

When do co-located firms selling identical products thrive?

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Abstract

When consumers only see prices once they visit stores, and some consumers have time to comparison shop, co-location commits stores to lower prices that draw consumers away from isolated stores. Profits of co-located firms are a single-peaked function of the number of shoppers—co-located firms thrive when there are some shoppers, but not too many. When consumers know in advance whether they have time to shop, these effects are enhanced: co-located stores may draw enough shoppers to drive the expected price paid by a non-shopper below that paid when consumers do not know if they will have time to shop.

1 Introduction

Lincoln Square Mall (LSM), built in downtown Urbana in the mid 1960s, was one of the first downtown fully-enclosed malls in the United States.¹ Over the next 35 years, a series of three “anchor” department stores and two groceries failed, and since Bergner’s left in 1999, no major store has located there. Standard spatial theories of firm competition suggest that these stores *should* have thrived: LSM is located at a prime population center, and there were no nearby competing stores—these stores had extensive local monopoly power.

In sharp contrast, at the northern tip of neighboring Champaign, far from population centers, Walmart and Meijer profitably co-exist despite selling similar products on opposing sides of the same street, as do Lowes and Home Depot. Standard theory suggests that competition between closely-located firms selling almost identical products *should* drive their profits down to zero. More generally, in the canonical spatial model (Tirole 2001, p. 281), an increase in market share from locating closer to a rival is more than offset in profit terms by the resulting increased price competition—firms do best to separate maximally.

Why then despite ignoring this prescription, have Walmart and Meijer thrived? In short, how did they get it so right, by doing it so wrong? The phenomena illustrated by these anecdotes is widespread. A large literature highlights the hollowing out of town centers (e.g., Iaria 2014), and of ostensibly strongly competing firms successfully co-locating at strip malls (e.g., Page and Tessier 2007), with policy responses seeking to offset this (e.g., the 1996 Town Centres First policy in England).

Our paper asks: when do co-located firms selling identical products thrive? In our spatial model, one firm is located at one end of a line, and two firms are located at the other. The firms sell identical products. Were consumers to see the prices set by firms before making shopping decisions, competition between the co-located firms would drive their prices down to marginal cost, while the isolated firm would set a higher price and profit from closely-located consumers. However, in practice, consumers rarely see prices before making shopping decisions—most consumers only see the actual prices once they enter a store. But, then the isolated firm’s customers are, in effect, captured, leading the isolated firm to monopoly price.

In contrast, co-located firms compete on price to attract a greater share of consumers who shop at their location: when prices are unobserved, co-location serves as a commitment

¹http://deadmalls.com/malls/lincoln_square_mall.html

to lower prices that makes consumers willing to travel further to shop. The isolated firm may want to commit to lower prices to attract more consumers, but cannot. Our paper characterizes how price competition between the co-located firms—which hinges on the attribute composition of consumers who go to the cluster—interacts with the price-elasticity of shopping decisions that determines consumer shopping location choices to determine the profits of isolated and co-located firms.

In our model, some consumers who visit the co-located stores have time to comparison shop, and buy at the store offering the lower price, while the others only have time to visit one store. Competition for comparison shoppers causes the co-located stores to adopt mixed pricing strategies, offering stochastically greater price improvement when the share of comparison shoppers is higher (Stahl 1989). Lower travel costs make more-distantly located consumers willing to go to the co-located stores, raising their profits at the expense of the isolated store. Thus, secular declines in travel costs can drive the observed downward spiral in the profits of a town-center store, and increase profits of competing stores that co-locate far away.

The economics are more subtle than just what this observation suggests, hinging on how many consumers comparison shop. To see why, observe that if no one has time to comparison shop, then the co-located stores monopoly price, each earning half the profit of the isolated store. We prove that from this base, as one increases the fraction of consumers who are “shoppers”, profits of co-located stores *always* rise even though they compete more aggressively on price. With few shoppers, distance is the primary determinant of where individuals shop. As a result, the measure of consumers drawn to the cluster by increased price competition is initially very elastic—the marginal consumers have similar travel costs regardless of where they go. Thus, the price competition due to more shoppers draws enough customers to more than offset the reduced profit per customer, causing co-located firm profits to rise. We identify conditions under which profits of co-located stores are a concave, single-peaked function of the fraction of consumers who are shoppers. However, as the proportion of shoppers goes to one, co-located stores compete all profits away, earning less than the isolated store. Thus, co-located stores thrive when there are some comparison shoppers, but not too many.

In our base setting, consumers choose where to go before learning whether they have time to comparison shop. This eases analysis at the possible expense of realism. We then consider the possibility that consumers know whether they have time to comparison shop before choosing where to go. When this is so, the share of shoppers at the cluster rises because shoppers have more to gain—they always go to the store offering the lowest price.

In turn, this enhances price competition between co-located firms, which draws more consumers overall. We identify conditions under which even a consumer who knows he lacks time to comparison shop is more willing to go to the cluster than when consumers do not know whether they will have time to shop prior to making travel decisions. In particular, when there are few shoppers in the population, the expected price paid by a consumer who knows he is a non-shopper falls below that paid by consumers who do not know whether they will have time to shop, and hence might turn out to be shoppers.

Literature. The vast spatial industrial organization literature in which greater spatial separation enhances firm profit is well-known. Our model builds on the search-cost literature that gives rise to price dispersion (Burdett and Judd, 1983; Stahl, 1989, 1996; Ellison and Ellison, 2009; Ellison and Wolitzky, 2012; Ronanye 2017). Armstrong (2016) surveys recent advances in directed search models with heterogeneous sellers. Our paper integrates spatial features to costly search models and investigates the consequences of information structures for consumers (what they know about prices and their abilities to search when making shopping decisions). Constantinou and Bernhardt (2018) show how, when consumers do not see prices before making shopping decisions, firms may have incentives to offer price-matching guarantees that result in a prisoner’s dilemma in which guarantees hurt all firms.

Fischer and Harrington (1996) spawned an agglomeration literature in which firms selling heterogeneous products can either co-locate or locate far from each other, when consumers do not know their product valuations before visiting a store. They argue that firms selling more differentiated products have greater incentives to co-locate. Subsequent work by Non (2010) and Parakhonyak and Titova (2018) identifies trade-offs between margins and traffic in the presence of search costs when consumers have identical travel costs.² Parakhonyak and Titova analyze a directed search, matching model with differentiated goods, multiple clusters, and consumers with identical search costs. Bigger clusters offer greater variety and lower prices, so they are searched first, allowing them to earn higher profits. Non compares clustering vs isolation when shoppers can costlessly search all firms, and non-shoppers incur fixed travel and store-entering costs. In contrast to these papers, we focus on the spatial travel structure that is central to town-center considerations, analyzing how consumers’ search and location characteristics influence the profits of the isolated and co-located firms. In our model, stores selling identical products cluster, and the heterogeneous travel costs of different consumers that underlie the price elasticity of shopping decisions drive equilibrium outcomes.

²See also Dudey (1990), and Page and Tassier (2007) who build an ‘ecological model’ of location dynamics in which firms co-locate because they ‘fit together’ for some reason.

2 Model

A continuum of consumers, indexed by their locations x , is uniformly distributed on the unit interval. Three profit-maximizing stores sell a homogeneous good. Stores L_1 and L_2 are co-located at 0, and store R is located at 1. Marginal costs of production are normalized to zero. Consumers choose whether to go to location 0 or 1, and when there, how much to buy. Traveling distance d costs $\alpha t(d)$, with $\alpha > 0$, $t(0) = 0$, and $t'(\cdot) > 0$, $t''(\cdot), t'''(\cdot) \geq 0$. Consumers do not see prices set by stores until they visit. Fraction $\mu \in [0, 1]$ of consumers are shoppers who have time to costlessly visit both co-located stores and purchase from the store offering the lower price. The remaining fraction $1 - \mu$ are non-shoppers who only have time to visit one store. Each consumer has a continuously differentiable, strictly decreasing demand $D(p)$. We assume that revenues $R(p) = pD(p)$ are strictly concave with a global maximum at the monopoly price p^m . Consumer surplus at price p^m is assumed to be high enough that even consumers located at one-half want to visit a store; and we assume that consumer surplus given price 0 is finite.

When stores L_1 and L_2 set prices p_1 and p_2 and store R sets p_R , a shopper who travels distance x to location 0 gets payoff $\max\{\int_{p_1}^{\infty} D(p)dp, \int_{p_2}^{\infty} D(p)dp\} - \alpha t(x)$, while a non-shopper who visits L_i gets $\int_{p_i}^{\infty} D(p)dp - \alpha t(x)$, and a customer who visits R gets $\int_{p_R}^{\infty} D(p)dp - \alpha t(1-x)$.

Timing. First, stores set (unobserved) prices and consumers select shopping locations. Consumers who go to location 0 learn whether they are shoppers: shoppers search both L_1 and L_2 , see prices and make purchases; non-shoppers (possibly randomly) select a store, see price and then make purchases. Consumers who visit store R see price and then make purchases.

Strategies. A (possibly mixed) strategy of store $j \in \{L_1, L_2, R\}$ is a cdf $F_j(p)$ over the price set $p \in [0, \infty)$. A strategy for consumer is a function mapping her location x into a choice of where to shop, and a probability of going to firm L_1 if she is a non-shopper at location 0.

We focus on symmetric Perfect Bayesian equilibria in which stores L_1 and L_2 employ the same pricing strategy $F(p) = F_1(p) = F_2(p)$, and non-shoppers at location 0 mix with equal probability over which store to visit. Thus, store R charges the monopoly price p^m , and collects revenue $R^m \equiv R(p^m)$ from each of its customers.

We begin with a lemma characterizing pricing by co-located stores. Their pricing strategy depends on the share of shoppers at location 0. Because consumers choose location before knowing their types, the fraction of shoppers is μ . This result mirrors that in Stahl (1989).

