GOVERNMENT BONDS, TAXES, AND FISCAL POLICY

I Bond Finance, Basic Issues

Bonds allow for the possibility of deficits. In principle, the government may alter the flow of spending and investment by, say, giving a tax cut, financing with a deficit to be paid back later. Alternatively, the government could run a surplus and hold extra spending to be given as a tax cut later. This could smooth the business cycle in principle.

1. How do tax cuts and deficit spending affect behavior?

   - Undergraduate principle of crowding out: does government borrowing replace or crowd out private borrowing?
   - All government budget decisions are linked through the government budget constraint. Suppose a bond/deficit financed cut in wage taxes. Is the resulting increase in labor supply caused by the tax cut or the deficit? To determine the impact of deficits, we need to give a deficit financed lump sum tax. The lump sum tax does not affect behavior.
   - Can the government smooth the business cycle?

2. Steady state deficits. Can the government run a deficit indefinitely? What size deficits must eventually result in default?

3. Optimal deficit policy.

II More on Monetary and Fiscal Policy

Since monetary and fiscal policies are coordinated through the same budget constraint, they are linked in a certain sense. Further open market operations involve a coordinated effort of both monetary and fiscal policy.

- The monetarist school states that bond financed spending is not inflationary like inflation financed deficits. Will the increase in aggregate demand result in inflation?
- Open market operations versus other methods of injecting money (helicopter drop).
- Default through seniorage: by printing dollars and using those dollars to repay bonds, the government is essentially paying back the bonds with worthless dollars. This can be considered a "default".

III Social Security and Redistribution

The running of deficits and social security allows redistribution across generations.

- Optimality of transfers across generations.
- Insolvency of social security.

STRUCTURE OF GOVERNMENT BONDS, TAXES, AND INFLATION

We will use the money in the utility framework. Your homework uses the Cooley-Hansen CIA model.

I Government Bonds

A Bond Prices and interest rate

Suppose the existence of 1-period 0-coupon nominal bonds sold at price Q_t, redeemable for \$1 in $t+1$.

Let B_t denote the total stock of bonds held per person, redeemable at period t. Let b_t denote the stock held by an individual at time t.

Define a net nominal rate of return R_{t+1} as:

$$Q_t + R_{t+1}Q_t = 1, \tag{1.1.1}$$

$$R_{t+1} \equiv \frac{1 - Q_t}{Q_t}. \tag{1.1.2}$$

This is the cash return on a dollar invested in t, paying net nominal return R_{t+1} in $t+1$. The bond price and interest rate are inversely related, both are used on Wall Street. Thus we may work with R_{t+1} or Q_t, here we will generally use R_{t+1}. Clearly R_{t+1} is determined in period t, is a function of the states in period t, and is known in period t.

B Bond revenue and redemption values

The cash value of bonds held by an individual is $1 \cdot b_t$. Since bonds pay in cash, the real bond holdings are $\frac{b_t}{p_t}$. We normalize using $\frac{b_t}{\bar{M}_t} \equiv \hat{b}_t$ (ala Cooley-Hansen). Thus $\frac{\hat{b}_t}{\hat{p}_t}$ is the real bond holdings. Let $\hat{B}_t = B_t/\bar{M}_t$ denote the aggregate bond stock.

The dollar value of bond purchases is $B_{t+1}Q_t$. Note that $Q_t = \frac{1}{1+R_{t+1}}$, and hence the cost to all households (or revenue raised by the government) is:

$$\text{govt rev } = B_{t+1}\frac{1}{1+R_{t+1}} \tag{1.2.3}$$

$$\text{real rev } = \frac{B_{t+1}}{p_t}\frac{1}{1+R_{t+1}} \tag{1.2.4}$$

Normalized:

$$= \frac{B_{t+1}}{\bar{M}_{t+1}}\frac{\bar{M}_{t+1}}{\bar{M}_t}\frac{\bar{M}_t}{P_t}\frac{1}{1+R_{t+1}} = \frac{\hat{B}_{t+1}}{\hat{p}_t}\frac{1+\mu}{1+R_{t+1}} \tag{1.2.5}$$

The government raises less revenue than the face value of the bonds since $Q_t < 1$ is paid for each \$1 bond. Note that $1 + r_t^b = \frac{1+R_{t+1}}{1+\pi_t}$ is the real rate of return: We must divide the interest rate by $1 + \pi_t > 1$ to account for the fact that the dollars we are paid back with buy less goods.

