Consequences of Modeling Habit Persistence

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Abstract

In this paper, we study the stationary and non-stationary equilibria of a deterministic, pure exchange, two-period overlapping generations model with habit persistence. We show that preferences with multiplicative habits can lead to quite different equilibrium outcomes compared to subtractive ones. The two most commonly adopted habit specifications can differ in terms of homotheticity, gross substitutability, and intertemporal marginal rates of substitution. We study the differences in savings behavior across the two with respect to steady state equilibria, indeterminacy, and local dynamics.

JEL Classification: E52, E63.
Keywords: Multiplicative and subtractive habit persistence, multiple equilibria, equilibrium dynamics.

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1 Introduction

Understanding how optimal savings decisions change according to different preference specifications is of paramount importance since consumption and saving decisions are essential for both short- and long-run macroeconomic analysis. In the short run, spending dynamics are crucial for business cycle analysis and the management of monetary policy. And in the long run, aggregate saving determines the aggregate capital stock, with consequences for wages, interest rates, and the standard of living.

A recent strand of the economic literature has analyzed theoretical and empirical implications of endogenous preferences, that is preferences that depend on time, personal experience, or social conditions. Among the different forms of endogenous preferences those displaying habits have received particular attention.

One of the first works explicitly addressing habits is Pollak [17] which shows that when past consumption choices affect current ones, so that preferences are “intertemporally dependent,” the long-run and short-run demand functions differ. Becker and Murphy [3], in their seminal paper, employ habits in consumption to study addiction. As a result of habit persistence, apparently irrational behavior can be explained using rational choice theory.

Habits have also been explored in the economic growth literature. In particular, Ryder and Heal [18] consider habit persistence in a growth model and show that the optimal path can be quite different with capital overshooting the equilibrium value before settling down to its steady state. Carroll and Weil [9], on the other hand, consider the effect of a negative shock to capital in an endogenous growth model with habit persistence and show how savings and growth rates temporarily fall in response to the shock.

Departures from standard preferences have helped explain some important economic phenomena, such as the equity premium puzzle. Abel [1] and Constantinides [12] show how the equity premia predicted by habit formation models are as large as the historical observations in the United States.

The literature on habit persistence specifies preferences either using subtractive (SH) or multiplicative habits (MH), according to the terminology introduced by Carroll [7].

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1Adam Smith in his Theory of Moral Sentiments [19] was probably the first to suggest the notion of internal habit persistence. He did so by introducing the concept of “customary” consumption levels. Households are used to a certain amount of consumption and over time they may then derive less and less utility from the same constant level of consumption.

2Some examples of MH are Abel [1] and Carroll, Overland, and Weil [10] and for SH, Campbell and
though habits have helped shed some light on economic phenomena, there has not been much attention paid to the economic consequences of these two alternative formulations. For example, Abel [1] and Constantinides [12] reach similar conclusions using the two alternative formulations. This may seem to suggest, prima facie, that the qualitative properties of models with SH and MH are pretty similar. As a matter of fact, this is the main finding of Bunzel [5]. However, Wendner [21], shows that the two most commonly used habit specifications may easily come to opposite implications regarding household savings behavior. He shows that in response to an increase in the strength of habits, young households may increase savings in the case of SH, while they may lower savings in the case of MH. Furthermore, Carroll [7] points out that under plausible parameters SH may give rise to a not well defined utility in stochastic environments. Thus further analysis is warranted.

The objective of this paper is to examine the differences between the two alternative formulations adopted in the literature. We explore the consequences of specifying different forms of habit in terms of multiplicity of steady states, dynamic indeterminacy, and local dynamics in a pure exchange, two-period lived overlapping generations model. Habit persistence in this setting has been previously explored in the literature. Lahiri and Puhakka [16] consider subtractive habits in a pure exchange overlapping generations framework and show that increasing the strength of habits raises desired savings and endogenous cycles are possible. In a similar framework, Bunzel [5] argues that both SH and MH yield results that are qualitatively very similar. If the choice of how to model savings behavior in the presence of habits is not innocuous, the predictions and policy prescriptions based on the two types of habit persistence modeling choices could be quite different. An investigation of the underlying properties of the savings function under the two habit formulations is very relevant.

The main result of our paper is to show that modeling preferences with MH or SH yields predictions that are not necessarily equivalent. In contrast to Bunzel [5], the class of preferences we consider (HARA) is more general and subject to less criticism than the one she adopts (CRRA). Furthermore, we do not restrict agents to have zero endowments

Cochrane [6] and Constantinides [12].