Lemma 1 (Stahl 1989) *In any symmetric equilibrium, if $\mu \in (0, 1)$, then stores at location 0 use a mixed pricing strategy over $p \in [b, p^m]$ with cumulative distribution function:*

$$F(p) = 1 - \left[\left(\frac{1 - \mu}{2\mu} \right) \left(\frac{R^m}{R(p)} - 1 \right) \right],$$

where b is the unique solution to $R(b) = \left[\frac{1 - \mu}{1 + \mu} \right] R(p^m)$. If $\mu = 0$, then both stores set $p = p^m$; if $\mu = 1$, then both stores set $p = 0$.³

Let $A(\mu)$ be the consumer surplus gain from visiting the co-located stores rather than store R :

$$A(\mu) = \int_{b(\mu)}^{p^m} F(p; \mu) D(p) dp + \mu \int_{b(\mu)}^{p^m} F(p; \mu) (1 - F(p; \mu)) D(p) dp. \quad (1)$$

The first term is the consumer surplus that all consumers (both shoppers and non-shoppers) gain because cluster stores charge less than the monopoly price. The second term is the added surplus that shoppers get because when one store has a lower price than the other, shoppers buy at the lower price.

A consumer located at x goes to location 0 if and only if

$$A(\mu) \geq \alpha T(x), \quad \text{where } T(x) \equiv t(x) - t(1 - x).$$

Let $\bar{\alpha}(\mu) = A(\mu)/t(1) > 0$ be the lower bound on travel costs such that all customers go to the cluster. Thus, the marginal consumer going to location 0 is located at

$$x^*(\mu, \alpha) = \begin{cases} 1 & ; \alpha \leq \bar{\alpha}(\mu) \\ T^{-1}(A(\mu)/\alpha) & ; \alpha > \bar{\alpha}(\mu). \end{cases}$$

Because $T(x)$ and $A(\mu)$ are increasing, $x^*(\mu, \alpha)$ is increasing in μ and decreasing in α . With no shoppers, $x^*(0, \alpha) = T^{-1}(A(0)/\alpha) = T^{-1}(0) = 1/2$.

We now compare the profits of the monopolist and cluster stores. The monopolist gets

$$\pi_R = (1 - x^*)R^m.$$

Because cluster stores use a mixed strategy, a cluster store's expected profit must be the same at all prices prescribed by that strategy, including the monopoly price p^m . But if a cluster

³Rosenthal (1980) and Varian (1980) derive related mixing pricing results in environments where some consumers know prices at all stores and other consumers are captive.

store charges p^m , only non-shoppers who visit that store first will buy. A fraction $x^*(\mu, \alpha)$ of all consumers arrive at co-located stores, a fraction $(1 - \mu)$ of them will be non-shoppers, and half of them will visit each store. Thus, the expected profit of a co-located store is

$$\pi_L(p^m) = x^*(\mu, \alpha) \frac{1 - \mu}{2} R^m.$$

The profit of a cluster store exceeds that of the monopolist ($\pi_L(p^m) > \pi_R$) if and only if

$$x^*(\alpha, \mu) > \frac{2}{3 - \mu} \in (2/3, 1). \quad (2)$$

The inequality looks simple because the complexities of search and profit maximization by firms are embedded in x^* , which depends on $A(\mu)$, which, in turn, depends on $F(p)$ in Lemma 1.

Proposition 1 (1) *Increases in the share of shoppers (μ) raise a cluster store's profit $\pi_L(\mu)$ when there are few shoppers ($\mu \approx 0$), but reduces its profit when there are many shoppers ($\mu \approx 1$). When demand is linear, $\pi_L(\mu)$ is a concave, single-peaked function of μ .*

(2) *For any $\mu \in (0, 1)$, there exists an $\underline{\alpha}(\mu) \in (0, \bar{\alpha}(\mu))$, such that the profit of a co-located store is strictly higher than that of the isolated store if and only if $\alpha < \underline{\alpha}(\mu)$. When demand is linear, $\underline{\alpha}(\mu)$ is a single-peaked function of μ .*

Increasing the share μ of consumers who are shoppers has two effects on cluster store profits. The direct effect is to reduce their profits by increasing their competition with each other. The indirect effect is to draw more customers away from isolated store R . In sum, more shoppers raise competition at the cluster, but reduce the external competition.

The proposition shows that the strategic effect dominates when there are few shoppers—increasing the proportion of shoppers raises co-located store profits by increasing their incentives to undercut each other, which, in turn, draws more consumers. If no one has time to comparison shop, the co-located firms monopoly price, each earning half the profit of isolated firm R . From this base, increasing the number of shoppers *always* raises profits of cluster firms precisely *because* they compete more aggressively on price. With few shoppers, distance is the key determinant of where individuals go, and the marginal consumers have similar travel costs regardless of where they shop. As a result, the measure of consumers drawn to the cluster by increased price competition is initially very elastic, even though pricing remains close to monopoly. This price competition due to more shoppers draws

enough customers to more than offset the second-order reduced profit per customer, causing the profits of co-located stores to rise. When, instead, almost all consumers are shoppers, pricing approaches marginal costs at the cluster, and their profits go to zero.

Known Types. We now analyze a more realistic scenario in which consumers know whether they have time to comparison shop *before* deciding whether to go to isolated store R or the co-located stores, and show how qualitative findings are altered. To ease analysis, we assume:

Assumption 1 *Travel costs are linear or quadratic, $t(x) = \alpha x$ or $t(x) = \alpha x^2$.*

Assumption 2 (Stahl 1989) *$p R'(p)/R^2(p)$ is decreasing.*

If the co-located stores believe that a fraction μ' of their customers are shoppers, then their pricing is given by Lemma 1 with distribution $F(p, \mu')$ and boundary $b(\mu')$, which depend on μ' , not μ ; however, as shown below, the equilibrium value of μ' is determined by μ and travel cost parameter α .

A shopper located at x goes to location 0 if and only if

$$A_s(\mu') = \int_{b(\mu')}^{p^m} [1 - (1 - F(p; \mu'))^2] D(p) dp \geq \alpha T(x) = \alpha(2x - 1), \quad (3)$$

and a non-shopper located at x goes to location 0 if and only if

$$A_n(\mu') = \int_{b(\mu')}^{p^m} F(p; \mu') D(p) dp \geq \alpha T(x). \quad (4)$$

All consumers collect the same consumer surplus from going to store R . However, shoppers gain more surplus than non-shoppers from going to location 0, i.e., $A_s(\mu') > A_n(\mu')$, as shoppers pay the lowest price. That is, $1 - (1 - F(p; \mu'))^2 > F(p; \mu')$. Lemma 2 follows directly.

Lemma 2 *There exist $s(\mu', \alpha)$ and $n(\mu', \alpha) \geq \frac{1}{2}$ such that a shopper located at x goes to location 0 if and only if $x \leq s(\mu', \alpha)$, and a non-shopper goes if and only if $x \leq n(\mu', \alpha)$. Further, more shoppers visit the co-located stores: $s(\mu', \alpha) \geq \max\{n(\mu', \alpha), x^*(\mu, \alpha)\}$.*

Shoppers have more incentive to go to location 0 than non-shoppers and unknown types. From (3) and (4), the indifferent shopper $s(\mu', \alpha)$ and non-shopper $n(\mu', \alpha)$ are given by

$$s(\mu', \alpha) = \frac{\alpha + A_s(\mu')}{2\alpha} \quad \text{and} \quad n(\mu', \alpha) = \frac{\alpha + A_n(\mu')}{2\alpha} \quad (\text{indifference conditions}). \quad (5)$$

In equilibrium, beliefs are consistent with strategies. Thus, the fraction of consumers at location 0 who are shoppers is

$$\mu'(\mu, \alpha) = \frac{\mu}{\mu + (1 - \mu) n(\mu', \alpha)/s(\mu', \alpha)} \quad (\text{belief consistency}). \quad (6)$$

Definition 1 *An equilibrium is given by a belief–price distribution pair, $\{\mu' \equiv \mu^*(\mu, \alpha), F(p, \mu')\}$, that satisfy equations (5) and (6). At $\mu^*(\mu, \alpha)$, both $s^*(\mu, \alpha) \equiv s(\mu^*(\mu, \alpha), \alpha)$ and $n^*(\mu, \alpha) \equiv n(\mu^*(\mu, \alpha), \mu)$ are uniquely defined.*

A lower ratio of the indifferent marginal non-shopper to the indifferent marginal shopper ($n(\mu', \alpha)/s(\mu', \alpha)$) raises the ratio of shoppers to non-shoppers (μ') at the co-located stores, reducing prices—in the sense of the first order stochastic dominance. Lower prices, in turn, increase the incentives of all consumers to visit the co-located stores. Conversely, if shoppers respond more sharply than non-shoppers to reductions in prices at the co-located stores, the ratio of shoppers to non-shoppers rises at the co-located stores (n/s falls, so that μ' rises). Thus, we can think of the game as one between a representative co-located store, which chooses prices, and a representative consumer who chooses n/s . The actions of the representative co-located store and representative consumer feature strategic complements, and the question is whether those strategic complementarities can result in multiple equilibria—e.g., an equilibrium in which relatively few shoppers go to location zero, resulting in relatively high prices, and confirming the optimality of relatively few shoppers going to location zero; and an equilibrium in which the opposite holds.

We next establish that shoppers are never so much more sensitive to price changes than non-shoppers that multiple equilibria can arise.

