1. (a) Find the optimal consumption levels as well as the optimal savings for this household.

\[
\max_{(c_1, c_2) \geq 0} \left\{ \log c_1 + \frac{1}{3} \log c_2 \right\} \quad \text{subject to} \quad \begin{align*}
C_1 + S_1 &= 20 \\
C_2 &= (1 + r)S_1.
\end{align*}
\]

\[
\Leftrightarrow \max_{(S_1) \geq 0} \left\{ \log(20 - S_1) + \frac{1}{3} \log((1 + r)S_1) \right\}
\]

FOC:
\[
\frac{-1}{20 - S_1} + \frac{1}{3 (1 + r) S_1} = 0 \Rightarrow \frac{1}{3} \frac{1}{S_1} = \frac{1}{20 - S_1} \Rightarrow 20 - S_1 = 3S_1 \Rightarrow S_1 = 5.
\]

From the budget constraints,
\[
C_1 = 20 - S_1 = 15,
C_2 = (1 + r)5.
\]

(b) Write down the investment demand function for the firm.

\[
\max_{(Y, I) \geq 0} \{ Y - (1 + r)I \} \quad \text{subject to} \quad Y = 10I^{1/2}
\]

\[
\Leftrightarrow \max_{(I) \geq 0} \{ 10I^{1/2} - (1 + r)I \}
\]

The first order condition for profit maximization gives
\[
10 \frac{1}{2} I^{-1/2} = 1 + r \Rightarrow 5I^{-1/2} = 1 + r \Rightarrow \left( \frac{5}{1 + r} \right)^2 = I .
\]

(c) Find the equilibrium in the loanable funds market.

To find the equilibrium interest rate, notice that this is a closed economy and so \( S = I \) in equilibrium.
\[
\frac{25}{(1 + r)^2} = 5 \Rightarrow 5 = (1 + r)^2 \Rightarrow \sqrt{5} - 1 = r^*.
\]

Hence, \( I^* = S^* = 5 \).
2. The budget constraint in the first period:

\[ C_1 + S_1 = 10 \]

The budget constraint in the second period:

\[ C_2 = 5 + (1 + 0.20)S_1 \Leftrightarrow C_2 = 5 + 1.20S_1 \]

(b) To find the optimal savings and consumption levels for this household, we need to solve for the consumer’s problem:

\[
\max \left\{ \log C_1 + \beta \log C_2 \right\} \text{ subject to } C_1 + S_1 = 10 \text{ and } C_2 = 1.2S_1 + 5.
\]

Substitute \( C_1 \) and \( C_2 \) into the utility function:

\[
\max \left\{ \log(10 - S_1) + \beta \log(1.2S_1) \right\}
\]

The first order condition:

\[
-\frac{1}{10 - S_1} + \beta \frac{1.2}{1.2S_1 + 5} = 0
\]

\[
\Rightarrow \beta \frac{1.2}{1.2S_1 + 5} = \frac{1}{10 - S_1}
\]

\[
\Rightarrow \beta 1.2 (10 - S_1) = 1.2S_1 + 5
\]

\[
\Rightarrow \beta 12 - \beta 1.2S_1 = 1.2S_1 + 5
\]

\[
\Rightarrow \beta 12 - 5 = \beta 1.2S_1 + 1.2S_1
\]

\[
\Rightarrow \beta 12 - 5 = 1.2S_1 (\beta + 1)
\]

\[
\Rightarrow (\beta 12 - 5) / (1.2(\beta + 1)) = S_1
\]

So, for \( \beta = 0.5 \), the household saves \( S_1 = 0.55 \).

From the budget constraints, we can find the optimal consumption levels in each period for the household:

\[ 10 = C_1 + S_1 \Rightarrow C_1 = 10 - S_1 = 10 - 0.55 = 9.45 \]

\[ C_2 = (1.2)S_1 + 5 = 1.2 * 0.55 + 5 = 5.66. \]
(c) \( I = 5 - (r_0)^2 = 5 - (0.2)^2 = 4.96. \)
Since savings in part (b) are equal to 0.55, this country is running a trade deficit of 4.96-0.55=4.41.

