Dynamic Macroeconomic Theory
Notes

David L. Kelly

Department of Economics
University of Miami
Box 248126
Coral Gables, FL 33134
dkelly@miami.edu

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I Introduction

A Basic Principles of Modern Economics and Macro

Basic principle: individuals and firms act in their own best interest. They strive to make the best decisions given constraints and respond to incentives. This may seem ridiculous or obvious. To those who are right now thinking of examples where people do not act in their own best interest, remember that only a few people on the margin need to act in their own best interest for the overall economy to perform as if everyone acts in their own best interest. Secondly, is you example of someone acting not in their interest really the case, or do you just not understand their incentives and constraints? To those who think the principle is obvious, listen to the news and see how many contradictions you find.

As far as macroeconomics, we use "micro-foundations" style of analysis. Bottom-up models start with individual agents making the best decisions given constraints as above, then aggregate across all individuals to get economy wide aggregates. Top down models, which you are probably familiar with from undergraduate classes examine aggregate markets and try to find empirical relationships and make implications about individual behavior as well as aggregate behavior ("consumption is an increasing function of income").

Mathematically, what we have is maximization of interests subject to constraints. Macro differs from micro primarily in that we maximize a sequence of decisions over time.

Macro is not "forecasting interest rates/stock prices"– something that cannot be done as we shall see. Macro is not about how the FED and Government control the economy – we will see that they do not. Instead we will ask: "why do some countries grow faster than others?" or "why do interest rates tend to rise in booms and fall in recessions?" or "why does investment vary more than output?"

B Topics and Models

1. Growth Theory (most important).
    (a) Why do countries grow and what determines the growth rate?
        - Optimal Growth (standard model).
    (b) Why do some countries not grow? What is the convergence rate of incomes (if any)?
        - Endogenous Growth.

(c) What is the relationship between the business cycle and other variables (eg. interest rates and total hours)?
   - Stochastic Growth.

2. Topics in Monetary Theory and Policy.
   (a) What is the value of money?
   (b) What is the optimal monetary policy?
   (c) What is the relationship between monetary aggregates and the business cycle?
   (d) Inflation taxation.
      - Asset pricing models (money as an asset).
      - Money-in-utility function Model.
      - Cash-in-advance Models.
      - Overlapping Generations model with money.
      - Price rigidities.

3. Fiscal Policy.
   (a) Optimal taxation.
   (b) Bonds and deficits: Ricardian Equivalence and sustainability of deficits.
   (c) Social Security.
   (d) Government Spending.
   (e) Fiscal Policy and the business cycle.