Proposition 2 *A unique equilibrium exists. Also, $\mu^*(\mu, \alpha)$, $s^*(\mu, \alpha)$ and $n^*(\mu, \alpha)$ weakly increase in μ and decrease in α , strictly so for $s^*(\mu, \alpha) < 1$. As travel costs rise, the ratio of shoppers to non-shoppers visiting the co-located stores falls: $s^*(\mu, \alpha)/n^*(\mu, \alpha)$ decreases in α .*

The proof uses Assumption 2 to bound the relative gains from being a shopper. The gain associated with being a shopper—more choice—is greatest for low prices with the lowest probability (i.e., prices close to $b(\mu')$). This reflects that $A_s(\mu') - A_n(\mu') = \int_{b(\mu')}^{p^m} [F(p; \mu') - F(p; \mu')^2] D(p) dp$, whose argument takes the form $z - z^2$, which has a derivative $1 - 2z$ that decreases in z . Assumption 2 bounds the degree of convexity of demand, which captures the

marginal benefit associated with getting a lower price, and hence the relative sensitivity of shoppers vs. non-shoppers to the improved prices associated with an increased μ' .

The proposition continues to establish that as the population share μ of shoppers rises, so does the proportion $\mu^*(\mu, \alpha)$ of consumers at the cluster who are shoppers. In turn, prices at the cluster fall in a first order stochastic dominance sense, inducing both more shoppers and non-shoppers to visit the cluster. Conversely, increasing travel costs, α , reduces the shopping price elasticity, increasing the impact of consumer location on the choice of where to shop, reducing the share of shoppers at the cluster.

As before, the profit of a cluster store is

$$\pi_L(p^m) = n^*(\mu, \alpha) \frac{1 - \mu}{2} R^m, \quad (7)$$

while the isolated store earns

$$\pi_R = (1 - \mu s^*(\mu, \alpha) - (1 - \mu) n^*(\mu, \alpha)) R^m. \quad (8)$$

Since $s^*(\mu, \alpha)$ and $n^*(\mu, \alpha)$ increase in μ , the isolated store R 's profit falls in μ . Indeed, store R is hurt if consumers know their types before deciding where to shop: it collects the same profit per consumer but its consumer base falls, i.e., $\mu s^*(\mu, \alpha) + (1 - \mu) n^*(\mu, \alpha) > x(\mu, \alpha)$ for $\mu > 0$. This reflects that the co-located stores draw a higher mix of shoppers when consumers know their own types, leading to stochastically better prices, and hence more consumers.

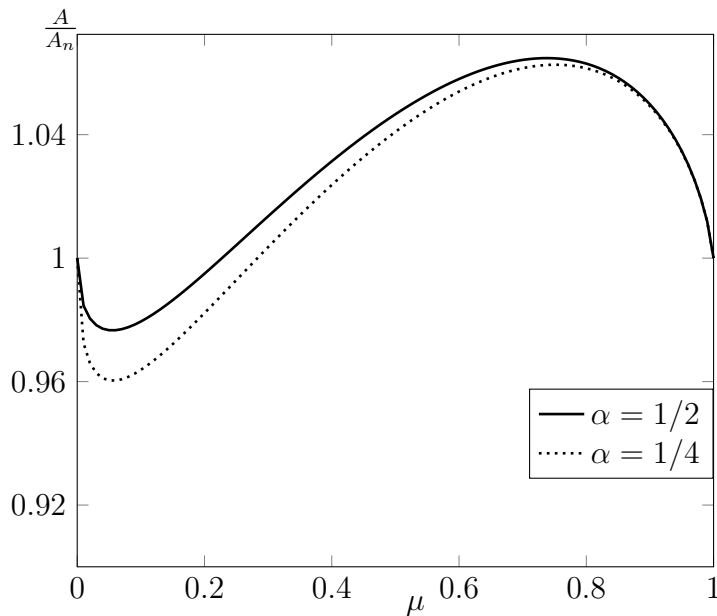
The profits of co-located stores are higher when consumers know whether they are shoppers at the outset than when they do not if and only if $n^*(\mu, \alpha) > x(\mu, \alpha)$.

Proposition 3 *A cluster store's profit (π_L) increases in μ if there are few shoppers ($\mu \approx 0$), but decreases if there are many shoppers ($\mu \approx 1$).*

If there are few shoppers ($\mu \approx 0$), then more non-shoppers than unknown types visit the cluster: $n(\mu^, \alpha) > x(\mu, \alpha)$. Conversely, if there are many shoppers ($\mu \approx 1$), then $n^*(\mu, \alpha) < x(\mu, \alpha)$.*

Figure 1 depicts $A(\mu)/A_n(\mu^*(\mu, \alpha))$ for different values of μ and α , for linear demand $D(p) = \beta - (\beta/2)p$. A single-crossing property holds: there exists a $\bar{\mu} \in (0, 1)$ such that $A(\mu) < A_n(\mu^*(\mu, \alpha))$ if and only if $\mu < \bar{\mu}$, a stronger statement than Proposition 3, which only requires, via Assumption 2, that the demand not be too convex.

Figure 1: $A(\mu)/A_n(\mu^*(\mu))$ with linear demand $D(p) = \beta - (\beta/2)p$.



For co-located stores, similar direct (increased price competition) and indirect effects (reduced external competition) exist as in our base setting via the positive effect of μ on $\mu^*(\mu, \alpha)$. Once more, a cluster store's profit first rises in μ when there are few shoppers, reflecting the high sensitivity to increased price competition of a consumer's choice of where to shop when travel distances to stores are very similar. The proposition reveals that the qualitative implications are reinforced if consumers know in advance whether they have time to comparison shop. In fact, with few shoppers, price competition is so enhanced by the higher endogenous share of shoppers at the cluster relative to μ that the expected price paid at a cluster store by a consumer who *knows* he does not have time to comparison shop falls below that paid by a consumer in the base setting who could turn out to have time to comparison shop. This means that when μ is small, even a consumer who knows he does not have time to comparison shop is willing to travel farther than a consumer in the base setting, i.e., $n^*(\mu, \alpha) > x(\mu, \alpha)$. However, with enough shoppers, this inequality is reversed—the heightened value of likely securing the lowest price more than offsets the higher expected prices that obtain when consumers do not know whether they will have time to shop.

Establishing that $n^*(\mu, \alpha) > x(\mu, \alpha)$ when μ is small is challenging—we must show that $A_n(\mu^*(\mu)) < A(\mu)$ in an open neighborhood of $\mu = 0$, even though $A_n(\mu^*(\mu)) = A(\mu)$ at

$\mu = 0$. In effect, we must sign the derivative $A'(\mu) - A'_n(\mu^*(\mu)) \frac{d\mu^*(\mu)}{d\mu}$ at $\mu = 0$. This is tricky because (1) μ^* is an equilibrium object, with the properties that $\lim_{\mu \rightarrow 0} \frac{\mu^*(\mu)}{\mu} = \lim_{\mu \rightarrow 0} \frac{d\mu^*(\mu)}{d\mu} = 1$; and (2) $\lim_{\mu \rightarrow 0} A'(\mu) = \lim_{\mu \rightarrow 0} A'_n(\mu^*(\mu)) = \infty$. Thus, to sign the derivative, one must identify the rates of convergence. A key is to show that $0 < \lim_{z \rightarrow 0} \sqrt{z} A'_n(z) < \lim_{z \rightarrow 0} \sqrt{z} A'_s(z) < \infty$.

3 Conclusion

Standard spatial theory suggests that firms selling similar products maximize profits by separating maximally. Nonetheless, in recent years, stores like Lowes and Home Depot that sell very similar products have thrived despite co-locating (at fringes), while stores in city centers that face limited local competition have had troubles. Our paper observes that when most consumers only see prices once they visit a store and some consumers have time to comparison shop then co-location commits stores to lower prices that draw more consumers.

Our central finding is that co-located firms thrive when there are some shoppers, but not too many. With few shoppers, the measure of consumers drawn to the co-located stores is very price elastic because travel costs differ only modestly for the marginal consumer. Thus, the marginal value of commitment to slightly lower prices is high. These effects are enhanced if consumers know in advance whether they will have time to comparison shop. Indeed, price competition at co-located stores may rise by enough that the expected price paid by a non-shopper falls below that paid when consumers do not know if they will have time to shop. The flip side is that with too many shoppers, price competition grows so fierce that the high numbers of customers drawn fail to offset the reduced profit per customer.

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5 Appendix

Proof of Proposition 1: We first prove two lemmas.

Lemma 3 (i) The consumer surplus gain from visiting the co-located stores rather than store R is given by equation (1); (ii) $A'(\mu) > 0$ with $\lim_{\mu \rightarrow 0} A'(\mu) = \infty$ and $\lim_{\mu \rightarrow 1} A'(\mu) = 0$; (iii) $A(\mu)$ is strictly concave when demand is linear, $D(p) = \beta_1 - \beta_2 p$.

Proof: (i) If $X_1, X_2 \sim iidF$, then $\min\{X_1, X_2\} \sim 1 - [1 - F(p)]^2$. Thus, expected consumer surplus at location 0 is

$$(1 - \mu) \int_b^\infty F(p)D(p)dp + \mu \int_b^\infty [1 - (1 - F(p))^2] D(p)dp. \quad (9)$$

Because $(1 - \mu)F + \mu [1 - (1 - F(p))^2] = F + \mu F(1 - F)$, equation (9) simplifies to

$$\int_b^{p^m} F(p)D(p)dp + \int_{p^m}^\infty D(p)dp + \mu \int_b^{p^m} F(p)(1 - F(p))D(p)dp - \alpha t(x), \quad (10)$$

exploiting the fact that $F(p) = 1$ for $p \geq p^m$. The expected consumer surplus at location 1 is

$$\int_{p^m}^\infty D(p)dp. \quad (11)$$

Subtracting (11) from (10) yields $A(\mu)$ in the text.