II Deficits

A Government budget constraint

The government must pay B_t dollars to current bond holders, this is an expense. The government has real revenue from the sale of next periods bonds, which is less than the total number of bonds issued.

$$\text{redemption value } = \frac{B_t}{p_t} = \frac{B_t}{\bar{M}_t}\frac{\bar{M}_t}{p_t} = \frac{\hat{B}_t}{\hat{p}_t} \tag{2.1.1}$$

The government budget constraint sets expenditures of spending, transfers, and bond redemptions equal to income of bond sale revenue plus seniorage:

$$g + \frac{\hat{B}_t}{\hat{p}_t} + TR_t = \frac{\hat{B}_{t+1}(1+\mu)}{\hat{p}_t}\frac{1}{1+R_{t+1}} + \frac{\mu}{\hat{p}_t} \tag{2.1.2}$$

Rewriting gives:

$$g - (-TR_t) = \frac{\hat{B}_{t+1}(1+\mu)}{\hat{p}_t}\frac{1}{1+R_{t+1}} - \frac{\hat{B}_t}{\hat{p}_t} + \frac{\mu}{\hat{p}_t} \tag{2.1.3}$$

The government now has 4 instruments: μ, $\hat{B}_{t+1}$, TR_t, and g. Three of these must be given government policies, and the fourth must be determined so that the government budget constraint holds. In reality, we often see g, TR, and μ set by policy, and borrowing is whatever is left over. Here, we wish to see the effects of a change in deficit policy, so we will have g, μ, and $\hat{B}_{t+1}$ set by policy with TR_t being whatever is left over.

The definition with wage taxes would be something like:

$$g - \tau_t w_t h_t = \frac{\hat{B}_t\,(1+\mu)}{\hat{p}_t}\frac{1}{1+R_{t+1}} - \frac{\hat{B}_t}{\hat{p}_t} + \frac{\mu}{\hat{p}_t} \tag{2.1.4}$$

B Primary deficit

Definition 28 *The* Primary Deficit *is the deficit excluding interest payments:*

$$D_t^p = g + TR_t \tag{2.2.5}$$

Here we just have spending less taxes, or:

$$D_t^p = \frac{\hat{B}_{t+1}\,(1+\mu)}{\hat{p}_t}\frac{1}{1+R_{t+1}} - \frac{\hat{B}_t}{\hat{p}_t} + \frac{\mu}{\hat{p}_t} \tag{2.2.6}$$

The primary deficit does not include interest payments as an expense. Note that seniorage revenue is also not included, making the primary deficit look bigger, since seniorage revenue is in fact tax revenue from a tax on money.

Ignoring seniorage ($\mu = 0$), a primary deficit implies the debt level increases by the difference between government spending and taxes, plus interest on next period's bond stock. That is, let $\mu = 0$, then:

$$\frac{\hat{B}' - \hat{B}}{\hat{p}} = \Delta\ \mathrm{debt}_t = \frac{\hat{B}_{t+1}}{\hat{p}_t} - \frac{\hat{B}_t}{\hat{p}_t} = D_t^p + \frac{\hat{B}_{t+1}}{\hat{p}_t}\frac{R_{t+1}}{1+R_{t+1}}. \tag{2.2.7}$$

The government must borrow to finance the current primary deficit. It must also roll over the existing debt. So the government must borrow to pay interest as well.

C Conventional Deficit

Definition 29 *The* Conventional Deficit *is the deficit including interest payments:*

$$D_t^c = g + TR_t + (1 - Q_{t-1})\,\frac{\hat{B}_t}{p_t} \tag{2.3.8}$$

Including interest payments as an expense gives:

$$D_t^c = g + TR_t + \frac{\hat{B}_t}{\hat{p}_t}\frac{R_t}{1+R_t} \tag{2.3.9}$$

Hence:

$$D_t^c = g + TR_t + \frac{\hat{B}_t}{\hat{p}_t}\frac{R_t}{1+R_t} = D_t^p + \frac{\hat{B}_t}{\hat{p}_t}\frac{R_t}{1+R_t} \tag{2.3.10}$$

$$= \frac{\hat{B}_{t+1}(1+\mu)}{\hat{p}_t}\frac{1}{1+R_{t+1}} - \frac{\hat{B}_t}{\hat{p}_t}\frac{1}{1+R_t} + \frac{\mu}{\hat{p}_t} \tag{2.3.11}$$

The conventional deficit is the one typically reported in the press. The conventional deficit also does not include seniorage revenues. Ignoring seniorage, the government's debt level may rise or fall if the conventional deficit is positive, depending on the difference between R_t and R_{t+1}.