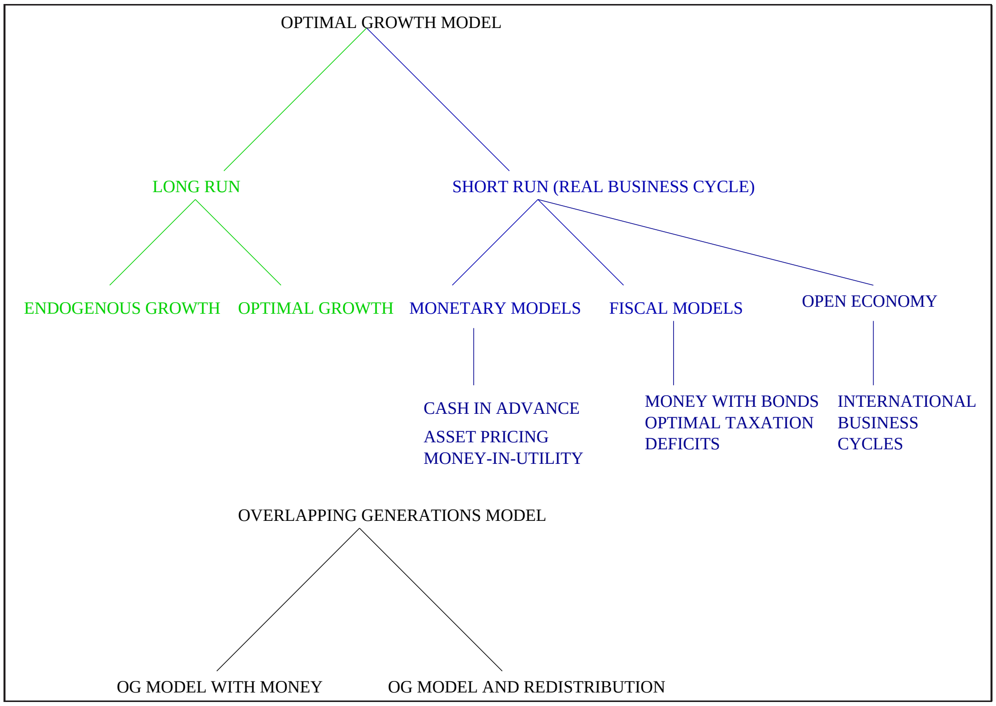


Figure 1: Macroeconomic Models

C Mathematical Tools

We are going to use exactly the same set of tools throughout the first two semesters: Discrete time dynamic programming. The main alternative is to use continuous time, which is often exactly analogous in terms of results. We will learn some continuous time in the third semester. Within discrete time, one can use dynamic programming or look at the Euler equations. Here dynamic programming is most common these days, but I will in the beginning show how the two methods relate.

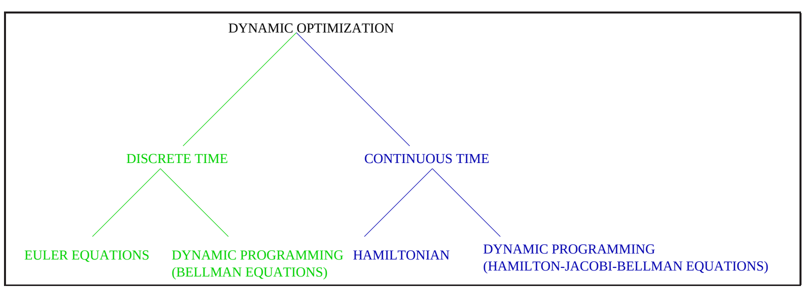


Figure 2: Mathematical Tools of Macroeconomic Analysis

MATHEMATICAL PRELIMINARIES

I Unconstrained Optimization

A Maximization

Recall individuals 'maximize,' or act in their own best interest. Graphical examples:

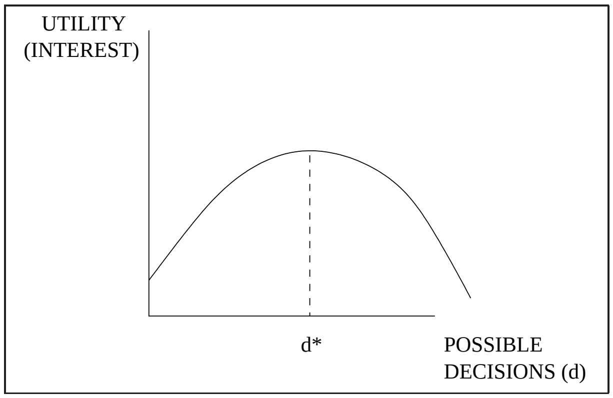


Figure 3: Utility or self interest or objective

How do we compute d^*, the optimal decision? Notice that the slope of the utility curve is zero at the maximum.

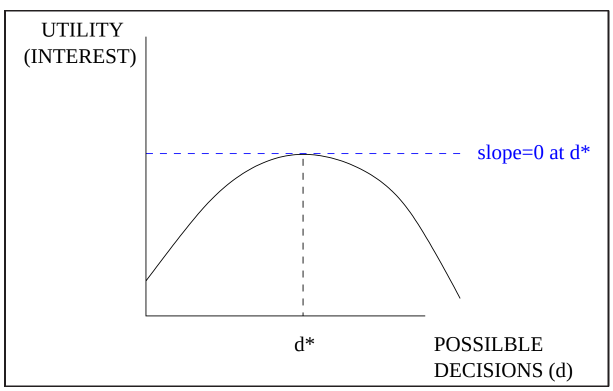


Figure 4: Decision which maximizes objective.