(ii) Differentiating $A(\mu)$ and recognizing that $F(b(\mu)) = 0$, yields:

$$\begin{aligned} \frac{dA(\mu)}{d\mu} &= \int_b^{p^m} \frac{d[F(p; \mu)(1 + \mu - \mu F(p; \mu))]}{d\mu} D(p)dp \\ &= \int_b^{p^m} \left[\frac{dF(p; \mu)}{d\mu} (1 + \mu - 2\mu F(p; \mu)) \right] D(p)dp + \int_b^{p^m} F(p; \mu)(1 - F(p; \mu)) D(p)dp. \end{aligned}$$

From Lemma 1,

$$\frac{dF(p; \mu)}{d\mu} = \frac{1}{2\mu^2} \left(\frac{R^m}{R(p)} - 1 \right) = \frac{1 - F(p; \mu)}{\mu(1 - \mu)} > 0. \quad (12)$$

Thus,

$$\begin{aligned}
\frac{dA(\mu)}{d\mu} &= \int_b^{p^m} (1 - F(p)) \left(\frac{1 + \mu - 2\mu F(p)}{\mu(1 - \mu)} + F(p) \right) D(p) dp \\
&= \int_b^{p^m} (1 - F(p)) \left(\frac{1 + \mu - 2\mu F(p) + \mu F(p) - \mu^2 F(p)}{\mu(1 - \mu)} \right) D(p) dp \\
&= \int_b^{p^m} (1 - F(p)) \left(\frac{1 + \mu - \mu(1 + \mu)F(p)}{\mu(1 - \mu)} \right) D(p) dp. \\
&= \frac{1 + \mu}{\mu(1 - \mu)} \int_{b(\mu)}^{p^m} (1 - F(p))(1 - \mu F(p)) D(p) dp > 0.
\end{aligned}$$

Next, we show that $\lim_{\mu \rightarrow 0} A'(\mu) = \infty$.

$$\lim_{\mu \rightarrow 0} \frac{dA(\mu)}{d\mu} = \lim_{\mu \rightarrow 0} \frac{1}{\mu} \int_{b(\mu)}^{p^m} (1 - F(p)) D(p) dp. \quad (13)$$

Applying L'Hôpital's rule yields

$$\begin{aligned}
\lim_{\mu \rightarrow 0} A'(\mu) &= \lim_{\mu \rightarrow 0} \frac{d}{d\mu} \int_{b(\mu)}^{p^m} (1 - F(p)) D(p) dp \\
&= \lim_{\mu \rightarrow 0} -[1 - F(b(\mu))] D[b(\mu)] b'(\mu) - \lim_{\mu \rightarrow 0} \int_{b(\mu)}^{p^m} \frac{dF(p; \mu)}{d\mu} D(p) dp \\
&= -D(p^m) \lim_{\mu \rightarrow 0} b'(\mu) - \lim_{\mu \rightarrow 0} A'(\mu) \quad (\text{using (13) and (12)}).
\end{aligned}$$

Thus,

$$\lim_{\mu \rightarrow 0} \frac{dA(\mu)}{d\mu} = -\frac{D(p^m)}{2} \lim_{\mu \rightarrow 0} b'(\mu).$$

From Lemma 1,

$$\lim_{\mu \rightarrow 0} b'(\mu) = \lim_{\mu \rightarrow 0} \frac{-\frac{2R^m}{(1+\mu)^2}}{R'(b(\mu))} = \lim_{\mu \rightarrow 0} \frac{-2R^m}{R'(b(\mu))} = -\infty, \quad (14)$$

because p^m is an interior maximum, and hence as $p \rightarrow p^m$ from the left, $R'(p)$ approaches 0 from above. Combining these results yields

$$\lim_{\mu \rightarrow 0} A'(\mu) = \infty.$$

A similar application of L'Hôpital's rule yields

$$\lim_{\mu \rightarrow 1} A'(\mu) = 0.$$

(iii) We take the second derivative of $A(\mu)$, given linear demand $D(p) = \beta_1 - \beta_2 p$. Mathematica calculations yield

$$A(\mu) = \frac{\beta_1^2}{4\beta_2} \frac{4\mu^2 + \sqrt{2\mu(1+\mu)}(1+3\mu) - (1-\mu)^2 \tanh^{-1}\left(\sqrt{\frac{2\mu}{1+\mu}}\right)}{8\mu}.$$

Moreover,

$$\frac{dA(\mu)}{d\mu} = \left(\frac{\beta_1^2}{4\beta_2}\right)^2 \frac{1}{18\mu^2} \left\{ \sqrt{\frac{2\mu}{1+\mu}} (-2 + 3\mu + 7\mu^2) + 8\mu^2 + 2(1-\mu^2) \tanh^{-1}\left(\sqrt{\frac{2\mu}{1+\mu}}\right) \right\}.$$

Next, we show $\frac{d^2A(\mu)}{d\mu^2} < 0$.

$$18 \left(\frac{\beta_1^2}{4\beta_2}\right)^{-2} \frac{d^2A(\mu)}{d\mu^2} = \frac{\sqrt{\left(\frac{\mu}{1+\mu}\right)^3} \left\{ \sqrt{2\mu}(8+9\mu+3\mu^2) - 8\sqrt{\frac{\mu}{1+\mu}} (1+\mu)^2 \tanh^{-1}\left(\sqrt{\frac{2\mu}{1+\mu}}\right) \right\}}{2\mu^5}$$

It suffices to show that the curly bracket is negative. To show this, we observe that a Taylor expansion together with strict concavity of $\tanh^{-1}(\mu)$ for $\mu > 0$ imply that

$$\tanh^{-1}(\mu) > \mu + \frac{\mu^3}{3}, \text{ for } \mu > 0.$$

Substituting $\mu + \frac{\mu^3}{3}$ for $\tanh^{-1}(\mu)$ yields

$$\begin{aligned} & \sqrt{2\mu}(8+9\mu+3\mu^2) - 8\sqrt{\frac{\mu}{1+\mu}} (1+\mu)^2 \tanh^{-1}\left(\sqrt{\frac{2\mu}{1+\mu}}\right) \\ & < \sqrt{2\mu}(8+9\mu+3\mu^2) - 8\sqrt{\frac{\mu}{1+\mu}} (1+\mu)^2 \left(\sqrt{\frac{2\mu}{1+\mu}} + \frac{1}{3} \left(\sqrt{\frac{2\mu}{1+\mu}}\right)^3 \right) \\ & = \frac{\sqrt{2}}{3} \mu^2(9\mu - 13) < 0. \end{aligned}$$

Thus, $\frac{d^2A(\mu)}{d\mu^2} < 0$ for $\mu \in (0, 1)$. □

Lemma 4 For $x^* < 1$, $x^*(\mu)$ is a strictly increasing function of μ , with $\lim_{\mu \rightarrow 0} \frac{dx^*(\mu)}{d\mu} = \infty$ and $\lim_{\mu \rightarrow 1} \frac{dx^*(\mu)}{d\mu} = 0$. If demand is linear, then $x^*(\mu)$ is strictly concave.

Proof: Recall that $\alpha T(x^*(\mu)) = A(\mu)$ when $x^* < 1$. Thus,

$$\frac{dx^*(\mu)}{d\mu} = \frac{1}{\alpha T'(x^*(\mu))} \frac{dA(\mu)}{d\mu} > 0, \tag{15}$$

and

$$\frac{d^2 x^*(\mu)}{d\mu^2} = \frac{1}{T'(x^*(\mu))} \left(\frac{1}{\alpha} \frac{d^2 A(\mu)}{d\mu^2} - T''(x^*(\mu)) \left(\frac{dx^*(\mu)}{d\mu} \right)^2 \right), \quad (16)$$

Moreover, $T''(x) = t''(x) - t''(1-x)$. Because $t'''(x) \geq 0$ and $x^* \geq 1/2$, we have $T''(x^*) \geq 0$. The results then follow from Lemma 3. \square

We now use these lemmas to prove Proposition 1.

Part 1 of Proposition 1. Observe that

$$\frac{d\pi_L(\mu)}{d\mu} = \frac{R^m}{2} \left(\frac{dx^*(\mu)}{d\mu} (1-\mu) - x^*(\mu) \right). \quad (17)$$

Thus, using Lemma 4,

$$\begin{aligned} \lim_{\mu \rightarrow 0} \frac{d\pi_L(\mu)}{d\mu} &= \frac{R^m}{2} \left(\lim_{\mu \rightarrow 0} \frac{dx^*(\mu)}{d\mu} - x^*(0) \right) = \infty \\ \lim_{\mu \rightarrow 1} \frac{d\pi_L(\mu)}{d\mu} &= \frac{R^m}{2} (0 - x^*(1)) < 0. \end{aligned}$$

Moreover, differentiating (17),

$$\frac{d^2 \pi_L(\mu)}{d\mu^2} = \frac{R^m}{2} \left(\frac{d^2 x^*(\mu)}{d\mu^2} (1-\mu) - 2 \frac{dx^*(\mu)}{d\mu} \right).$$

From Lemma 4, $x^*(\mu)$ increases in μ . Thus, for μ sufficiently small that $x^*(\mu) < 1$, from Lemma 4, with linear demand, $\frac{d^2 x^*(\mu)}{d\mu^2} < 0$, and hence $\frac{d^2 \pi_L(\mu)}{d\mu^2} < 0$. Otherwise, $x^*(\mu) = 1$, and thus $\frac{d^2 \pi_L(\mu)}{d\mu^2} = 0$.

Part 2 of Proposition 1. At $\underline{\alpha}(\mu)$, $x(\mu, \alpha(\mu)) = 2/(3-\mu)$. By the implicit function theorem (evaluated at $x^*(\mu, \alpha)$)

$$\alpha'(\mu) = \frac{1}{\partial x^*/\partial \alpha} \left(\frac{2}{(3-\mu)^2} - \frac{\partial x^*}{\partial \mu} \right). \quad (18)$$

We claim that $\alpha'(\mu)$ satisfies a single crossing property: $\alpha'(\mu) > 0$ for $\mu < \tilde{\mu}$, and $\alpha'(\mu) < 0$ for $\mu > \tilde{\mu}$.