Carroll [7] p. 68 states that: "CRRA utility in combination with the subtractive formulation...has several theoretical problems, the gravest of which is that for microeconomically plausible parameterizations of consumption variation the equation [of consumption] accumulation can easily lead to a zero or negative argument to the [utility] function...generating infinite negative utility."
when old. Even when we do so, we find multiple steady states under MH while unique steady state under SH. Moreover, we show that modeling habit persistence with MH may not always yield concave or homothetic preferences over consumption. This modeling choice can give rise to multiple monetary equilibria. Therefore, endogenous volatility and stationary sunspots equilibria may arise. Moreover, the stability properties of an economy with MH change drastically as we vary the strength of habits. Thus the the resulting qualitative properties of the economy under SH and MH can be quite different.

In the next section, we introduce the general model with MH and SH and study the corresponding concavity and homotheticity properties of these preferences. Then, we analyze the steady state equilibria and their associated dynamics. We also give a general result on the difference in the dynamics under the classical case. Finally, we turn to some numerical exercises that suggest -once again- non negligible discrepancies in the local stability properties of the steady states across MH and SH. The last section summarizes the main findings and concludes. The proofs can be found in Appendix.

2 The Model

This paper builds on Gale [14] and Lahiri and Puhakka [16] by considering a pure exchange overlapping generations model. Complex dynamics and endogenous cycles have been shown to emerge in overlapping generation models with production and/or a variety of different assumptions concerning market imperfections. Here we abstract from technology to reinforce the point this study tries to make: with habits, differences in dynamics and in steady states properties need not be driven by the production side of the economy, but rather by the modeling choice of preferences alone. Economic activity takes place over infinite discrete time. Each generation lives for two periods and has perfect foresight. Agents are endowed with $w_1$ units of the single good when young and with $w_2$ units when old. Utility is derived from consumption in both periods. However, due to the presence of habit formation, utility

\footnote{See Boldrin and Woodford (1990).}
of a given level of consumption when old depends on consumption when young. Formally:

$$V(c_1, c_2) = u(c_1) + \beta u(c_1, c_2; \gamma)$$

where $\gamma \in (0, 1)$ denotes the strength of habits in the instantaneous utility function and $\beta$ is the discount factor. We assume that the function $u(.)$ is well behaved, i.e. strictly increasing, strictly concave, homogeneous, and twice continuously differentiable. If one considers SH, then preferences are typically given by $u(c_1, c_2; \gamma) = u(c_2 - \gamma c_1)$ as in Lahiri and Puhakka [16]. If, instead, the instantaneous utility is specified with MH, we have $u(c_1, c_2; \gamma) = u(c_2/c_1^\gamma)$. Regardless of the specification considered, the importance of past consumption in determining the utility derived by the “effective” consumption is increasing with $\gamma$.

Each agent maximizes utility subject to budget constraints; $c_1 = w_1 - s_t$ when young, and $c_2 = w_2 + R_t s_t$ when old. $s_t$ and $R_t$ denote savings when young and the gross nominal interest rate on savings at time $t$, respectively. In what follows, we first study the concavity, the homothetic properties and the steady state equilibria that emerge in each case. We then explore the local dynamic properties of the equilibria under the two habit specifications.

### 2.1 Preferences

One of the most important differences across alternative habit specifications is in terms of their implications on the underlying preferences. Alonso-Carrera, Caballé, and Raurich [2] show that when habits are introduced multiplicatively in a capital accumulation model, the consumers’ objective function might fail to be concave. In this paper, we find a similar result for pure exchange overlapping generation models as stated in Proposition 1.

**Proposition 1 (Concavity)** Consider a pure exchange two-period lived overlapping generations model. Then: (i) under SH persistence, the utility function is strictly concave in consumption, and (ii) under MH persistence, concavity is not always ensured.

**Proof.** See Appendix. $\blacksquare$

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5Specifically, both the absolute level of consumption in the second period and the change in consumption between the two periods are important. The more the consumption when young, the more consumption in the following period is required to derive a given level of utility.
Proposition 1 implies that one needs to be extra cautious when solving for the consumption plan that maximizes consumers’ utility under MH. The next proposition outlines another crucial implication in terms of the savings function resulting from the consumer’s problem. The adoption of MH or SH can greatly affect the savings behavior, even if other fundamentals remain the same.

**Proposition 2 (Homotheticity)** If the instantaneous utility function \( u(,) \) is homogeneous of degree \( n \), then the SH specification yields homothetic preferences whereas the MH specification does not.

**Proof.** See Appendix.

It is well known that, in overlapping generations models, the saving function is linear in income/endowments if the lifetime utility function is homothetic.\(^6\) Thus, we expect to observe substantial differences in underlying dynamics between SH and MH specifications. This particular comparison is analyzed in section 2.3.