III Consumer's Budget Constraint

Redeemed bonds count as income. Purchased bonds are spending. Let b_t denote individual bond demand, with normalization $\hat{b}_t = \frac{b_t}{M_t}$. Hence we have:

$$\begin{aligned} c_t \quad &+ \quad k_{t+1} - (1-\delta)k_t + \frac{\hat{m}_{t+1}(1+\mu)}{\hat{p}_t} + \frac{1+\mu}{1+R_{t+1}}\frac{\hat{b}_{t+1}}{\hat{p}_t} \\ &= \quad r_t k_t + w_t h_t + \frac{\hat{b}_t}{\hat{p}_t} + \frac{\hat{m}_t}{\hat{p}_t} + TR_t. \end{aligned} \tag{3.0.1}$$

We set an inflation tax μ using government policy and used transfers/taxes to balance the government constraint. Now we can use either bonds or taxes to balance the government constraint. Substituting for government policy variable TR_t gives:

$$\begin{aligned} c_t \quad &+ \quad k_{t+1} - (1-\delta)k_t + \frac{\hat{m}_{t+1}(1+\mu)}{\hat{p}_t} + \frac{1+\mu}{1+R_{t+1}}\frac{\hat{b}_{t+1} - \hat{B}_{t+1}}{\hat{p}_t} \\ &= \quad r_t k_t + w_t h_t + \frac{\hat{b}_t - \hat{B}_t}{\hat{p}_t} + \frac{\hat{m}_t + \mu}{\hat{p}_t} - g \end{aligned} \tag{3.0.2}$$

IV Problem and Equilibrium

Here is the recursive version of the problem, assuming constant money growth (this assumption does not matter).

$$\begin{aligned} v\left(z, k, K, \hat{b}, \hat{B}, \hat{m}\right) \quad = \quad & \max_{h,k',\hat{m}',\hat{b}'} \left\{ u \left[e^z F_k\left(K,H\right) k + e^z F_h\left(K,H\right) h + \frac{\hat{b} - \hat{B}}{\hat{p}} + \right. \right. \\ & \frac{\hat{m} + \mu}{\hat{p}} + (1-\delta) k - k' - \frac{\hat{m}'\left(1+\mu\right)}{\hat{p}} - \frac{1+\mu}{1+R'} \frac{\hat{b}' - \hat{B}'}{\hat{p}} \\ & \left. \left. -g, \frac{\hat{m}}{\hat{p}} \right] - \psi h + \beta \mathrm{E}\left[v\left(\rho z + \epsilon', k', K', \hat{b}', b', \hat{m}'\right)\right] \right\} \end{aligned} \tag{4.0.1}$$

The maximization is subject to the general equilibrium restrictions:

$$\hat{b} = \hat{B}, \quad \hat{m} = 1, \quad k = K \ \Rightarrow \ \hat{b}' = \hat{B}', \quad h = H, \quad k' = K', \quad \hat{m}' = 1 \tag{4.0.2}$$

Notice that I have assumed that none of the proceeds from the bond sales can be used for shopping in period t. Further, the transfer also cannot be used for shopping (both of these happen towards the end of the period).

Let $s = \left[z, k, K, \hat{b}, \hat{B}, \hat{m}\right]$ and $S = \left[z, K, K, \hat{B}, \hat{B}, 1\right]$. Then a *Recursive Competitive Equilibrium* given individual states s, aggregate states S, and government policies $\left[\mu, g, \hat{B}'\right]$, is a set of individual decisions $c\left(s\right)$, $h\left(s\right)$, $k'\left(s\right)$, $\hat{m}'\left(s\right)$, $\hat{b}'\left(s\right)$, aggregate decisions $C\left(S\right)$, $H\left(S\right)$, $K'\left(S\right)$, prices $\hat{p}\left(S\right)$, $R\left(S\right)$, $w\left(S\right)$, $r\left(S\right)$, a government decision TR, and a value function $v\left(s\right)$, such that four first order conditions hold, the four first order conditions hold in equilibrium, one budget constraint holds, one resource constraint holds, one government budget constraint holds, two firm first order conditions hold, and the Bellman equation holds.

We have 14 equations and unknowns.

V First order conditions and envelope equations

A First order conditions

The first order conditions are:

$$u_c\left(c, \frac{\hat{m}}{\hat{p}}\right) = \beta \mathrm{E}\left[v_k\left(\rho z + \epsilon', k', K', \hat{b}', \hat{B}', \hat{m}'\right)\right] \tag{5.1.1}$$

$$u_c\left(c,\frac{\hat{m}}{\hat{p}}\right)e^zF_h\left(K,H\right)=\psi \tag{5.1.2}$$

$$\frac{1+\mu}{\hat{p}}u_c\left(c,\frac{\hat{m}}{\hat{p}}\right)=\beta\mathrm{E}\left[v_{\hat{m}}\left(\rho z+\epsilon',k',K',\hat{b}',\hat{B}',\hat{m}'\right)\right] \tag{5.1.3}$$

$$\frac{1+\mu}{1+R'}\frac{1}{\hat{p}}u_c\left(c,\frac{\hat{m}}{\hat{p}}\right)=\beta\mathrm{E}\left[v_{\hat{b}}\left(\rho z+\epsilon',k',K',\hat{b}',\hat{B}',\hat{m}'\right)\right] \tag{5.1.4}$$

The only new first order condition is the one for bonds. Note that the real cost of a bond which pays one unit of consumption next period is less than one unit of consumption (specifically, $\frac{1}{1+R}$). This is the right hand side, the consumption lost when purchasing a bond. The right hand side is the expected marginal value of the bond.