To see this, first notice that in equilibrium $x^*(\mu, \alpha) = T^{-1}(A(\mu)/\alpha)$, and thus

$$\frac{\partial x^*(\mu, \alpha)}{\partial \alpha} = -\frac{A(\mu)}{\alpha^2 T'(A(\mu)/\alpha)} < 0,$$

since $T'(x) > 0$. Moreover, $dx^*(x, \alpha)/d\mu > 0$ (from (15)), and when $A(\mu)$ is concave and $T(x)$ is convex, we have $d^2 x^*(x, \alpha)/d\mu^2 < 0$ (from (16)).

Thus, the denominator in (18) is negative such that $\alpha'(\mu) > 0$ if $2/(3-\mu)^2 - \partial x^*/\partial\mu < 0$; and $\alpha'(\mu) < 0$ otherwise. The first term $2/(3-\mu)^2$ increases in μ , while $\partial x^*/\partial\mu$ decreases in μ (since $d^2x^*/d\mu^2 < 0$). Thus, $2/(3-\mu)^2 - \partial x^*/\partial\mu$ increases in μ : if $2/(3-\tilde{\mu})^2 - \partial x^*/\partial\mu|_{\mu=\tilde{\mu}} = 0$ at $\tilde{\mu}$, then $2/(3-\mu)^2 - \partial x^*/\partial\mu < 0$ for all $\mu < \tilde{\mu}$, and $2/(3-\mu)^2 - \partial x^*/\partial\mu > 0$ for all $\mu > \tilde{\mu}$.

For $\mu \approx 0$, we have $\alpha'(\mu) > 0$, since $\partial x^*/\partial\mu \rightarrow \infty$ (from (15)) and $2/(3-\mu)^2 \rightarrow 2/9$ as $\mu \rightarrow 0$. If $\tilde{\mu} \in (0, 1)$, then $\alpha(\mu)$ increases for $\mu < \tilde{\mu}$, peaks at $\tilde{\mu}$, and decreases for $\mu > \tilde{\mu}$. If $\tilde{\mu} \notin (0, 1)$, then $\alpha'(\mu) > 0$ for all μ and the peak is reached at $\mu = 1$. \square

Proof of Proposition 2: From equation (5),

$$\frac{s(\mu', \alpha)}{n(\mu', \alpha)} = \frac{\alpha + A_s(\mu')}{\alpha + A_n(\mu')}. \quad (19)$$

From equation (6),

$$\frac{s(\mu', \alpha)}{n(\mu', \alpha)} = \frac{\mu'(1-\mu)}{(1-\mu')\mu}. \quad (20)$$

Combining equations (19) and (20) yields that μ' is consistent with equilibrium if and only if

$$g(\mu'; \alpha, \mu) = \frac{\mu'}{1-\mu'} \frac{1-\mu}{\mu} - \frac{\alpha + A_s(\mu')}{\alpha + A_n(\mu')} = 0.$$

Observe that $g(0; \alpha, \mu) < 0 < g(1; \alpha, \mu)$. Thus, at least one solution exists. To show that the solution is unique, we prove that $\partial g(\mu'; \alpha, \mu) / \partial \mu' > 0$.

Lemma 5 $\partial g(\mu'; \alpha, \mu) / \partial \mu' > 0$.

Proof: Since $s(\mu', \alpha) \geq n(\mu', \alpha)$, Bayes rules (6) implies $\mu' \geq \mu$. Moreover, from Lemma 1

$$\frac{dF(p; \mu')}{d\mu'} = \frac{1 - F(\mu')}{\mu'(1 - \mu')}. \quad (21)$$

Next, recall that from equations (3) and (4),

$$A_s(\mu') = \int_{b(\mu')}^{p^m} [F(p; \mu') + F(p; \mu')(1 - F(p; \mu'))] D(p) dp \quad (22)$$

$$> A_n(\mu') = \int_{b(\mu')}^{p^m} F(p; \mu') D(p) dp > 0. \quad (23)$$

Differentiating (23) with respect to μ' , and using (21), yields:

$$A'_n(\mu') = \int_{b(\mu')}^{p^m} \frac{1 - F(p; \mu')}{\mu'(1 - \mu')} D(p) dp, \quad (24)$$

using $F(b(\mu'); \mu') = 0$. Similarly from (21) and (22),

$$A'_s(\mu') = \int_{b(\mu')}^{p^m} 2 \frac{dF(p; \mu')}{d\mu'} (1 - F(\mu')) D(p) dp = 2 \int_{b(\mu')}^{p^m} \frac{(1 - F(p; \mu'))^2}{\mu'(1 - \mu')} D(p) dp. \quad (25)$$

Moreover, from (24) and (25)

$$2A'_n(\mu') - A'_s(\mu') = 2 \int_{b(\mu')}^{p^m} \frac{dF(p; \mu')}{d\mu'} F(\mu') D(p) dp > 0. \quad (26)$$

Combining these results, yields

$$\begin{aligned} \frac{\partial g(\mu'; \alpha, \mu)}{\partial \mu'} &= \frac{1 - \mu}{(1 - \mu')^2 \mu} + \frac{(\alpha + A_s(\mu')) A'_n(\mu') - (\alpha + A_n(\mu')) A'_s(\mu')}{(\alpha + A_n(\mu'))^2} \quad (27) \\ &> \frac{1 - \mu}{(1 - \mu')^2 \mu} + \frac{(\alpha + A_n(\mu')) A'_n(\mu') - (\alpha + A_n(\mu')) A'_s(\mu')}{(\alpha + A_n(\mu'))^2} \quad (\text{from (23)}) \\ &= \frac{1 - \mu}{(1 - \mu')^2 \mu} + \frac{A'_n(\mu') - A'_s(\mu')}{\alpha + A_n(\mu')} \\ &> \frac{1 - \mu}{(1 - \mu')^2 \mu} - \frac{A'_n(\mu')}{\alpha + A_n(\mu')} \quad (\text{from (26)}) \\ &> \frac{1 - \mu}{(1 - \mu')^2 \mu} - \frac{A'_n(\mu')}{A_n(\mu')} \quad (\text{because } \alpha > 0) \\ &\geq \frac{1 - \mu'}{(1 - \mu')^2 \mu'} - \frac{A'_n(\mu')}{A_n(\mu')} \quad (\text{because } \mu' \geq \mu) \\ &= \frac{1}{(1 - \mu') \mu'} - \frac{A'_n(\mu')}{A_n(\mu')}. \quad (28) \end{aligned}$$

Now, combining (23) and (24) yields:

$$\frac{A'_n(\mu')}{A_n(\mu')} = \frac{1}{\mu'(1 - \mu')} \frac{\int_{b(\mu')}^{p^m} (1 - F(p; \mu')) D(p) dp}{\int_{b(\mu')}^{p^m} F(p; \mu') D(p) dp}. \quad (29)$$

Comparing (29) and (28) reveals that $\partial g(\mu'; \alpha, \mu) / \partial \mu' > 0$ if

$$\int_{b(\mu')}^{p^m} F(p; \mu') D(p) dp \geq \int_{b(\mu')}^{p^m} (1 - F(p; \mu')) D(p) dp. \quad (30)$$

To see that (30) holds for $\mu' > 0$, substitute $F(p; \mu') = 1 - \frac{(1 - \mu')}{2\mu'} (R^m / R(p) - 1)$ and re-arrange it as

$$\int_{b(\mu')}^{p^m} D(p) dp - \left(\frac{1 - \mu'}{\mu'} \right) \int_{b(\mu')}^{p^m} \left(\frac{R^m}{R(p)} - 1 \right) D(p) dp \geq 0.$$

Substitute $R(p) = pD(p)$, re-arrange and multiply through by $\mu' > 0$ to obtain

$$\int_{b(\mu')}^{p^m} D(p)dp - R^m (1 - \mu') \int_{b(\mu')}^{p^m} \frac{1}{p} dp \geq 0.$$

Integrating the last term yields

$$h(\mu') = \int_{b(\mu')}^{p^m} D(p)dp - R^m (1 - \mu') (\ln(p^m) - \ln(b(\mu'))) \geq 0. \quad (31)$$

Lemma 6 For $z > 0$ define $h(z) = \int_{b(z)}^{p^m} D(p)dp - R^m (1 - z) (\ln(p^m) - \ln(b(z)))$, with $R(b(z)) = \frac{1-z}{1+z} R^m$. Then,

1. $\lim_{z \rightarrow 0} h(z) = \lim_{z \rightarrow 0} h'(z) = 0$.
2. $h''(z) > 0$ for $z > 0$.