### 2.2 Steady State Equilibria

In this section, we show that if one adopts SH then there exists a unique monetary steady state while under MH multiple monetary equilibria cannot be ruled out. The optimal savings function for a young agent in this model with MH is defined as follows:

\[
s^* = \arg \max u(w_1 - s) + \beta u \left( \frac{Rs + w_2}{(w_1 - s)^\gamma} \right).
\]

Thus, the implicit function for optimal savings can be written as:

\[
F(s) = -u'(w_1 - s) + \beta u' \left( \frac{Rs + w_2}{(w_1 - s)^\gamma} \right) \left( \frac{R(w_1 - s) + \gamma(Rs + w_2)}{(w_1 - s)^{\gamma+1}} \right) = 0.
\]

The implicit function for optimal savings under SH, on the other hand, is given by the following equation:

\[
G(s) = -u'(w_1 - s) + (R + \gamma) \beta u' ((R + \gamma)s + w_2 - \gamma w_1) = 0.
\]

\(^6\)REFERENCE.
To close the model and analyze the steady states, we introduce an outside asset into the economy. Following Lahiri and Puhakka [16], we assume that there is a government that borrows from and lends to the public. This approach is clearly equivalent to injecting valueless fiat money in the economy. Government liabilities in period \( t \) are denoted by \( b_t \), and the real deficit by \( d_t \). The government’s budget constraint for period \( t \) is then given by:

\[
d_t + R_{t-1} b_{t-1} = b_t.
\]

If \( b_t > 0 \), then the economy is in the Samuelson case according to the terminology first coined by Gale [14]. Instead, if \( b_t < 0 \), then the economy is in the Classical case. In Proposition 4 and Lemma 1 below, we derive a general result on the dynamics under MH and SH in the classical case. The analysis and the numerical examples of section 2.3 consider both cases.

Asset market equilibrium requires that \( b_t = s_t \) for all \( t \). Setting the government’s deficit to zero, the law of motion for this economy is defined as follows:

\[
s_{t+1} = s_t R_t
\]

which also represents the offer curve for this economy.

Using a general instantaneous utility function does not always yield an explicit expression for optimal savings, nor the gross interest elasticity of savings. Thus, it is not possible to explicitly analyze equation (6). However, we can characterize a crucial difference between the two alternative specifications in a general setting where no explicit functional form assumptions regarding the utility are made. The following proposition states that the standard results on uniqueness do not necessarily apply to the case with MH. This result on the uniqueness of equilibria will then have crucial implications for the local dynamics of the economy.

**Proposition 3 (Uniqueness)** Under SH, there always exists a unique monetary steady state. Under MH, on the other hand, multiplicity of monetary steady states cannot be ruled out.

**Proof.** See Appendix. □

The economic intuition behind this result is related to the possible lack of gross sub-
stitutability in the MH case. It is well known that when consumption in the first and the second periods are weak gross substitutes, then the steady state is unique. This condition can be verified by studying the sign of the intertemporal elasticity of substitution (IES). Under the subtractive specification, IES is always positive, whereas under the multiplicative case it can be negative. Formally, the IES between consumption when young and old can be written as:

\[
\Psi(c_1, c_2) = \frac{1}{c_1 \frac{\partial V}{\partial c_1}} + \frac{1}{c_2 \frac{\partial V}{\partial c_2}} - \frac{\partial^2 V}{\partial c_1 \partial c_2} + 2 \frac{\partial^2 V}{\partial c_1 \partial c_2} - \frac{\partial^2 V}{\partial c_1 \partial c_2} \frac{\partial^2 V}{\partial c_2^2}
\]

The crucial term in \( \Psi \) is \( \frac{\partial^2 V}{\partial c_1 \partial c_2} \), if it is negative then gross substitutability could break down. One can verify that under SH:

\[
\frac{\partial^2 V}{\partial c_1 \partial c_2} = -\gamma\beta u''(c_2 - c_1) > 0.
\]

When we consider MH, instead, we obtain the following:

\[
\frac{\partial^2 V}{\partial c_1 \partial c_2} = -\gamma\beta c_1^{-\gamma-1} [u'(c_2c_1^{-\gamma}) + c_2c_1^{-\gamma} u''(c_2c_1^{-\gamma})]
\]

where the sign of this expression depends on the underlying parameter values of the model.

Under intertemporally non-separable preferences, the elasticity of substitution between current and future consumption is time variant. However, different functional forms for the outer flow utility imply different degrees of risk aversion. A subtractive specification implies that consumption in the first and the second periods are always gross substitutes, whereas this is not necessarily true when a multiplicative formulation is adopted. Due to lack of gross substitutability, it is possible for multiple monetary steady states to arise. This clearly has important implications on the dynamics of the economy.

In the next section, we highlight the differences between MH and SH modeling choices using numerical examples for the Samuelsonian and Classical cases as well as for different strengths of habits.

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7change reference See, for example, Azariadis, Bullard, and Ohanian [?].
2.3 Dynamics

One clear message of the previous section is that we should expect differences in the local dynamics between the two specifications due to homotheticity or lack thereof. Before we proceed, let us establish a general result for the classical case which is a direct consequence of a theorem by Gale [14]. The following proposition together with Proposition 2 suggest that alternative habit formulations can yield drastically different results in terms of dynamic indeterminacy in the classical case.