(d) When \( \beta = 0.9 \), the solution to the household problem gives
\[
\left( \beta * 12 - 5 \right) / \left( 1.2 (\beta + 1) \right) = S_1
\]
\[
\Rightarrow S_1 = 2.54
\]
Therefore, consumption in the first period will be \( C_1=10-2.54=7.46 \), and the second period consumption will be \( C_2=5+1.2*2.54=8.04 \).

We see that as beta increases, household consumes less in the first period of lifetime, saves more, and consumes more in the second period. Note that beta shows how patient the household is: as beta gets bigger, the role of second period consumption in the lifetime utility increase. Therefore, consumption is the second period becomes more preferable for the household.

3. The Effects of Taxes on Savings and Consumption

(a) Write down the lifetime budget constraint of the household by using the specific numbers given above.
\[
C_1 + S_1 = 20
\]
\[
C_2 = (1 + r)S_1
\]

(b) Find the optimal consumption levels as well as the optimal savings for this household.
\[
\max \left\{ \log \left( C_1 \right) + \log \left( C_2 \right) \right\} \ s.t \ C_1 + S_1 = 20
\]
\[
C_2 = (1 + r)S_1
\]

The first budget constraint gives us \( C_1 = 20 - S_1 \) and the second budget constraint gives us \( C_2 = 1.02S_1 \) (since \( r=0.02 \)). Substituting them in the utility function gives
\[
\max \left\{ \log \left( 20 - S_1 \right) + \log \left( 1.02S_1 \right) \right\}
\]

\[
\text{FOC:} \quad \frac{-1}{20 - S_1} + \frac{1.02}{1.02S_1} = 0
\]
\[ \frac{-1}{20 - S_i} + \frac{1}{S_i} = 0 \iff \frac{1}{S_i} = \frac{1}{20 - S_i} \]

\[ \Rightarrow 20 - S_i = S_i \]

\[ \Rightarrow 20 = 2S_i \]

\[ \Rightarrow 10 = S_i \]

Given that the individual has 20 units of income in the first period and he saves 10 units, he must consume 10 units in the first period and 10*1.02=10.2 in the second period.

(c) Suppose that the household pays 4 units of taxes out of his income when young and receives 6 units in benefits when old. What is the new lifetime budget constraint?

The new lifetime budget constraint is

\[ C_1 + \frac{C_2}{1 + r} = (Y_1 - 4) + \frac{Y_2 + 6}{1 + r} \iff C_1 + \frac{C_2}{1.02} = 16 + \frac{6}{1.02} \]

\[ \iff C_1 + \frac{C_2}{1.02} = 21.8 \]

(d) In part (c) how much will the household save?

\[ \max_{C_1, C_2} \left\{ \log(C_1) + \log(C_2) \right\} \quad s.t \quad C_1 + S_i = 16 \]

\[ C_2 = (1 + r)S_i + 6 \]

Substitution gives

\[ \max_{S_i} \left\{ \log(16 - S_i) + \log(1.02S_i + 6) \right\} \]

foc:

\[ \frac{-1}{16 - S_i} + \frac{1.02}{1.02S_i + 6} = 0 \]

\[ \Rightarrow \frac{1}{16 - S_i} = \frac{1.02}{1.02S_i + 6} \]

\[ 1.02S_i + 6 = 1.02(16 - S_i) \]

\[ 1.02S_i + 6 = 16.32 - 1.02S_i \]

\[ 2.04S_i = 10.32 \]

\[ S_i = 5.05 \]
4.
(a) $PDV = \frac{1000}{1.05}$

(b) $PDV = \frac{1000}{(1.05)^2}$

(c) $PDV = \frac{1000}{1.05} + \frac{1000}{(1.05)^2} + \frac{1000}{(1.05)^3}$

(d) $PDV = \frac{1000}{0.05}$