Proof: From the definition of $b(z)$, $\lim_{z \rightarrow 0} b(z) = p^m$. Thus,

$$\lim_{z \rightarrow 0} h(z) = \int_{p^m}^{p^m} D(p)dp - R^m (\ln(p^m) - \ln(p^m)) = 0.$$

Differentiating $h(z)$ with respect to z ,

$$h'(z) = -b'(z)D(b(z)) + R^m (\ln(p^m) - \ln(b(z))) + R^m(1 - z) \frac{b'(z)}{b(z)}.$$

Recall that $R(p) = pD(p)$. Thus, $D(b(z)) = \frac{1}{b(z)} \frac{1-z}{1+z} R^m$. Substituting for $D(b(z))$ yields

$$h'(z) = R^m \left(\ln(p^m) - \ln(b(z)) + \left(\frac{1-z}{1+z} \right) \frac{z b'(z)}{b(z)} \right). \quad (32)$$

As $z \rightarrow 0$, we have

$$\begin{aligned} \lim_{z \rightarrow 0} h'(z) &= \frac{R^m}{p^m} \lim_{z \rightarrow 0} z b'(z) \\ &= \frac{R^m}{p^m} \lim_{z \rightarrow 0} z \frac{1}{R'(b(z))} \frac{-2R^m}{(1+z)^2} \quad (\text{from definition of } b'(z) \text{ in (14)}) \\ &= \frac{-2(R^m)^2}{p^m} \lim_{z \rightarrow 0} \frac{1}{R''(b(z)) b'(z)} \quad (\text{by L'Hôpital's Rule}) \\ &= \frac{-2(R^m)^2}{p^m} \lim_{z \rightarrow 0} \frac{R'(b(z))}{R''(b(z))} \frac{(1+z)^2}{-2R^m} \quad (\text{from definition of } b'(z) \text{ in (14)}) \\ &= D(p^m) \frac{R'(p^m)}{R''(p^m)} = 0. \end{aligned}$$

This concludes part 1.

For part 2, differentiating (32) yields

$$\begin{aligned} h''(z)/R^m &= -\frac{b'(z)}{b(z)} + \frac{1-2z-z^2}{(1+z)^2} \frac{b'(z)}{b(z)} + \frac{z(1-z)}{1+z} \left(\frac{b''(z)}{b(z)} - \frac{b'(z)^2}{b(z)^2} \right) \\ &= \frac{-2z(2+z)b'(z)b(z) - z(1-z^2)b'(z)^2 + z(1-z^2)b''(z)b(z)}{(1+z)^2b(z)^2}. \end{aligned}$$

Let $N(z)$ be the numerator. It suffices to show that $N(z) > 0$. Observe that

$$b'(z) = -\frac{2R^m}{(1+z)^2 R'(b(z))} < 0 < b''(z) = 2R^m \left(\frac{2R'(b(z)) + (1+z)b'(z)R''(b(z))}{(1+z)^3 R'(b(z))^2} \right). \quad (33)$$

Substitute for $b'(z)$ and $b''(z)$ in $N(z)$ and simplifying yields

$$N(z) \propto \frac{bR'(b)}{R(b)} - \frac{1}{3} - \frac{1}{3} \frac{bR''(b)}{R'(b)}.$$

Thus, if $bR'(b)/R(b) > 1/3$, then $N(z) > 0$. Now, recall that $R'(p) = D(p)(1+\epsilon)$, where $\epsilon = pD'(p)/D(p)$ is the price elasticity of demand. Thus, $bR'(b)/R(b) > 1/3$ is equivalent to $\epsilon = bD'(b)/D(b) > -2/3$. By our assumptions that $R(p)$ is concave with a maximum at p^m , we already have $\epsilon > -1$. The weaker necessary and sufficient condition for $N(z) > 0$ is

$$3bR'(b)^2 - R(b)R'(b) - bR(b)R''(b) > 0, \quad (34)$$

which is less strict than requiring $pR'(p)/R(p)^2$ to be decreasing as in Stahl (1989). Stahl's requirement (our Assumption 2) that $pR'(p)/R(p)^2$ be decreasing in p for $p < p^m$ implies that

$$2pR'(p)^2 - R(p)R'(p) - pR(p)R''(p) > 0. \quad (35)$$

Since $b < p^m$, (34) is implied by (35). \square

Lemma 6 implies that $h(\mu') > 0$ for all $\mu' > 0$ and $h(\mu') = 0$ for $\mu' = 0$, which, in turn, implies $\partial g(\mu; \alpha, \mu)/\partial \mu' > 0$ for all $\mu \geq 0$. Thus, a unique equilibrium cutoff $\mu^*(\mu, \alpha)$ exists.

The comparative statics with respect to μ are:

$$\frac{\partial \mu^*(\mu, \alpha)}{\partial \mu} = -\frac{\partial g/\partial \mu}{\partial g/\partial \mu'} \Big|_{\mu'=\mu^*(\mu, \alpha)} = \frac{\mu^*/((1-\mu^*)\mu^2)}{\partial g/\partial \mu' |_{\mu'=\mu^*(\mu, \alpha)}} > 0 \quad (\text{since } \partial g/\partial \mu' > 0). \quad (36)$$

Since $A_s(\mu')$ and $A_n(\mu')$ strictly increase in μ' , $s(\mu^*(\mu), \alpha)$ and $n(\mu^*(\mu), \alpha)$ increase in μ .

The comparative statics with respect to α are:

$$\frac{\partial \mu^*(\mu, \alpha)}{\partial \alpha} = - \frac{\partial g / \partial \alpha}{\partial g / \partial \mu'} \Big|_{\mu' = \mu^*(\mu, \alpha)} = - \frac{(A_s(\mu^*) - A_n(\mu^*)) / (\alpha + A_n(\mu^*))^2}{\partial g / \partial \mu' |_{\mu' = \mu^*(\mu, \alpha)}} < 0,$$

because we have established that $\partial g / \partial \mu' > 0$ and $A_s(\mu) - A_n(\mu) > 0$. From (5), $s(\mu^*(\alpha), \alpha) = 1/2 + A_s(\mu^*(\alpha)) / 2\alpha$ and $n(\mu^*(\alpha), \alpha) = 1/2 + A_n(\mu^*(\alpha)) / 2\alpha$. Because $\mu^*(\alpha)$ decreases in α and $A_s(\mu)$ and $A_n(\mu)$ both increase in μ ,

$$\frac{\partial s(\mu^*(\alpha), \alpha)}{\partial \alpha}, \frac{\partial n(\mu^*(\alpha), \alpha)}{\partial \alpha} < 0.$$

Finally, in equilibrium $\frac{s(\mu^*(\alpha), \alpha)}{n(\mu^*(\alpha), \alpha)} = \frac{\mu^*(\alpha)}{1 - \mu^*(\alpha)} \frac{1 - \mu}{\mu}$ from Bayes rule (6). Thus,

$$\frac{\partial s(\mu^*(\alpha), \alpha) / n(\mu^*(\alpha), \alpha)}{\partial \alpha} = \frac{(1 - \mu)}{\mu(1 - \mu^*)^2} \frac{\partial \mu^*(\alpha)}{\partial \alpha} < 0. \quad \square$$

Proof of Proposition 3: We first prove a lemma.

Lemma 7 (i) $\lim_{\mu \rightarrow 0} \frac{\mu^*(\mu)}{\mu} = 1$ and $\lim_{\mu \rightarrow 1} \frac{\mu^*}{\mu} = 1$. (ii) $\lim_{\mu \rightarrow 0} \frac{d\mu^*(\mu)}{d\mu} = \lim_{\mu \rightarrow 1} \frac{d\mu^*(\mu)}{d\mu} = 1$.

Proof: (i) Because $n, s \in [1/2, 1]$, from (6), $\lim_{\mu \rightarrow 0} \mu^*(\mu) = 0$ and $\lim_{\mu \rightarrow 1} \mu^*(\mu) = 1$. Moreover, $\lim_{\mu \rightarrow 0} n(\mu^*(\mu)) / s(\mu^*(\mu)) = n(0) / s(0) = \frac{1/2}{1/2} = 1$. Thus,

$$\lim_{\mu \rightarrow 0} \frac{\mu^*(\mu)}{\mu} = \lim_{\mu \rightarrow 0} \frac{1}{\mu + (1 - \mu)(n(\mu^*(\mu)) / s(\mu^*(\mu)))} = \frac{1}{n(0) / s(0)} = 1.$$

(ii) Because $\lim_{z \rightarrow 0} b(z) = p^m$, from (24) and (25),

$$\lim_{z \rightarrow 0} z A'_n(z) = \lim_{z \rightarrow 0} \int_{b(z)}^{p^m} \frac{(1 - F(p; z))}{1 - z} D(p) dp = 0 \quad (37)$$

$$\lim_{z \rightarrow 0} z A'_s(z) = 2 \lim_{z \rightarrow 0} \int_{b(z)}^{p^m} \frac{(1 - F(p; z))^2}{1 - z} D(p) dp = 0.$$

Moreover, substituting from (27) into (36) yields

$$\begin{aligned} \lim_{\mu \rightarrow 0} \frac{\partial \mu^*(\mu, \alpha)}{\partial \mu} &= \lim_{\mu \rightarrow 0} \frac{\frac{\mu^*}{1 - \mu^*} \frac{1}{\mu^2}}{\frac{1 - \mu}{\mu(1 - \mu^*)^2} - \frac{(\alpha + A_n(\mu^*)) A'_s(\mu^*) - (\alpha + A_s(\mu^*)) A'_n(\mu^*)}{(\alpha + A_n(\mu^*))^2}} \\ &= \lim_{\mu \rightarrow 0} \frac{(\mu^* / \mu)^2}{\mu^* / \mu - \mu^* (A'_s(\mu^*) - A'_n(\mu^*)) / \alpha} \quad (\text{since } A_n(0) = A_s(0) = 0) \\ &= \lim_{\mu \rightarrow 0} \frac{1}{1 - \mu^* (A'_s(\mu^*) - A'_n(\mu^*)) / \alpha} \quad (\text{since } \lim_{\mu \rightarrow 0} \mu^* / \mu = 1 \text{ from part (i)}) \\ &= 1 \quad (\text{from (37)}). \end{aligned} \quad (38)$$

The result for $\mu \rightarrow 1$ is analogous, and uses $\lim_{z \rightarrow 1} (1-z)A'_s(z) = \lim_{z \rightarrow 1} (1-z)A'_n(z) = 0$. \square

Part 1. From (7),

$$\frac{\partial \pi_L}{\partial \mu} = \frac{R^m}{2} \left((1-\mu) \frac{\partial n(\mu^*(\mu), \alpha)}{\partial \mu^*} \frac{d\mu^*(\mu)}{d\mu} - n(\mu^*, \alpha) \right). \quad (39)$$

