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ECO 301
Macroeconomic Theory
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MATH REVIEW

The notes I provide here are not meant to be a complete review of the algebra you have learned so far. Rather, my aim is to provide you examples of some very important algebraic definitions and rules that we will make use of throughout our class. For interested students the two references below are excellent sources of further information:

Chiang, A. "Fundamental Methods of Mathematical Economics," Third Edition, McGraw Hill Publishers.

Takayama, A. "Mathematical Economics," Second Edition, Cambridge University Press, 1985.

Percentage change in a variable

Suppose that we would like to calculate how much GDP per capita grew in the US in the last 20 years. If GDP per capita was $22,000 in 1985 and $36,000 in 2005, then the growth would be

$$\frac{36,000-22,000}{22,000}\times 100=\frac{14,000}{22,000}\times 100=63\%$$

We can generalize this formula as follows: let x_o be the beginning value and x_1 be the ending value of variable *x*. Then, the percentage change in the value of *x* is

$$\frac{ending\ value-beginning\ value}{beginning\ value}\times 100=\frac{x_1-x_o}{x_o}\times 100$$

Working with exponentials

Suppose that we have four real numbers *x*, *y*, *n*, and *m*. Below are the rules that we follow when we work with powers:

1. $\frac{x^n}{x^m} = x^{n-m}$

2. $x^n . x^m = x^{n+m}$

3. $\frac{x^n}{y^n} = \left(\frac{x}{y}\right)^n$

4. $x^n . y^n = (x.y)^n$

5. If $x^n = y^m$, then $x = y^{m/n}$ and $y = x^{n/m}$

6. $(x^m)^n = x^{m.n}$

7. $x^{-m} = \frac{1}{x^m}$

Examples:

$\frac{5^3}{5^8} = 5^{3-8} = 5^{-5}$

$\frac{x^4}{x^2} = x^{4-2} = x^2$

$x^2 . x^4 = x^{2+4} = x^6$

$2^x 2^y = 2^{x+y}$

$5^6 = x^2 \Rightarrow x = 5^{6/2} = 5^3 = 125$

$2^{\frac{1}{3}} = y^2 \Rightarrow y = 2^{\frac{1}{3}/2} = 2^{\frac{1}{6}}$

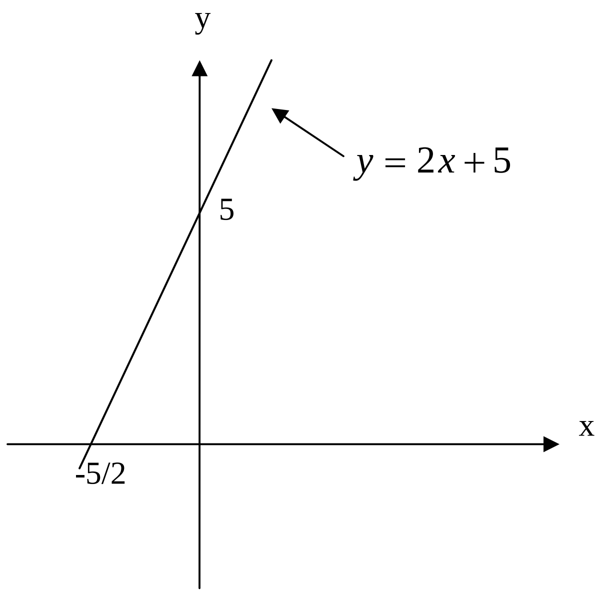
$\left(x^2\right)^3 = x^{2..3} = x^6$

$\left(4^3\right)^{3/2} = 4^{3.3/2} = 4^{9/2}$

Linear functions and their graphs

The equation of a linear function is given by $y = ax + b$. Here, *a* is called the “slope” and *b* is called the “y-axis intercept.” To draw the graph of this function, we determine the points at which the line crosses the x- and y-axes and the connect those two points. Let’s see an example:

Example: Suppose that the equation of our linear function is $y = 2x + 5$. To find the point at which the line crosses the y-axis, we set x=0. This gives y=2.0+5=5. Hence, the y-intercept is 5. Then, we find the point at which the line crosses the x-axis. To do this we set y=0. Hence, 0=2x+5, which means that our x-intercept is x=-5/2. We finish by connecting those two points with a straight line:



Solving one equation in one unknown:

If, in an in equation, there is only one unknown variable, there are a couple of strategies we can use to solve for the value of the unknown. Let's see some examples:

I. Solving Quadratic Equations

Quadratic equations have the general form $ax^2 + bx + c = 0$, where a, b, and c are real numbers. In order to solve quadratics, there is a simple strategy: we divide *c* into two factors c_1 and c_2 and *a* into two factors a_1 and a_2 such that $a_1c_2 + a_2c_1 = b$.

Examples:

1. $x^2 + 4x + 4 = 0$

$1.x \quad +2$

$1.x \quad +2$

In the above case, a=1, b=4, and c=4. We divided a=1 into two factors: 1 and 1. We divided c=4 into two factors: 2 and 2. Then, the above equation can be written as $(x + 2)(x + 2) = 0$

After factorization, we equate each component to zero to find the value of x:
$x+2=0 \Rightarrow x=-2$

2. $x^2-4x+4=0$

1.x -2

1.x -2

In the above case, a=1, b=-4, and c=4. We divided a=1 into two factors: 1 and 1. We divided c=4 into two factors: -2 and -2. Then the above equation can be written as
$(x-2)(x-2)=0$
After factorization, we equate each component to zero to find the value of x:
$x-2=0 \Rightarrow x=2$

3. $2x^2-8x+6=0$

2.x -2

x -3

In the above case, a=2, b=-8, and c=6. We divided a=2 into two factors: 2 and 1. We divided c=6 into two factors: -2 and -3. Then the above equation can be written as
$(2x-2)(x-3)=0$
After factorization, we equate each component to zero to find the value of x:
$(2x-2)=0 \Rightarrow 2x=2 \Rightarrow x=1$
$(x-3)=0 \Rightarrow x=3$

4. $5x^2+7x-6=0$

x +2

$5x$ -3

In the above case, a=5, b=7, and c=-6. We divided a=5 into two factors: 5 and 1. We divided c=-6 into two factors: 2 and -3. Then the above equation can be written as
$(x+2)(5x-3)=0$
After factorization, we equate each component to zero to find the value of x:
$(x+2)=0 \Rightarrow x=-2$
$(5x-3)=0 \Rightarrow x=-3/5$

II. Cross Multiplication

When the equality to be solved involves a fraction, we use “cross multiplication,” i.e, we multiply denominators and numerators of opposite sides of the equality. Let’s see examples:

1. $\frac{1}{x+5} = \frac{2}{3} \Rightarrow 1.3 = 2(x+5) \Rightarrow 3 = 2x+10 \Rightarrow 3-10 = 2x \Rightarrow -7 = 2x \Rightarrow x = -7/2$

2. $\frac{x^2}{3.x^5} = 2 \Rightarrow x^2 = 2.3.x^5 \Rightarrow x^2 = 6.x^5 \Rightarrow \frac{1}{6} = \frac{x^5}{x^2} \Rightarrow \frac{1}{6} = x^3 \Rightarrow x = \left(\frac{1}{6}\right)^{1/3}$

3. $\frac{1}{x^2+4x} = -\frac{1}{4} \Rightarrow 4.1 = (-1)(x^2+4x) \Rightarrow 4 = -x^2-4x \Rightarrow x^2+4x+4 = 0$

$x \quad +2$

$x \quad +2$

$\Rightarrow (x+2)(x+2) = 0$

$\Rightarrow x = -2$

4. $\frac{2x-1}{x+7} = \frac{x}{2} \Rightarrow (2x-1)2 = x(x+7) \Rightarrow 4x-2 = x^2+7x \Rightarrow x^2+7x-4x+2 = 0$

$\Rightarrow x^2+3x+2 = 0$

$x \quad +2$

$x \quad +1$

$\Rightarrow (x+2)(x+1) = 0$

$\Rightarrow (x+2) = 0$ or $(x+1)$

$\Rightarrow x = -2$ or x=-1

Derivatives of functions of one variable:

First Order Derivative

Let $f: R \to R$, $g: R \to R$ and $h: R \to R$ be functions of one variable, *x*, where *R* denotes the set of real numbers. We denote the first derivative of *f* by $\frac{d\left(f(x)\right)}{dx}$ or $f'(x)$. The following rules will be the useful throughout this class:

1. The derivative of a constant is zero.
Example: $f(x) = 5$. Then, $f'(x) = 0$.

2. If $f(x) = \alpha x$, where α is a constant, then $f'(x) = \alpha$.

3. If $f(x) = x^n$, then $f'(x) = nx^{n\text{-}1}$. (*n* is a constant)

4. If $f(x) = \ln x$, then $f'(x) = 1/x$.

5. If $f(x) = \ln g(x)$, then $f'(x) = g'(x)/g(x)$

Examples:

$f(x) = \ln 5x \Rightarrow f'(x) = 5/(5x) = 1/x$

$f(x) = \ln x^2 \Rightarrow f'(x) = 2x/x^2 = 2/x$

$f(x) = \ln(5x-1) \Rightarrow f'(x) = 5/(5x-1)$

6. Summation rule: If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.

Example: $f(x) = x^3 + 2x$, then $f'(x) = 3x^2 + 2$.

Similarly, $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$.

7. If $f(x) = [g(x)]^n$, then $f'(x) = n[g(x)]^{n-1} g'(x)$.

Example: $f(x) = (5-2x)^4$. Then, $f'(x) = 4[5-2x]^3(-2) = -8[5-2x]^3$.

8. The Product Rule: : If $f(x) = g(x)h(x)$, then $f'(x) = g'(x)h(x) + g(x)h'(x)$.

Example: $f(x) = (2x+1)(x-5)$.

Then, $f'(x) = 2(x-5) + (2x+1)(-5)$

$=2x - 10 - 10x - 5$

=-8x-15.

9. The Quotient Rule: $f(x) = \dfrac{g(x)}{h(x)}$, then $f'(x) = \dfrac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}$.

Example: $f(x) = (2x+1)/(x-5)$.

Then, $f'(x) = \dfrac{2(x-5) - (2x+1)(-5)}{(x-5)^2}$

$= \dfrac{2x - 10 + 10x + 5}{(x-5)^2}$

$= \dfrac{12x-5}{(x-5)^2}$

Second Order Derivative

Second order derivative is the derivative of the first derivative. So, in order to find the second derivative of a function of one variable, you must find the first derivative, then take its derivative. The second derivative is denoted by $\frac{d^2 f(x)}{dx^2}$ or $f''(x)$.

Examples:

1. $f(x) = x^3 + 2x$. Then $f'(x) = 3x^2 + 2$, and $f''(x) = 6x$.

2. $f(x) = \ln x$. Then, $f'(x) = 1/x$, and $f''(x) = -1/x^2$ (Using the quotient rule).

3 $f(x) = (5-2x)^4$. Then, $f'(x) = 4\left[5-2x\right]^3(-2) = -8\left[5-2x\right]^3$
and $f''(x) = (-8)3\left[5-2x\right]^2(-2) = -48\left[5-2x\right]^2$.

Partial Derivatives of Functions of Two-Variables

Let $f : R^2 \to R$ be a function of two variables, *x* and *y*. Here, I will review first- and second-order partial derivatives of *f*.

First-order partial derivative:

The first-order partial derivative of *f* with respect to x is denotes by $\frac{\partial f(x,y)}{\partial x}$ or f_x.
Similarly, the partial derivative of *f* with respect to *y* is denoted by $\frac{\partial f(x,y)}{\partial y}$ or f_y.

When taking the partial derivative with respect to *x*, we treat *y* like a "constant." We do the reverse if we want the partial derivative with respect to *x*. Below are some examples.

Examples:

1. $f(x,y) = 2x + 3y$. Then, $f_x = 2$ and $f_y = 3$.

2. $f(x,y) = xy$. Then, $f_x = y$ and $f_y = x$.