Proposition 4 (Gale, 1973) In the classical case, if the utility function is either (i) separable, or (ii) homothetic then there exist a unique path approaching the steady state given an initial condition.

Proof. See the proof of Theorem 5 in Gale ([14], p.25).

Weak gross substitutability implies that, if it exists, there is at most one equilibrium price sequence which converges to the steady state. Proposition 4 and Lemma 1 suggest that this is not the case for MH.

Lemma 1 In the classical case, SH lead to determinate dynamic equilibrium whereas MH display dynamic indeterminacy.

Proof. Recall that both habit specifications imply non-separable preferences. In addition, Proposition 2 showed that preferences are non-homothetic under MH while they are homothetic under SH. These facts together with Proposition 4 above completes the proof.

In general, the local dynamic properties are determined by the slope of the offer curve given by:

$$\frac{dS_{t+1}}{dS_t} = R_t \left( 1 + \frac{1}{\epsilon} \right)$$

where $\epsilon$ is the gross interest rate elasticity of savings. Since we cannot obtain a closed form solution for the offer curve under MH nor SH, it is not possible to study its behavior analytically. For this reason, we provide numerical examples that show how the dynamics can differ in the Samuelsonian as well as in the Classical case. In particular, we show that multiple monetary steady states are possible under MH and endogenous volatility is possible under SH and MH, but not necessarily at the same time and for the same parameter values.
For our numerical examples, let us consider the class of preferences with hyperbolic absolute risk aversion (HARA). This class includes as special cases the family of utility functions with a constant coefficient of relative risk aversion (CRRA), the one with a constant absolute risk aversion (CARA), and the quadratic utility. The HARA family is commonly used in the finance literature\textsuperscript{8}. This fact is relevant since the two seminal papers of Abel [1] and Constantinides [12], which adopt the alternative functional forms of habit persistence to explain the risk premium, are widely cited in the financial literature. Moreover, Carroll and Kimball [8] show how CARA and CRRA specifications imply savings behavior that is qualitatively very different from the one corresponding to a HARA formulation. In particular, Carroll and Kimball [8] show that under HARA preferences consumption rules are concave, which is consistent with the marginal propensity to consume out of wealth or transitory income declines with the level of wealth, but linear when CARA and CRRA specifications are considered. Thus the underlying choice of preferences is not innocuous. The utility function takes the following form:

\[ u(c) = \frac{1}{1 - \sigma} \left[ \sigma \left( A + \frac{\alpha c}{\sigma} \right)^{1-\sigma} \right] \]  

where \( A \) and \( \sigma \) are real numbers, and \( \alpha > 0 \). Incorporating MH in the HARA family yields the following von Neumann-Morgenstern utility function for our example:

\[ V(c_1, c_2) = \frac{1}{1 - \sigma} \left[ \sigma \left( A + \frac{\alpha c_1}{\sigma} \right)^{1-\sigma} \right] + \beta \frac{1}{1 - \sigma} \left[ \sigma \left( A + \frac{\alpha c_2}{\sigma c_1} \right)^{1-\sigma} \right] \]  

Typically, to characterize the dynamical properties of this economy, we need to determine the slope of the offer curve at each of the steady states. The marginal rate of substitution at the initial endowment point should be studied as well. However, the HARA utility function may not always be defined at the origin. Formally, the utility function is defined only when \( A + \frac{\alpha c}{\sigma} \geq 0 \). By substituting the budget constraint for each period into the respective flow utilities we obtain an upper and a lower bound on savings:

\[ A + \frac{\alpha (w_1 - s_1)}{\sigma} \geq 0 \quad (B1) \]

\textsuperscript{8}See for example Ingersoll [15].
and

\[ A + \frac{\alpha (w_2 + R_t s_t)}{\sigma (w_1 - s_t)^\gamma} \geq 0 \]  \hspace{1cm} (B2)

Therefore, these additional constraints need to be taken into account when building the offer curve.

Throughout our numerical analysis we set \( \beta = 0.55 \) which corresponds to an annual discount rate of 0.976 when the “generation” considered lasts 25 years. Moreover, we choose the following parameter values for the utility function: \( \sigma = 0.39 \), \( A = -1 \), and \( \alpha = 0.79 \). The findings corresponding to this specific parameterization are not unique and similar qualitative properties are found with different parameter values. In particular, we find that by varying the strength of habits, \( \gamma \), the dynamical properties of the steady state can be greatly affected.

The tables corresponding to the examples provide information regarding steady state allocations for savings \( (s^{s,s}) \), consumption when young \( (c_1^{s,s}) \) and old \( (c_2^{s,s}) \), the gross interest rate elasticity of savings \( \epsilon \) and the slope of the offer curve evaluated at the steady states, \( (1 + \frac{1}{\epsilon}) \). Finally, our tables also report the principal minor \( (PM) \) and the determinant of the associated Hessian at steady state \( (detH) \) as well. We do so in light of Proposition 1 which warns of the potential lack of concavity under the MH case. We want to be sure that the steady states that we find are equilibria indeed.