We first show that $\lim_{\mu \rightarrow 0} \frac{\partial \pi_L}{\partial \mu} > 0$. From (5), $\partial n(\mu^*(\mu), \alpha) / \partial \mu^* = A'_n(\mu^*) / 2\alpha$. From part (i) of Lemma 7, $\lim_{\mu \rightarrow 0} n(\mu^*(\mu), \alpha) = n(0, \alpha) = 1/2$. From part (ii) of Lemma 7, $\lim_{\mu \rightarrow 0} \frac{d\mu^*(\mu)}{d\mu} = 1$. Substituting these into equation (39) yields

$$\lim_{\mu \rightarrow 0} \frac{\partial \pi_L}{\partial \mu} = \frac{R^m}{2} \left(\lim_{\mu \rightarrow 0} \frac{A'_n(\mu^*)}{2\alpha} - \frac{1}{2} \right).$$

Thus, it suffices to show that $\lim_{\mu \rightarrow 0} \frac{A'_n(\mu^*)}{2\alpha} = \infty$. From part (i) of Lemma 7,

$$\begin{aligned} \lim_{\mu \rightarrow 0} A'_n(\mu^*(\mu)) &= \lim_{\mu \rightarrow 0} A'_n(\mu) = \lim_{\mu \rightarrow 0} \int_{b(\mu)}^{p^m} \frac{(1-F(p; \mu))}{\mu(1-\mu)} D(p) dp \quad (\text{by (24)}) \\ &= \lim_{\mu \rightarrow 0} \int_{b(\mu)}^{p^m} \frac{(1-F(p; \mu))}{\mu} D(p) dp \\ &= \infty \quad (\text{by (13)}). \end{aligned}$$

Next, consider $\mu \rightarrow 1$. From part (i) of Lemma 7, $\lim_{\mu \rightarrow 1} \mu^* = 1$. Thus, $\lim_{\mu \rightarrow 1} n(\mu^*, \alpha) = n(1, \alpha) \geq 1/2$. Moreover, from (5), $\partial n(\mu^*, \alpha) / \partial \mu^* = A'_n(\mu^*) / 2\alpha$. Thus,

$$\lim_{\mu \rightarrow 1} \frac{\partial \pi_L}{\partial \mu} = \frac{R^m}{2} \left(\lim_{\mu \rightarrow 1} (1-\mu) \frac{A'_n(\mu)}{2\alpha} - n(1, \alpha) \right).$$

Now, from (24),

$$\lim_{\mu \rightarrow 1} (1-\mu) A'_n(\mu) = \lim_{\mu \rightarrow 1} \int_{b(\mu)}^{p^m} \frac{(1-F(p; \mu)) D(p) dp}{\mu} = 0 \quad (\text{by (24) and Lemma 1}).$$

Thus,

$$\lim_{\mu \rightarrow 1} \frac{\partial \pi_L}{\partial \mu} = -\frac{R^m}{2} n(1, \alpha) < 0.$$

Part 2. First, we prove the case of $\mu \approx 0$. Then, we prove the case of $\mu \approx 1$.

Case of $\mu \approx 0$. We show that there exists $\epsilon > 0$ such that if $\mu \in (0, \epsilon)$, then $A(\mu) - A_n(\mu^*(\mu)) < 0$. Observe that $A(0) - A_n(\mu^*(0)) = 0$. Thus, it suffices to show that

$$\lim_{\mu \rightarrow 0} \left\{ \Delta(\mu) \equiv A'(\mu) - A'_n(\mu^*) \frac{d\mu^*(\mu)}{d\mu} \right\} < 0. \quad (40)$$

From (1), (3), and (4), we have:

$$A(\mu) = \mu A_s(\mu) + (1 - \mu)A_n(\mu), \quad (41)$$

which implies

$$A'(\mu) = A_s(\mu) - A_n(\mu) + \mu A'_s(\mu) + (1 - \mu)A'_n(\mu). \quad (42)$$

Substituting from (42) into (40), we have:

$$\Delta(\mu) = A_s(\mu) - A_n(\mu) + \mu(A'_s(\mu) - A'_n(\mu)) + A'_n(\mu) - A'_n(\mu^*) \frac{d\mu^*(\mu)}{d\mu}. \quad (43)$$

Next, we substitute for $\frac{d\mu^*(\mu)}{d\mu}$ into (43) and simplify. From equations (36) and (38),

$$\frac{d\mu^*(\mu)}{d\mu} = - \left. \frac{\partial g / \partial \mu}{\partial g / \partial \mu'} \right|_{\mu' = \mu^*(\mu, \alpha)} = \frac{\frac{\mu^*}{1 - \mu^*} \frac{1}{\mu^2}}{\frac{1 - \mu}{\mu(1 - \mu^*)^2} - \frac{(\alpha + A_n(\mu^*))A'_s(\mu^*) - (\alpha + A_s(\mu^*))A'_n(\mu^*)}{(\alpha + A_n(\mu^*))^2}}. \quad (44)$$

To ease exposition, we define $\delta(\mu)$ to be the denominator of the above expression:

$$\delta(\mu) \equiv \left. \frac{\partial g}{\partial \mu'} \right|_{\mu' = \mu^*(\mu, \alpha)} = \frac{(1 - \mu)}{\mu(1 - \mu^*)^2} + \frac{(\alpha + A_s(\mu^*))A'_n(\mu^*) - (\alpha + A_n(\mu^*))A'_s(\mu^*)}{(\alpha + A_n(\mu^*))^2}.$$

Using (37), and $\lim_{\mu \rightarrow 0} \frac{d\mu^*(\mu)}{d\mu} = 1$ and $\lim_{\mu \rightarrow 0} \frac{\mu^*(\mu)}{\mu} = 1$ from Lemma 7, we have:

$$\lim_{\mu \rightarrow 0} \mu^* \delta(\mu) = 1. \quad (45)$$

Next, substitute equation (44) into (43), and use $\delta(\mu)$ to obtain:

$$\begin{aligned} \delta(\mu) \Delta(\mu) &= \delta(\mu) \left\{ A_s(\mu) - A_n(\mu) + \mu(A'_s(\mu) - A'_n(\mu)) \right\} \\ &+ \frac{1 - \mu}{\mu(1 - \mu^*)^2} A'_n(\mu) - \frac{\mu^*}{1 - \mu^*} \frac{1}{\mu^2} A'_n(\mu^*) \\ &- A'_n(\mu) \frac{(\alpha + A_n(\mu^*))A'_s(\mu^*) - (\alpha + A_s(\mu^*))A'_n(\mu^*)}{(\alpha + A_n(\mu^*))^2} \end{aligned} \quad (46)$$

Our goal is to show that $\lim_{\mu \rightarrow 0} \delta(\mu) \Delta(\mu) < 0$. From (41), $\lim_{\mu \rightarrow 0} A_n(\mu) = \lim_{\mu \rightarrow 0} A_s(\mu) = 0$. From (37), $\lim_{\mu \rightarrow 0} \mu A'_n(\mu) = \lim_{\mu \rightarrow 0} \mu A'_s(\mu) = 0$. Thus, (46) simplifies to:

$$\begin{aligned} \lim_{\mu \rightarrow 0} \mu^* \delta(\mu) \Delta(\mu) &= \lim_{\mu \rightarrow 0} \left\{ \mu^* \frac{1 - \mu}{\mu(1 - \mu^*)^2} A'_n(\mu) - \frac{\mu^{*2}}{1 - \mu^*} \frac{1}{\mu^2} A'_n(\mu^*) \right\} \\ &- \lim_{\mu \rightarrow 0} \mu^* A'_n(\mu) \frac{(\alpha + A_n(\mu^*))A'_s(\mu^*) - (\alpha + A_s(\mu^*))A'_n(\mu^*)}{(\alpha + A_n(\mu^*))^2}. \end{aligned}$$

Rearranging μ^* terms, write $\lim_{\mu \rightarrow 0} \mu^* \delta(\mu) \Delta(\mu)$ as:

$$\begin{aligned} & \lim_{\mu \rightarrow 0} \left\{ \frac{\mu^*(1-\mu)}{\mu(1-\mu^*)^2} A'_n(\mu) - \frac{1}{1-\mu^*} \left(\frac{\mu^*}{\mu} \right)^2 A'_n(\mu^*) \right\} \\ & - \lim_{\mu \rightarrow 0} \sqrt{\frac{\mu^*}{\mu}} \sqrt{\mu} A'_n(\mu) \frac{(\alpha + A_n(\mu^*))\sqrt{\mu^*} A'_s(\mu^*) - (\alpha + A_s(\mu^*))\sqrt{\mu^*} A'_n(\mu^*)}{(\alpha + A_n(\mu^*))^2}. \end{aligned} \quad (47)$$

Lemma 8 below shows that $\lim_{z \rightarrow 0} \sqrt{z} A'_n(z) = K_n$ and $\lim_{z \rightarrow 0} \sqrt{z} A'_s(z) = K_s$, with $0 < K_n < K_s$. Moreover, recall from part (i) of Lemma 7 that $\lim_{\mu \rightarrow 0} \frac{\mu^*(\mu)}{\mu} = 1$. Thus, (47) simplifies to:

$$\begin{aligned} \lim_{\mu \rightarrow 0} \mu^* \delta(\mu) \Delta(\mu) &= \lim_{\mu \rightarrow 0} \left\{ \frac{\mu^*(1-\mu)}{\mu(1-\mu^*)^2} A'_n(\mu) - \frac{1}{1-\mu^*} \left(\frac{\mu^*}{\mu} \right)^2 A'_n(\mu^*) \right\} - \frac{K_n(K_s - K_n)}{\alpha} \\ &\leq \lim_{\mu \rightarrow 0} \left\{ \frac{\mu^*}{\mu(1-\mu^*)} \left(A'_n(\mu) - \left(\frac{\mu^*}{\mu} \right) A'_n(\mu^*) \right) \right\} - \frac{K_n(K_s - K_n)}{\alpha} \\ &= \lim_{\mu \rightarrow 0} \left\{ \frac{\mu^*}{\mu^2(1-\mu^*)} \left(\mu A'_n(\mu) - \mu^* A'_n(\mu^*) \right) \right\} - \frac{K_n(K_s - K_n)}{\alpha}. \end{aligned}$$