Hence we might suppose that for a continuously differentiable function, $U(d)$, it is "necessary" (in the sense that the maximum has this property) to find a d such that:

$$\frac{\partial U(d)}{\partial d_i} = 0 \ \forall i \tag{1}$$

However, remember that minima have the same property. Hence we need a “sufficient” condition.

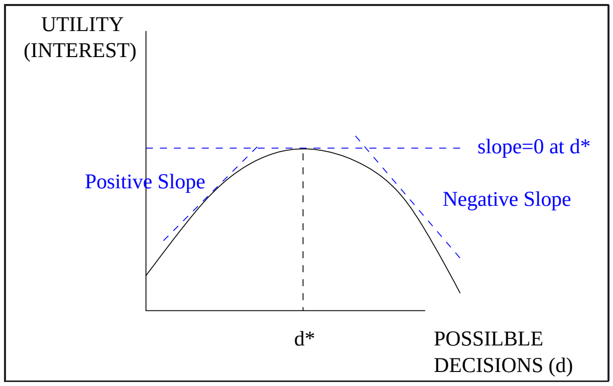


Figure 5: Slope of objective function is zero at the maximum.

Hence we need the slopes to decrease or:

$$H \equiv \frac{\partial^2 U(d^*)}{\partial d^2} = \begin{bmatrix} \frac{\partial^2 U(d^*)}{\partial d_1^2} & \cdots & \frac{\partial^2 U(d^*)}{\partial d_1 d_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 U(d^*)}{\partial d_n d_1} & \cdots & \frac{\partial^2 U(d^*)}{\partial d_n^2} \end{bmatrix} < 0 \tag{2}$$

For one variable, negativity is easy.

$$\frac{\partial^2 U(d^*)}{\partial d_1^2} < 0 \tag{3}$$

For many variables, we use the determinant test for negative definiteness.

$$(-1)^i |H_i| > 0 \qquad\qquad i = 1 \ldots n \tag{4}$$

Here H_i is a sub matrix consisting of the first i rows and columns of H.

We still may only have a relative maximum. For a global maximum, we need the second order condition to hold for all d.

$$\frac{\partial^2 U(d)}{\partial d^2} < 0 \ \forall d \tag{5}$$

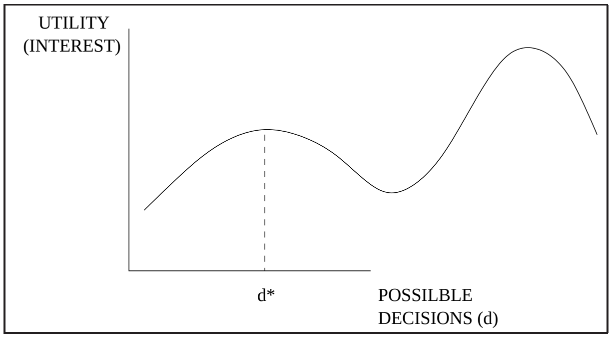


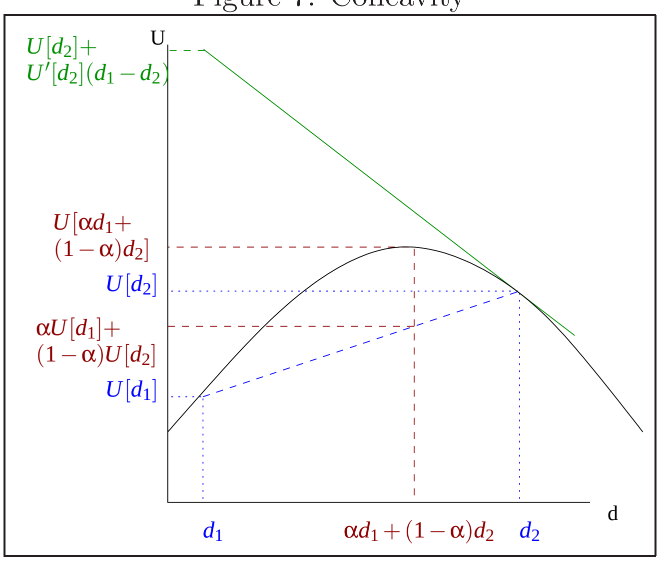
Figure 6: Slope decreases, moving from positive to negative for a maximum.