2.3.1 Examples for the Samuelson Case

In this subsection we examine the consequences of increasing the strength of habits. Our examples show that as we increase the strength of habits the number of monetary equilibria and the corresponding local dynamics of the two habit specifications can be quite different.

**Example 1** When \( \gamma = 0.5 \), \( w_1 = 3 \), and \( w_2 = 1 \), we find a unique monetary steady state for both habit specifications. These steady states have similar local dynamics.

Table 1 indicates that stability properties of the steady states are qualitatively similar across specifications. Both monetary steady states are locally unstable and endogenous volatility is not possible because \( (1 + \frac{1}{\epsilon}) \) is greater than one. The stylized offer curves corresponding to this example are shown in Figure 1.
Table 1: Unique equilibria for the two specifications and similar dynamics.

<table>
<thead>
<tr>
<th></th>
<th>SH</th>
<th>MH</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^{ss}$</td>
<td>1.195</td>
<td>0.192</td>
</tr>
<tr>
<td>$c_1^{ss}$</td>
<td>1.804</td>
<td>2.807</td>
</tr>
<tr>
<td>$c_2^{ss}$</td>
<td>2.195</td>
<td>1.192</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>0.068</td>
<td>3.129</td>
</tr>
<tr>
<td>$(1 + \frac{1}{\epsilon})$</td>
<td>15.680</td>
<td>1.319</td>
</tr>
<tr>
<td>$PM$</td>
<td>-0.204</td>
<td>-0.049</td>
</tr>
<tr>
<td>$detH$</td>
<td>0.028</td>
<td>0.018</td>
</tr>
</tbody>
</table>

Figure 1 here.

The offer curve plotted in Figure 1(a) suggests that SH introduces a source of indeterminacy, which results from the fact that parts of the offer curve are multi-valued. In other words, under SH we find that for a particular $s_t$, there might be more than one $s_{t+1}$; thus generating dynamic indeterminacy.

Example 2 When $\gamma = 0.7$, $w_1 = 3$, and $w_2 = 1$, we find a unique monetary steady state for both habit specifications. These steady states have distinct local dynamics.

Table 2: Unique Equilibria for the two specifications, but different dynamics.

<table>
<thead>
<tr>
<th></th>
<th>SH</th>
<th>MH</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^{ss}$</td>
<td>1.457</td>
<td>0.327</td>
</tr>
<tr>
<td>$c_1^{ss}$</td>
<td>1.543</td>
<td>2.673</td>
</tr>
<tr>
<td>$c_2^{ss}$</td>
<td>2.457</td>
<td>1.327</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>-0.033</td>
<td>1.156</td>
</tr>
<tr>
<td>$(1 + \frac{1}{\epsilon})$</td>
<td>-28.665</td>
<td>1.864</td>
</tr>
<tr>
<td>$PM$</td>
<td>-0.293</td>
<td>-0.051</td>
</tr>
<tr>
<td>$detH$</td>
<td>0.033</td>
<td>0.017</td>
</tr>
</tbody>
</table>

As we can see from Table 2, increasing the strength of habits changes the local dynamics. Both monetary steady states are locally unstable but the corresponding eigenvalue for MH is positive and greater than one, thus endogenous volatility is not possible. On the other hand, nondamped oscillations are observed under SH. In contrast to the previous example, the local dynamics of these two economies are vastly different. The stylized offer curves corresponding to the two cases of this example are shown in Figure 2.
Figure 2 here.

Figure 2 also reveals that dynamic indeterminacy is not possible since the offer curve is not multivalued for both SH and MH specifications.

**Example 3** When $\gamma = 0.85$, $w_1 = 3$, and $w_2 = 1$, we find two monetary steady states under MH and a unique monetary steady state under SH. Local dynamics associated with the two different set of steady states are quite different.

<table>
<thead>
<tr>
<th></th>
<th>SH</th>
<th>MH</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Steady State</td>
<td>Steady State I</td>
</tr>
<tr>
<td>$s^{s.s.}$</td>
<td>1.610</td>
<td>0.445</td>
</tr>
<tr>
<td>$c_1^{s.s.}$</td>
<td>1.389</td>
<td>2.554</td>
</tr>
<tr>
<td>$c_2^{s.s.}$</td>
<td>2.610</td>
<td>1.445</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>-0.067</td>
<td>0.542</td>
</tr>
<tr>
<td>$(1 + \frac{1}{\gamma})$</td>
<td>-13.896</td>
<td>2.845</td>
</tr>
<tr>
<td>$PM$</td>
<td>-0.374</td>
<td>-0.057</td>
</tr>
<tr>
<td>$detH$</td>
<td>0.038</td>
<td>0.015</td>
</tr>
</tbody>
</table>

Table 3: Multiplicity of Equilibria for the MH case, but not for the SH case.