Lemma 9 below shows that $zA'(z)$ is increasing when z is sufficiently small. This together with $\mu^* \geq \mu$ and $\lim_{\mu \rightarrow 0} \mu^*/\mu = 1$ implies that $\mu A'_n(\mu) - \mu^* A'_n(\mu^*) \leq 0$. Thus,

$$\lim_{\mu \rightarrow 0} \mu^* \delta(\mu) \Delta(\mu) < -\frac{K_n(K_s - K_n)}{\alpha} < 0 \Rightarrow \lim_{\mu \rightarrow 0} \delta(\mu) \Delta(\mu) = -\infty.$$

Lemma 8 (i) $\lim_{z \rightarrow 0} \sqrt{z} b'(z) = -2R^m K_0$, where $0 < K_0 < \infty$, (ii) $\lim_{z \rightarrow 0} \sqrt{z} A'_n(z) = K_n$ and $\lim_{z \rightarrow 0} \sqrt{z} A'_s(z) = K_s$, with $0 < K_n < K_s < \infty$ and $K_n = \frac{4}{3} R^m D(p^m) K_0$.

Proof: (i) Define $K_z \equiv \sqrt{z}/R'(b(z))$. Then,

$$\begin{aligned} \lim_{z \rightarrow 0} K_z &= \lim_{z \rightarrow 0} \frac{\sqrt{z}}{R'(b(z))} = \lim_{z \rightarrow 0} \frac{1}{2\sqrt{z}R''(b(z))b'(z)} \quad (\text{by L'Hôpital's Rule}) \\ &= \lim_{z \rightarrow 0} \frac{1}{2R''(b(z)) \left(-\frac{2\sqrt{z}R^m}{(1+z)^2 R'(b(z'))} \right)} \\ &= -\lim_{z \rightarrow 0} \frac{1}{4R^m R''(b(z)) K_z}. \end{aligned}$$

Cross-multiplying by K_z and then taking the square root to solve for K_z yields:

$$\lim_{z \rightarrow 0} K_z = \left(-\frac{1}{4R^m R''(p^m)} \right)^{1/2} = K_0 > 0,$$

since $R''(p^m) < 0$. Thus,

$$\lim_{z \rightarrow 0} \sqrt{z} b'(z) = \lim_{z \rightarrow 0} \sqrt{z} \frac{-2R^m}{(1+z)^2} \frac{1}{R'(b(z))} = -2R^m K_0.$$

(ii) We have

$$\begin{aligned} \lim_{z \rightarrow 0} \sqrt{z} A'_n(z) &= \frac{1}{\sqrt{z}} \int_{b(z)}^{p^m} (1 - F(p; z)) D(p) dp \quad (\text{by (24)}) \\ &= \lim_{z \rightarrow 0} \frac{-b'(z) (1 - F(b(z); z)) D(b(z)) + \int_{b(z)}^{p^m} \frac{\partial(1-F(p; z))}{\partial z} D(p) dp}{1/2 (z^{-1/2})} \\ &= \lim_{z \rightarrow 0} 2\sqrt{z} \left(-b'(z) D(p^m) - \int_{b(z)}^{p^m} \frac{1 - F(p; z)}{z(1-z)} D(p) dp \right) \\ &= \lim_{z \rightarrow 0} -2\sqrt{z} b'(z) D(p^m) - 2 \lim_{z \rightarrow 0} \sqrt{z} A'_n(z). \end{aligned}$$

Thus,

$$K_n \equiv \lim_{z \rightarrow 0} \sqrt{z} A'_n(z) = \frac{4}{3} R^m D(p^m) K_0 \quad (\text{by part (i)}).$$

By an analogous argument, $K_s \equiv \lim_{z \rightarrow 0} \sqrt{z} A'_s(z) = \frac{8}{5} R^m D(p^m) K_0 > K_n$. \square

Lemma 9 $\lim_{z \rightarrow 0} \frac{d(zA'_n(z))}{dz} > 0$.

Proof: From equation (37),

$$\begin{aligned} \frac{d(zA'_n(z))}{dz} &= -b'(z) \frac{D(b(z))}{1-z} + \int_{b(z)}^{p^m} \frac{1 - F(p; z)}{(1-z)^2} D(p) dp - \int_{b(z)}^{p^m} \frac{1 - F(p; z)}{z(1-z)^2} D(p) dp \\ &= \frac{1}{1-z} \left(-b'(z) D(b(z)) + \int_{b(z)}^{p^m} \frac{1 - F(p; z)}{1-z} - A'_n(z) \right), \end{aligned}$$

where we have used (21) and (24). Thus,

$$\begin{aligned} \lim_{z \rightarrow 0} \frac{d(zA'_n(z))}{dz} &= -\lim_{z \rightarrow 0} \{ b'(z) D(p^m) + A'_n(z) \} \\ &= -\lim_{z \rightarrow 0} \frac{\sqrt{z} b'(z) D(p^m) + \sqrt{z} A'_n(z)}{\sqrt{z}} \\ &= -\lim_{z \rightarrow 0} \frac{1}{\sqrt{z}} \left(-2R^m K_0 D(p^m) + \frac{4}{3} R^m D(p^m) K_0 \right) \quad (\text{by Lemma 8}) \\ &= \lim_{z \rightarrow 0} \frac{1}{\sqrt{z}} \frac{2}{3} R^m D(p^m) K_0 = \infty. \end{aligned}$$

\square

Case of $\mu \approx 1$. We first prove a lemma.

Lemma 10 $0 < \lim_{z \rightarrow 1} A'_s(z) < \lim_{z \rightarrow 1} A'_n(z) = \infty$.

Proof: From (25),

$$\begin{aligned}
\lim_{z \rightarrow 1} A'_s(z) &= \lim_{z \rightarrow 1} 2 \int_{b(z)}^{p^m} \frac{(1 - F(p; z))^2 D(p) dp}{z(1 - z)} \\
&= \lim_{z \rightarrow 1} 2 \left[\int_{b(z)}^{p^m} -2 \frac{\partial F(p; z)}{\partial z} \frac{(1 - F(p; z)) D(p) dp}{1 - 2z} - \frac{b'(z) D(b(z))}{1 - 2z} \right] \quad (\text{by L'Hôpital's Rule}) \\
&= \lim_{z \rightarrow 1} \left[\frac{-2A'_s(z) - 2b'(z) D(b(z))}{1 - 2z} \right] \quad (\text{by (22)}) \\
&= \lim_{z \rightarrow 1} [2A'_s(z) + 2b'(z) D(b(z))]
\end{aligned}$$

Thus, using b' from equation (33),

$$\lim_{z \rightarrow 1} A'_s(z) = -2 \lim_{z \rightarrow 1} b'(z) D(b(z)) = \frac{R^m D(b(1))}{R'(b(1))} < \infty.$$

Similarly, from (24)

$$\begin{aligned}
\lim_{z \rightarrow 1} A'_n(z) &= \lim_{z \rightarrow 1} \int_{b(z)}^{p^m} \frac{(1 - F(p; z)) D(p) dp}{z(1 - z)} \\
&= \lim_{z \rightarrow 1} \int_{b(z)}^{p^m} \frac{\left(1 - 1 + \frac{1-z}{2z} \left(\frac{R^m}{pD(p)} - 1\right)\right) D(p) dp}{z(1 - z)} \quad (\text{by Lemma 1}) \\
&= \lim_{z \rightarrow 1} \int_{b(z)}^{p^m} \frac{1}{2z^2} \left(\frac{R^m}{p} - D(p)\right) \\
&= \lim_{z \rightarrow 1} \left[\frac{1}{2z^2} \left(R^m \ln \left(\frac{p^m}{b(z)}\right) - \int_{b(z)}^{p^m} D(p) dp\right) \right] \\
&= \infty \quad (\text{by } \lim_{z \rightarrow 1} b(z) = 0 \text{ and } \int_0^{p^m} D(p) dp < \infty). \quad \square
\end{aligned}$$

Now, because $A(1) - A_n(\mu^*(1)) = 0$, it suffices to show that

$$\lim_{\mu \rightarrow 1} \left\{ \Delta(\mu) \equiv A'(\mu) - A'_n(\mu^*) \frac{d\mu^*(\mu)}{d\mu} \right\} < 0.$$

From (42) and $\lim_{\mu \rightarrow 1} (1 - \mu) A'_n(\mu) = 0$, we have $\lim_{\mu \rightarrow 1} A'(\mu) = \lim_{\mu \rightarrow 1} A'_s(\mu)$. From Lemma 7, $\lim_{\mu \rightarrow 1} d\mu^*/d\mu = 1$. From Lemma 10 above, $0 < \lim_{z \rightarrow 1} A'_s(z) < \lim_{z \rightarrow 1} A'_n(z) = \infty$. Combining these results, we have

$$\lim_{\mu \rightarrow 1} \Delta(\mu, \mu^*(\mu)) = \lim_{\mu \rightarrow 1} \left[A'_s(\mu) - A'_n(\mu^*) \frac{d\mu^*(\mu)}{d\mu} \right] = -\infty,$$

This concludes the proof of Proposition 3. □