Finally, let us take a moment to understand necessary and sufficient. If A is necessary for B then B is never present without A, or $B \Rightarrow A$. Ie. any maximum satisfies the first order conditions. If A is sufficient for B, then given A we know we have B or $A \Rightarrow B$. Hence necessary and sufficient is $A \Leftrightarrow B$. In the above example, B equals "d^* is a maximum and A equals "the first order condition equals zero at d^* or the second order condition is negative at d^*.

B Concavity

Many times economic functions are concave, which automatically satisfies the second order conditions. Consider the graph below:

Figure 7: Concavity



If we can draw a line underneath the curve, we should have a maximum. Mathematically, let d_1 and d_2 be any two decisions. Then:

THEOREM 1 *The following are equivalent statements:*

1. *U is (strictly) concave.*

2. *$U''(d)\,(<) \leq 0\ \forall d$.*

3. *$\alpha U(d_1) + (1-\alpha) U(d_2)\,(<) \leq U(\alpha d_1 + (1-\alpha) d_2)\ \ \forall d_1, d_2, \alpha \in (0,1)$*

4. *$U(d_1)\,(<) \leq U(d_2) + \frac{\partial U(d_2)}{\partial d}(d_1 - d_2)\ \ \forall d_1, d_2$*

EXERCISE: CONVINCE YOURSELF OF THESE FACTS.

C Time Dependent Maximization

Now expand to the time dependent case, everything should still go through. Let:

$$\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} d_t \\ d_{t+1} \\ \vdots \\ d_{t+n} \end{bmatrix} \tag{6}$$

A problem of the form:

$$\max_{\{d_0, d_1, \ldots d_n\}} \sum_{t=0}^{n} U(d_t) \tag{7}$$

is really no different.

Necessary conditions are:

$$\frac{\partial U(d_t)}{\partial d} = 0\ t = 0 \ldots n \tag{8}$$

Sufficient conditions are:

$$\frac{\partial^2 U(d_t)}{\partial d^2} < 0\ t = 0 \ldots n \tag{9}$$

Note that because the second derivative matrix is a diagonal matrix, the negative definite test simply requires each element of the diagonal to be negative.

II Equality Constraints

Consider a more complicated problem:

$$\max_{d_t} \sum_{t=0}^{n} U\left(d_t\right) \tag{1}$$

Subject to:

$$S_{t+1} = F\left(S_t\right) - d_t \tag{2}$$

$$S_{n+1} = \bar{S} \qquad\qquad S_0 \text{ given} \tag{3}$$

Here F might be production, which uses capital, S_t, and d_t might be consumption. S_{t+1} would correspond to investment.

A The Easy Way

Recognize that there are implicitly two decisions: investment and consumption. Once we know one decision, we know the other because of the constraint. So instead, let's remove one variable using the constraint:

$$\max_{S_{t+1}} \sum_{t=0}^{n-1} U\left(F\left(S_t\right) - S_{t+1}\right) + U\left(F\left(S_n\right) - \bar{S}\right) \tag{4}$$

Hence we have n decisions which correspond to $S_1 \ldots S_n$.

The necessary conditions are then:

$$-\frac{\partial U\left(F\left(S_t\right) - S_{t+1}\right)}{\partial S} + \frac{\partial U\left(F\left(S_{t+1}\right) - S_{t+2}\right)}{\partial S}\frac{\partial F\left(S_{t+1}\right)}{\partial S_{t+1}} = 0, \quad t = 0 \ldots n-1 \tag{5}$$

$$-\frac{\partial U\left(F\left(S_{n-1}\right) - S_n\right)}{\partial d} + \frac{\partial U\left(F\left(S_n\right) - \bar{S}\right)}{\partial d}\frac{\partial F\left(S_n\right)}{\partial S} = 0 \tag{6}$$

EXERCISE: DERIVE THE SUFFICIENT CONDITIONS FOR $n = 2$.

