more aerodynamic
- Red car paint causes drivers to drive faster
- Red car paint proxies for something else that correlates with driving speed
  - Sports cars are purchased by those who prefer, on average, to drive faster, and red is more popular for a sports car than green
  - So perhaps red correlates with the likelihood the car is a sports car

We could record data and then show statistical correlation, but that does not imply causality or that the underlying argument is sound “economically” speaking. We’ll discuss more of this later when we discuss statistics.

Logic and proofs

How could we validly prove $A \Rightarrow B$? (That A implies B?)

Here are two common methods (there are others)

Consider: A ≡ John is a human. B ≡ John is an animal.

Proof by construction (direct proof)

- Assume A is true and use A to show that B must be true
- Assume John is human. Human meets the scientific definition animal, and therefore John must be an animal.

Proof by contradiction

- Assume A is true. Assume B is not true. Use A and B to demonstrate a contradiction, that is, that both statements cannot be true at the same time)
- Assume John is human. Assume John is not an animal. But if John is a human, he meets the scientific definition of what defines an animal, so it cannot be true that John is not an animal. There is a contradiction.

Closing comments

Being mathematically rigorous & accurate is important in finance, but understanding the economics and intuition underlying what you are doing is equally important!

Throughout the program you will learn many tools and techniques, but to arrive at correct conclusions you must learn their assumptions, when and where they should be applied, and their limitations.

- Be an intuition "sponge" – soak up any knowledge that helps you build your intuition
- When learning some particular area, do not get so buried in the math and techniques that you lose sight of the overriding purpose of what you are learning!
- In coursework we often fail to emphasize the real-life uncertainty that may underlie the "inputs"
    - Understanding (and hopefully quantifying) the sensitivity of your answer and its implications to the input variables is extremely important and is part of the "artistic" side of finance