3. $f(x,y) = x^3 + 2xy + y^2$. Then, $f_x = 3x^2 + 2y$ and $f_y = 2x + 2y$.

Optimization

I. Unconstrained Optimization

Suppose that we would like to find the points at which a function reaches its maxima. Such points are referred to as “maximizers.” The way we find maximizers of functions are slightly different depending on whether we have a function of one variable or two variables.

Maximizers of a Function of One Variable:

Suppose that $f : R \to R$ is a function of one variable, *x*. For a point x_o to be a maximizer of *f*, the following sufficient conditions have to be satisfied:

1. First order condition (FOC): $f'(x_o) = 0.$
2. Second order condition (SOC): $f''(x_o) < 0.$

Maximizers of a Function of Two Variables:

Suppose that $f : R^2 \to R$ is a function of two variables, *x* and *y*. For a point (x_o, y_o) to be a maximizer of *f*, the following sufficient conditions have to be satisfied:

1. $f_x(x_o, y_o) = 0,\ f_y(x_o, y_o) = 0$,
2. $f_{x,x}(x_o, y_o),\ f_{y,y}(x_o, y_o) < 0$ and $f_{x,x}(x_o, y_o) f_{y,y}(x_o, y_o) - f_{x,y}(x_o, y_o) f_{y,x}(x_o, y_o) > 0.$

II. Constrained Optimization

Suppose that we would like to solve the problem of a utility maximizing individual who consumes two goods *x* and *y*. That is, we would like to find the optimal consumption bundle x* and y* subject to the budget constraint.

As an example, suppose that our consumer’s utility function is given by $U(x, y) = x.y$. The price of good *x* is 2 and the price of good *y* is 4. The consumer has 10 dollars to spend. Then, our problem is

$$\max_{x,y} U(x, y) \text{ subject to (the Budget Constraint)}$$

i.e.

$$\max_{x,y} \; xy \text{ subject to } 2x + 4y = 10.$$ [1]

To solve such a problem, we will use the Substitution Method.

The Substitution Method

Step 1. Find *y* in terms of *x* from the budget constraint.
Step 2. Substitute the value of *y* in the utility function to write down the utility in terms of *x* only.
Step 3. Take the first derivative of the Utility function with respect to x and equate it to zero. This will give you the value of x*. To find y* make use the relationship between *x* and *y* that you found in Step 1.

Example. Let's find the optimal consumption bundle for the consumer in our example above by following each step of the substitution method.

Step 1. The budget constraint is $2x + 4y = 10$. So, $y = 5/2 - x/2$.
Step 2. Substitute the value of *y* in the utility function: $U(x) = x(5/2 - x/2)$
Step 3. The derivative with respect to *x* is: $U'(x) = 5/2 - x$.
(I used the product rule to find the derivative.)
Equating the first derivative to zero will give us the optimal consumption of *x* for this consumer:

$$U'(x) = 0 \Leftrightarrow 5/2 - x = 0. \text{ Hence, } x^* = 5/2.$$

To find *y** we use $y = 5/2 - x/2$ from Step 1. Hence, $y^* = 5/2 - 5/4 = 5/4.$

Step 4. Make sure that the solution is indeed a maximizer, i.e, check the second order condition (SOC). SOC tells us to take the second derivative of *U(x)* and evaluate it at *x**=5/2. If the answer is less than zero, x*=5/2 is a maximizer:

$U''(x) = -1$. So, $U''(5/2) = -1 < 0$. This guarantees that our solution is a maximizer.

Example. Find the optimal quantities of textbooks (T) and movies (M) for a student whose preferences for the two goods are represented by the utility function

$$U(M,T) = \ln M + \ln T.$$

[1] The budget constraint of a consumer who wants to allocate his total income *I* between the two goods *x* and *y* is with prices p_x and p_y is $p_x x + p_y y = I$.

Suppose a movie is 2 dollars and a textbook is 1 dollar. The consumer has 4 dollars of income.