Table 3 shows that as we further increase the strength of habits multiple steady states are possible when MH are considered. This situation is non-generic since the only way to have multiple monetary steady states is for consumers to be indifferent between two (or more) levels of savings when $R=1$ holds. In other words, different local maxima in the consumer’s optimization problem must yield exactly the same utility level, which is clearly a non-generic situation. Even though this is the case, it is important to highlight that SH can never support multiple steady states as illustrated in section 2.2 above. Table 3 also highlights that all the steady states are unstable and that the steady state under SH yields endogenous arising volatility. The stylized offer curves corresponding to this example are shown in Figure 3. Notice that this time it is the economy under MH to exhibit indeterminacy since a part of the associated offer curve is multi-valued around the lower savings steady state.

Figure 3 here.

Example 3 also illustrates that sunspot equilibria may be observed under MH. In particular, sunspot equilibria can be constructed as a lottery between two deterministic steady
states as in Cass and Shell [11]. Furthermore, since multiple steady states may exist under MH, bifurcation phenomena are possible as we vary the strength of habits. By changing $\gamma$, we can move from a situation with a unique steady state to another where multiple steady states are possible.

### 2.3.2 Do endowment profiles matter?

In a recent paper, Bunzel [5] argues that, regardless of whether one adopts SH or MH, the dynamical properties of the pure exchange economy are very similar. Bunzel focuses on an outer utility function which is CRRA and considers zero second period endowments for tractability. In this subsection we want to determine if Bunzels irrelevance results are driven by the specific choice of the utility function of the particular endowment profile considered. The following example shows that Bunzels results are driven by the choice of the functional form of the outer utility. When HARA utility functions are considered the MH specification can indeed affect the qualitative properties of the dynamical system even when there is no second period endowment.

**Example 4** When $\gamma = 0.85$, $w_1 = 3$, and $w_2 = 0$, we find two monetary steady states under MH and a unique monetary steady state under SH. Local dynamics associated with the two different set of steady states are quite dissimilar.

<table>
<thead>
<tr>
<th></th>
<th>SH</th>
<th>MH</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Steady State</td>
<td>Steady State I</td>
</tr>
<tr>
<td>$s^{s.s.}$</td>
<td>1.956</td>
<td>1.225</td>
</tr>
<tr>
<td>$c_1^{s.s.}$</td>
<td>1.043</td>
<td>1.774</td>
</tr>
<tr>
<td>$c_2^{s.s}$</td>
<td>1.956</td>
<td>1.225</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>-0.204</td>
<td>-0.139</td>
</tr>
<tr>
<td>$(1 + \frac{1}{\gamma})$</td>
<td>-3.887</td>
<td>-6.183</td>
</tr>
<tr>
<td>$PM$</td>
<td>-0.737</td>
<td>-0.065</td>
</tr>
<tr>
<td>$detH$</td>
<td>0.148</td>
<td>0.020</td>
</tr>
</tbody>
</table>

Table 4: Multiplicity of Equilibria for the MH case, but not for the SH case.

As we can see from Table 4, all steady states are unstable under both habit specifications. Moreover, the local dynamics around the steady state under SH display nondamped oscillations. This is also the case for steady state I under MH. Steady state II has diverging-monotonic dynamics instead.
2.3.3 Examples for the Classical Case

In this subsection we check whether the differences in the dynamics for the two specifications are just observed in Samuelsonian economies. We find this not to be the case. Profound discrepancies in the local dynamics are possible even in the Classical Case when savings are negative.

Example 5 When $\gamma = 0.5$, $w_1 = 0$, and $w_2 = 3$, we find a unique monetary steady state for both habit specifications. These steady states have very different local dynamics.

<table>
<thead>
<tr>
<th></th>
<th>SH</th>
<th>MH</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^{s.s.}$</td>
<td>-1.330</td>
<td>-1.967</td>
</tr>
<tr>
<td>$c_1^{s.s.}$</td>
<td>1.330</td>
<td>1.967</td>
</tr>
<tr>
<td>$c_2^{s.s.}$</td>
<td>1.669</td>
<td>1.033</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>-0.784</td>
<td>-1.172</td>
</tr>
<tr>
<td>$(1 + \frac{1}{\gamma})$</td>
<td>-0.274</td>
<td>0.146</td>
</tr>
<tr>
<td>$PM$</td>
<td>-0.381</td>
<td>-0.086</td>
</tr>
<tr>
<td>$detH$</td>
<td>0.098</td>
<td>0.040</td>
</tr>
</tbody>
</table>

Table 5: Unique equilibria for SH and MH.