B The hard way

Exceptions to the easy way are cases where we are especially interested in the impact of the constraint on the problem, or where the constraint cannot easily be solved to eliminate one variable. In this case, we use the Lagrange form of the problem described in section (III.B).

III Inequality Constraints

Suppose we have the same problem with an inequality constraint.

$$\max_{d_t} \sum_{t=0}^{n} U(d_t) \tag{1}$$

Subject to:

$$S_{t+1} \leq F(S_t) - d_t \tag{2}$$

$$S_{n+1} = \bar{S} \qquad S_0 \text{ given} \tag{3}$$

A Substitution

Once again, we are doing things the easy way.

1. The first option is to show the constraint always holds with equality. Why would anyone throw away income? If the constraint always holds with equality, we are back in section II.

2. The second option is to show that the constraint is irrelevant (my consumption decision is not affected by the temperature of spit in Wichita. In this case we can ignore the constraint.

B Lagrange Form

Once in a while, the constraint binds occasionally, or we cannot prove one way or the other. In this case, we must set up the problem in Lagrange form. We write the Lagrange form of the problem as:

$$\mathcal{L} = \max \sum_{t=0}^{n} u(d_t) + \lambda_t \left[F(S_t) - d_t - S_{t+1} \right] \tag{4}$$

Notice that we set up the problem so that the term multiplied by λ is always greater than or equal to zero.

In this form, $\mathcal{L}$ is an adjusted measure of utility. In essence, we will be adding zero to the problem. Either the constraint will be zero, or λ will be zero. λ is a choice variable which represents the marginal utility of relaxing the constraint. In this case, the marginal utility obtained from a little more F.

1 Conditions on constraints for a unique maximum

We will still need concavity conditions to hold here for a maximum. But there are additional conditions on the constraint as well. These are conditions on the set of choices available, that is choices which do not violate the constraints.

1. INTERIORITY. The set of feasible decisions is not empty.

2. COMPACTNESS. If x_i is a feasible decision for all i and $x_i \to \bar{x}$, then $\bar{x}$ is also feasible. Closed and bounded.

3. CONVEXITY. If d^1 and d^2 are feasible then $\alpha d^1 + (1-\alpha)\, d^2$ is also feasible for all $\alpha \in (0,1)$.

Figure 8: Compactness

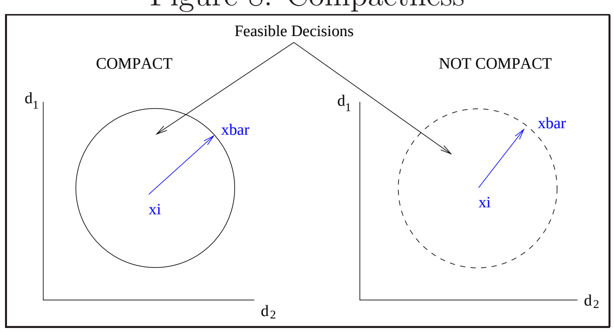
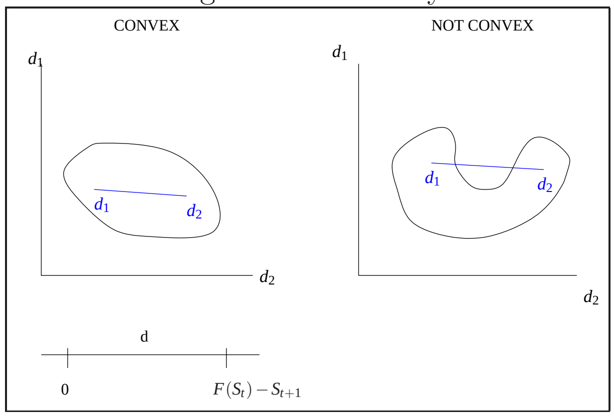


Figure 9: Convexity



Take our problem and suppose $d \geq 0$ is also a constraint. Then we have $F(S_t) - S_{t+1} \geq 0$ is compact, convex, and non-empty for $F(S_0)$ positive. It is easy to see that if interiority is not satisfied, there cannot exist a maximum. If compactness is not satisfied, a maximum also may not exist (to see this draw indifference curve). If convexity is not satisfied, multiple solutions may exist (to see this draw indifference curve).