Solution: The student's problem is

$$\max_{M,T}\left\{\text{ lnM+lnT}\right\} \text{ subject to } 1.\text{T} + 2M = 4.$$

We follow each step of the substitution method:

Step 1. The budget constraint is $T + 2M = 4$. So, $T = 4 - 2M$.
Step 2. Substitute the value of T in the utility function: $U(M) = \ln M + \ln(4 - 2M)$
Step 3. The derivative with respect to M is: $U'(M) = \frac{1}{M} + \frac{-2}{4-2M}$.
Equating the first derivative to zero will give us the optimal consumption of x for this consumer:

$$U'(M) = 0 \Leftrightarrow \frac{1}{M} + \frac{-2}{4-2M} = 0. \text{ Hence, } \frac{1}{M} = \frac{2}{4-2M} \Rightarrow \text{4-2M=2M} \Rightarrow \text{M}^* \text{=1 .}$$

To find T^* we use $T = 4 - 2M$ from Step 1. Hence, $T^* = 4 - 2.1 = 2.$

Step 4: Check the SOC to guarantee a maximizer:

$$U''(M) = 0 \Leftrightarrow \frac{-1}{M^2} + \frac{-(-2)(-2)}{\left(4-2M\right)^2} = \frac{-1}{M^2} + \frac{-4}{\left(4-2M\right)^2}$$

$$\text{Hence, } U''(1)\ \frac{-1}{1^2} + \frac{-4}{\left(4-2.1\right)^2} \text{=-1-1=-2<0 .}$$

Our solution is a maximizer.

Note: When we log-utility (i.e, U(x,y)= lnx+lny) , the SOC will always be satisfied. Hence, we may skip step 4!

The Problem of a Consumer who Chooses between Consumption and Leisure?

One of the interesting questions in macroeconomics is unemployment and the labor supply of individuals. Note that we, as consumers, make the decision on how much to work, how much time to spend on leisurely activities, and how many goods to buy with the wage we earn by working. So, how can we find the optimal balance of work, leisure and consumption for an individual?

Suppose that a consumer has to choose how many goods to consume (C) and how much time to spend on leisure each day. We are interested in the allocation of time between work and leisure in one day. Hence, if the individual works *L* fraction of the time, 1-*L* will be the fraction of the day spent on leisure. Assume that this individual's preferences over consumption and leisure are given by

$$U(C,L) = \underbrace{\ln C}_{\text{utility from consumption}} + \underbrace{\ln(1-L)}_{\text{utility from leisure}} .$$

The price of one consumption good is *p* dollars and the wage rate per unit of time worked is *w* dollars.

Before formalizing and solving this problem, let's first determine the budget constraint in this case. Remember: the budget constraint always tells us that total expenditure on consumption goods is equal to total income. In this example, when the consumer consumes *C* units of the good at per unit price *p*, his total consumption expenditure would be *p.C*. Similarly, if he works *L* fraction of the time earning wage *w* per unit of time worked, his total income would be *wL*. So, we can write the budget constraint as

$$pC = wL.$$

Now, we can formalize the problem:

$$\max_{(C,L)>0} \{\ln C + \ln(1-L)\} \text{ subject to } pC = wL .$$

This is a constrained optimization problem. So, we apply the steps of the substitution method:

Step 1: Use the budget constraint to eliminate *C*: $C = \frac{w}{p}L$.

Step 2: Substitute for C in the utility function:

$$\max_{(L)>0} \left\{ \ln\left(\frac{w}{p}L\right) + \ln(1-L) \right\}$$

Step 3: Write down the first order condition: Take the first order derivative with respect to L and equate it to zero (here *w* and *p* are treated as constants since we take the derivative with respect to variable *L*.).

FOC: $\dfrac{\frac{w}{p}}{\frac{w}{p}L} + \dfrac{-1}{1-L} = 0 \Rightarrow \dfrac{1}{L} + \dfrac{-1}{1-L} = 0 \Rightarrow \dfrac{1}{L} = \dfrac{1}{1-L} \Rightarrow 1-L = L \Rightarrow L^* = 1/2$.

Hence, this person spends half of the day working, i.e, he works for 12 hours a day.

To find how many goods he consumes, use $C = \frac{w}{p}L$:

$$C^* = \frac{w}{p}L^* = \frac{w}{p}\frac{1}{2}.$$