Under SH we find endogenous arising volatility that vanishes over time and under MH we observe monotonic dynamics converging to the steady state.

Example 6 When $\gamma = 0.85$, $w_1 = 0$, and $w_2 = 3$, we find two monetary steady states under MH and a unique monetary steady state under SH. Local dynamics associated with the two different set of steady states are quite different.

<table>
<thead>
<tr>
<th></th>
<th>SH</th>
<th>MH</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steady State</td>
<td>Steady State I</td>
<td>Steady State II</td>
</tr>
<tr>
<td>$s^{s.s.}$</td>
<td>-1.043</td>
<td>-0.608</td>
</tr>
<tr>
<td>$c_1^{s.s.}$</td>
<td>1.043</td>
<td>0.608</td>
</tr>
<tr>
<td>$c_2^{s.s.}$</td>
<td>1.956</td>
<td>2.391</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>-0.609</td>
<td>-0.198</td>
</tr>
<tr>
<td>$(1 + \frac{1}{\gamma})$</td>
<td>-0.641</td>
<td>-4.047</td>
</tr>
<tr>
<td>$PM$</td>
<td>-0.737</td>
<td>-2.162</td>
</tr>
<tr>
<td>$detH$</td>
<td>0.148</td>
<td>0.070</td>
</tr>
</tbody>
</table>

Table 6: Multiplicity of Equilibria for the MH case, but not for the SH case.
As we increase the strength of habits, it is interesting to note that whereas the steady state for SH is stable, the MH case has two steady states with opposite stability properties. Steady State I is unstable and exhibits nondamped oscillations. Steady state II, on the other hand, is stable with converging monotone dynamics.

The point we want to convey with all the previous examples is that stability and dynamical properties of an economy can change drastically as we vary the strength of habits for both SH and MH. This result is in line with the findings of Lahiri and Puhakka [16] for the subtractive framework. They show that switches between a Samuelsonian and a classical economy are possible as the strength of habits varies. As habits become stronger, different responses in savings behavior across MH and SH cases are possible. The response of savings to a variation in the gross interest rate depends crucially on the specification of habits. In general, with SH, the marginal utility of consumption when old is decreasing in $c_1$ at a decreasing rate. However, this might not be the case if we adopt the MH specification. In the latter case, the marginal utility of effective consumption when old is decreasing in $c_1$ at an increasing or at a decreasing rate depending on parameter values. This implies that, agents might value consumption growth more in the multiplicative specification than in the subtractive specification.

3 Conclusion

The objective of this paper is to examine the differences between two alternative formulations used in the habit persistence literature. In particular, we explore the consequences of adopting multiplicative versus subtractive habits in terms of multiplicity of steady states, dynamic indeterminacy, and local dynamics. We find that choosing MH as opposed to SH can result in very different outcomes in a pure exchange two period lived overlapping generations model. First, we show that in general under MH the optimization problem may not be concave and that preferences are not homothetic. Second, we find that multiple monetary steady states cannot be ruled out under MH. Third, the local dynamic properties of the economy crucially depend on the type of habits considered as well as their strength.

Finally, it is worthy to point out that we did not fully explore the differences in the global dynamics of the two alternative specifications. However, since we find discontinuities
in the offer curve under MH but not under SH because of the non-concavity issue, our educated guess is that SH and MH could yield very different time series patterns. Thus, the results on the irrelevance of modeling habits by Bunzel [5] do not hold under a more general specification of preferences.

Our paper suggests that there are important differences on how savings behave across the two most commonly adopted habit specifications. The choice of how to model savings behavior in the presence of habits is not as innocuous as it might seem. Hence, the predictions and policy prescriptions based on the two types of habit persistence modeling choices could be quite different.

References


Appendix

Proof of Proposition 1

Proof. (i) The principal minor of the Hessian associated with the SH specification is given by $u''(c_1) + \gamma^2 \beta u''(c_2 - \gamma c_1)$. Note that this expression is always negative. The determinant is given by $\beta u''(c_1)u''(c_2 - \gamma c_1)$ which is always positive. Therefore, the Hessian is negative definite and concavity is insured.

(ii) Repeating the process for the case of MH, we find that the first principal minor is negative if and only if:

$$u''(c_1) + \gamma \beta (\gamma + 1) \frac{c_2}{c_1+2}u'\left(\frac{c_2}{c_1}\right) + \gamma^2 \beta \frac{c_2^2}{c_1^2(\gamma+1)}u''\left(\frac{c_2}{c_1}\right) < 0$$

and the determinant is positive if and only if

$$u''\left(\frac{c_2}{c_1}\right)u''(c_1) + \gamma (1 - \gamma) \beta \frac{c_2}{c_1+2}u'\left(\frac{c_2}{c_1}\right) u''\left(\frac{c_2}{c_1}\right) > \gamma^2 \beta \frac{1}{c_1} \left(u'\left(\frac{c_2}{c_1}\right)\right)^2$$

Therefore, the Hessian is negative semi-definite only under certain parameter restrictions.