EXERCISE: GIVE AN EXAMPLE WHICH VIOLATES (2) AND HAS NO MAXIMUM.

2 Necessary conditions for a maximum and interpretation

For the Lagrange problem, the first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial d} = \frac{\partial U(d_t)}{\partial d} - \lambda_t = 0 \qquad t = 0 \dots n \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial S_{t+1}} = -\lambda_t + \lambda_{t+1} \frac{\partial F(S_{t+1})}{\partial S} = 0 \qquad t = 0 \dots n \tag{6}$$

$$\lambda_t \frac{\partial \mathcal{L}}{\partial \lambda_t} = \lambda_t (F(S_t) - d_t - S_{t+1}) = 0 \qquad t = 0 \dots n \tag{7}$$

The third equation is the "complementary slackness" (Kuhn-Tucker) condition. Notice equation (7) implies that if λ is greater than zero then the constraint holds with equality. Conversely, if the constraint does not hold with equality, λ must be zero.

In this case, equation (5) implies the $\lambda > 0$, if marginal utility is strictly positive. In this case, we can recover equations (5)-(6) by eliminating λ_t (the constraint holds with equality):

$$-\frac{\partial U\left(d_t\right)}{\partial d}+\frac{\partial U\left(d_{t+1}\right)}{\partial d}\frac{\partial F\left(S_{t+1}\right)}{\partial S}=0 \qquad t=0\ldots n \tag{8}$$

$$\frac{\partial U\left(d_t\right)}{\partial d}\left(F\left(S_t\right)-d_t-S_{t+1}\right)=0 \qquad t=0\ldots n \tag{9}$$

Conversely if the marginal utility is zero ("satiation"), then the equation (5) implies $\lambda=0$, and the constraint does not matter, and the first order conditions are the same as if we had ignored the constraint.

We can interpret λ as the marginal utility of income, or the marginal utility from relaxing the constraint. What could we do with a little extra income? We could consume or invest. These options must give identical utility, otherwise we could reallocate some existing income from the one which gives less utility to the one which gives more utility and be better off. Since both give equal utility, let us measure the marginal utility of income in terms of the extra consumption we get (first equation).

The second order conditions are again more complicated here, because the equation is linear in lambda. So we don't exactly need every determinant to be negative. To save time, let this be taught in Eco 512.

IV Analysis of Difference Equations

Our optimization often produces an equation governing the states that looks like:

$$S_{t+1}=h\left(S_t,S_{t-1},\ldots\right) \tag{1}$$

In fact, usually only one or two lags exist (equation 5 has two lags, for example). This equation determines the time series behavior of the economy, along with initial conditions (e.g. S_0 given).

For example, let:

$$S_{t+1}=1+\frac{1}{2}S_t \qquad S_0=0 \tag{2}$$

Then:

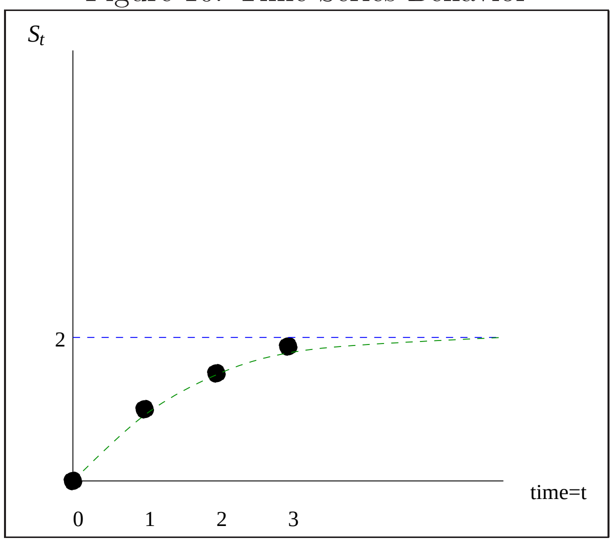
$$S_1=1+\frac{1}{2}\cdot 0=1 \tag{3}$$

$$S_2 = 1 + \frac{1}{2} \cdot 1 = 1.5 \tag{4}$$

$$S_3 = 1 + \frac{1}{2} \cdot 1.5 = 1.75 \tag{5}$$

$$S_4 = 1 + \frac{1}{2} \cdot 1.75 = 1.875 \tag{6}$$

Figure 10: Time Series Behavior



So the long run behavior of this economy is to have $S = 2$. Further, we expect S to increase at a decreasing rate starting from a small value.

A Long Run Behavior: Steady States

Where does the economy go to in the long run?

Definition 1 *A* steady state *or* balanced growth path *occurs when all variables grow at a constant rate.*

Definition 2 *A* stationary state *is a steady state where the growth rate is zero.*

All variables in the steady state satisfy $S_t = (1 + g)\, S_{t-1}$. The stationary state is the same with $g = 0$ so: $S_{t+1} = S_t = \bar{S}$

Typically the steady state is easy to compute. For the above example:

$$\bar{S} = 1 + \frac{1}{2}\bar{S} \tag{7}$$

$$\bar{S} = 2 \tag{8}$$

So that if $S_t = 2$, $S_{t+1} = 2$.

For more complicated possibly non-linear difference equations, the stationary state solves:

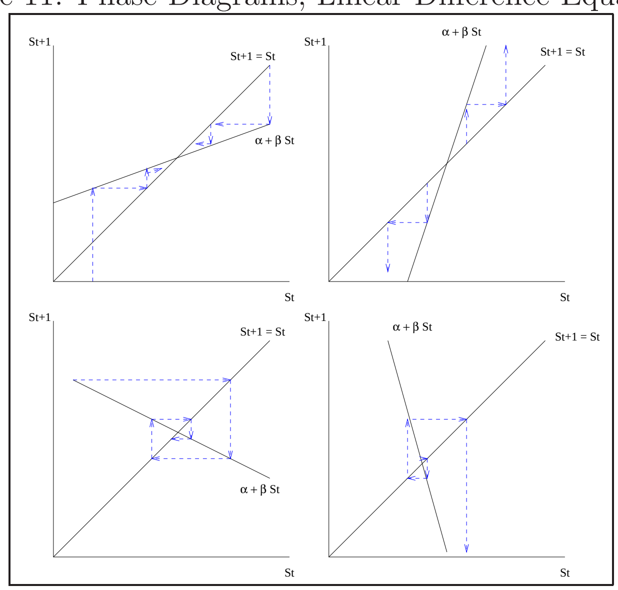
$$S_{t+1} = h\left(S_t, S_{t-1}, \ldots\right) \tag{9}$$

$$\bar{S} = h\left(\bar{S}, \bar{S}, \ldots\right) \tag{10}$$

B Convergence to the Steady State, 1 lag difference equations

Does the economy converges to the steady state? Not necessarily the case as seen by some examples:

Figure 11: Phase Diagrams, Linear Difference Equations



Stability seems to be related to the slope of the line: we want S_t to have a small effect on S_{t+1}. If so, the initial condition will die out, in the long run, the value of S_t will not depend on the initial conditions and will remain finite:

$$S_{t+1} = \alpha + \beta S_t \tag{11}$$

$$S_1 = \alpha + \beta S_0 \tag{12}$$

$$S_2 = \alpha + \beta \left(\alpha + \beta S_0\right) \tag{13}$$

$$S_2 = \alpha \left(1 + \beta\right) + \beta^2 S_0 \tag{14}$$

$$S_t = \alpha \sum_{i=0}^{t-1} \beta^i + \beta^t S_0 \tag{15}$$

So if $|\beta| > 1$, then S will go to infinity. But if $|\beta| < 1$ we have convergence. Specifically:

$\beta > 1$	diverges monotonically
$\beta = 1$	diverges monotonically for $\alpha \neq 0$
$0 < \beta < 1$	converges monotonically
$-1 < \beta < 0$	converges cyclically
$\beta = -1$	cycles around initial condition
$\beta < -1$	diverges cyclically

For non linear difference equations, we have only a local result. We need to consider the slope near the steady state.

THEOREM 2 *Hartman's theorem: The stability of a non-linear difference equation* $S_{t+1} = h\left(S_t\right)$ *is locally equivalent to:*

$$S_{t+1} = \bar{S} + \frac{\partial S_{t+1}\left(\bar{S}\right)}{\partial S_t} S_t \tag{16}$$

Hence we have a stable stationary state iff:

$$\left|\frac{\partial S_{t+1}\left(\bar{S}\right)}{\partial S_t}\right| < 1 \tag{17}$$

Here is the geometry of this result:

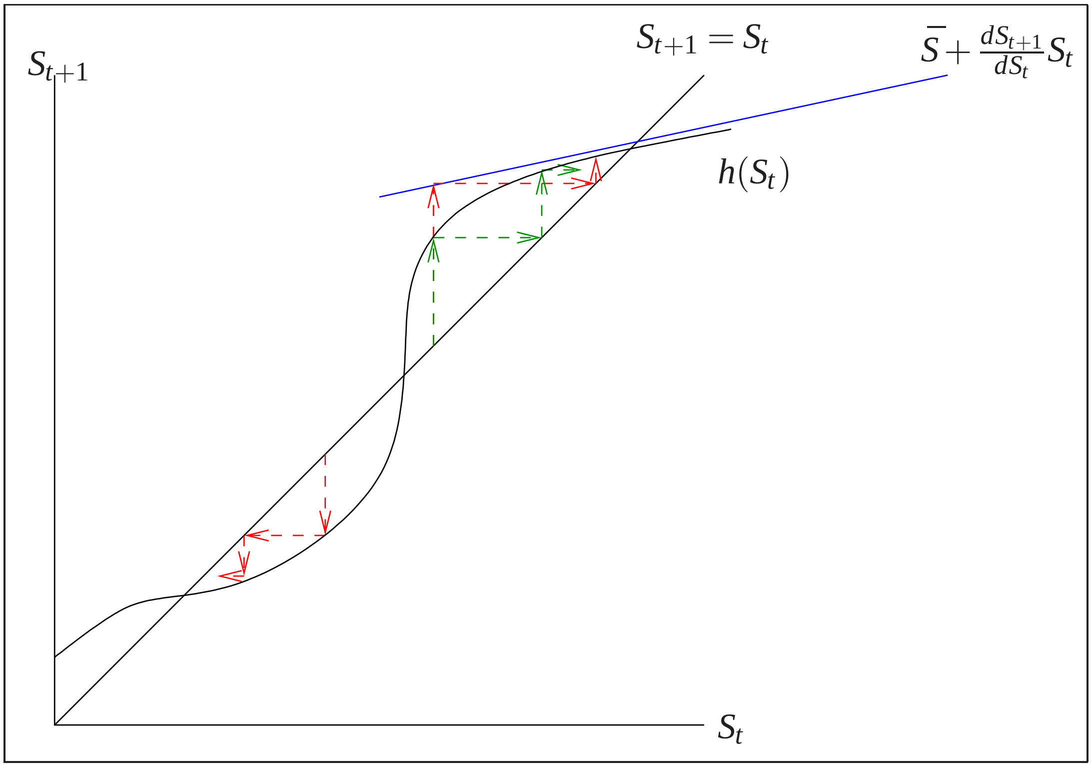


Figure 12: Hartman's Theorem