In order for the reader to be sure that these conditions are violated in some cases we need to provide at least one explicit example where the objective is not concave. Consider the HARA instantaneous utility together with MH: $V(c_1, c_2) = \frac{1}{1-\alpha} \left[\sigma \left(A + \frac{\alpha c_1}{\sigma}\right)^{1-\sigma}\right] + \beta \frac{1}{1-\sigma} \left[\sigma \left(A + \frac{\alpha c_2}{\sigma c_1}\right)^{1-\sigma}\right]$. Let $w_1 = 3, w_2 = 1, \gamma = 0.75, \sigma = 0.39, A = -1, \alpha = 0.79, \text{ and } \beta = 0.55$. Then for $s_t \in [0.974, 2.289]$ there is non concavity.

Proof of Proposition 2

Proof. It it sufficient to show that the von Neumann-Morgenstern utility function in the subtractive case is homogeneous of degree $n$, while the one in the multiplicative case is not.

Let $\lambda > 0$ be a scalar. If we multiply the first and second period consumption by $\lambda$ in the subtractive case we obtain: $u(\lambda c_1) + \beta u(\lambda c_2 - \gamma \lambda c_1) = u(\lambda c_1) + \beta u(\lambda (c_2 - \gamma c_1))$. Assuming that the felicity function $u(.)$ is homogeneous of degree $n$, then the overall utility function is clearly homothetic, i.e. $\lambda^n u(c_1) + \lambda^n \beta u(c_2 - \gamma c_1) = \lambda^n V(c_1, c_2)$.

For the multiplicative case, instead, we show that the utility function is not homogeneous of degree $n$ regardless of the degree of homogeneity of $u(.)$. Specifically:
\[ V(\lambda^n c_1, \lambda^n c_2) = u(\lambda^n c_1) + \beta u \left( \frac{\lambda^n c_2}{(\lambda^n c_1)\gamma} \right) = u(\lambda^n c_1) + \beta u \left( \frac{\lambda^{n(1-\gamma)} c_2}{c_1\gamma} \right) \]

\[ = \lambda^n u(c_1) + \lambda^{n(1-\gamma)} \beta u(c_2 - \gamma c_1) \neq \lambda^n V(c_1, c_2) \]

**Proof of Proposition 3**

**Proof.** The implicit function that defines the optimal level of savings, \( \hat{s} \), under SH is given by:

\[ \beta(R + \gamma)u'((R + \gamma)s + w_2 - \gamma w_1) = u'(w_1 - s) \]

Since both the left and the right hand side of the first order condition (FOC) are continuous and differentiable, we can study the sign of the derivatives. More precisely, we find the following expressions:

\[ \frac{\partial u'(w_1 - s)}{\partial s} = -u''(w_1 - s) > 0 \]

\[ \frac{\partial \beta(R + \gamma)u'((R + \gamma)s + w_2 - \gamma w_1)}{\partial s} = \beta(R + \gamma)^2 u''((R + \gamma)s + w_2 - \gamma w_1) < 0 \]

Note that the second expression (derivative for the left hand side of the FOC) is strictly decreasing in \( s \) and the first expression (derivative for the right hand side of the FOC) is strictly increasing in \( s \). By the intermediate value theorem, there exists a unique level of optimal savings, \( \hat{s} \). However, the same conclusion cannot be drawn under MH. The implicit function that defines \( s^* \), the optimal level of savings in the MH case, is given by:

\[ \beta u' \left( \frac{Rs + w_2}{(w_1 - s)\gamma} \right) \left( \frac{R(w_1 - s) + \gamma(Rs + w_2)}{(w_1 - s)\gamma} \right) = u'(w_1 - s) \]

While the right hand side is strictly increasing in \( s \), the sign of the slope for the left hand side depends on the underlying parameters of the economy. After simplifications, we have the following expression:
Thus, whether the left hand side is increasing or decreasing in $s$ depends on the following condition:

$$\frac{-u'(\frac{Rs+w_2}{(1-s)^\gamma})}{u''(\frac{Rs+w_2}{(1-s)^\gamma})} \geq \frac{R(w_1-s) + \gamma(Rs + w_2)(w_1-s)^{\gamma-1}}{(3\gamma + 1)(w_1-s)^{2\gamma-1}}$$

Therefore, we cannot rule out the possibilities for multiple or a continuum of equilibrium values of $s^*$. Note that we cannot rule out the possibility of no equilibria either.

**Figures**

![Figure 1: SH case (a) vs. MH case (b) for $\gamma = 0.5$.](image-url)
Figure 2: SH case (a) vs. MH case (b) for $\gamma = 0.7$

Figure 3: SH case (a) vs. MH case (b) for $\gamma = 0.85$