B Envelope equations

The envelope equations are:

$$v_k\left(s\right)=u_c\left(c,\frac{\hat{m}}{\hat{p}}\right)\left[e^zF_k\left(K,H\right)+1-\delta\right] \tag{5.2.5}$$

$$v_{\hat{m}}\left(s\right)=\frac{1}{\hat{p}}\left[u_c\left(c,\frac{\hat{m}}{\hat{p}}\right)+u_m\left(c,\frac{\hat{m}}{\hat{p}}\right)\right] \tag{5.2.6}$$

$$v_{\hat{b}}\left(s\right)=\frac{1}{\hat{p}}u_c\left(c,\frac{\hat{m}}{\hat{p}}\right) \tag{5.2.7}$$

The only new equation is the one for bonds. The marginal value of a bond is the amount of consumption goods the bond can be exchanged for.

C Equilibrium

Here are the equilibrium conditions.

$$u_c\left(C,\frac{1}{\hat{p}}\right)=\beta\mathrm{E}\left[v_k\left(\rho z+\epsilon',K',K',\hat{B}',\hat{B}',1\right)\right] \tag{5.3.8}$$

$$u_c\left(C,\frac{1}{\hat{p}}\right)e^zF_h\left(K,H\right)=\psi \tag{5.3.9}$$

$$\frac{1+\mu}{\hat{p}}u_c\left(C,\frac{1}{\hat{p}}\right)=\beta\mathrm{E}\left[v_{\hat{m}}\left(\rho z+\epsilon',K',K',\hat{B}',\hat{B}',1\right)\right] \tag{5.3.10}$$

$$\frac{1+\mu}{1+R'}\frac{1}{\hat{p}}u_c\left(C,\frac{1}{\hat{p}}\right)=\beta\mathrm{E}\left[v_{\hat{b}}\left(\rho z+\epsilon',K',K',\hat{B}',\hat{B}',1\right)\right] \tag{5.3.11}$$

$$C=e^zF\left(K,H\right)+\left(1-\delta\right)K-K'-g \tag{5.3.12}$$

$$g+TR=\frac{\hat{B}'}{\hat{p}}\frac{1+\mu}{1+R'}-\frac{\hat{B}}{\hat{p}}+\frac{\mu}{\hat{p}} \tag{5.3.13}$$

$$v_k\left(S\right)=u_c\left(C,\frac{1}{\hat{p}}\right)\left[e^zF_k\left(K,H\right)+1-\delta\right] \tag{5.3.14}$$

$$v_{\hat{m}}\left(S\right)=\frac{1}{\hat{p}}\left[u_c\left(C,\frac{1}{\hat{p}}\right)+u_m\left(C,\frac{1}{\hat{p}}\right)\right] \tag{5.3.15}$$

$$v_{\hat{b}}\left(S\right)=\frac{1}{\hat{p}}u_c\left(C,\frac{1}{\hat{p}}\right) \tag{5.3.16}$$

We have 9 equations for unknowns K', $\hat{B}'$, H, C, R', $\hat{p}$, v_k, $v_{\hat{m}}$, and v_b (with TR given as a government policy).

VI Ricardian Equivalence

The *Ricardian Equivalence Proposition* is:

THEOREM 11 *All equilibrium variables are unaffected by a bond-financed change in lump sum taxes.*

Consider an increase in $\hat{B}'$, financed by an increase in TR_t. From equations (5.3.8)-(5.3.16), only the government budget constraint (5.3.13) is affected. To see this, note that TR appears in none of the eight equations except the government constraint. The variable $\hat{B}'$ appears in the right hand side of (5.3.8)-(5.3.11), through the derivatives of v only. But it is clear from (5.3.14)-(5.3.16) that the derivatives of v are independent of $\hat{B}$. Thus $v_k\left(z,K,K,\hat{B},\hat{B},1\right)=v_k\left(z,K,K,1\right)$ and thus $v_k\left(S'\right)=v_k\left(z',K',K',1\right)$ is in fact independent of $\hat{B}'$. The same logic holds for v_b and v_m. So 8 equations are independent of $\hat{B}$ and TR. Further, only TR and $\hat{B}'$ change in the government constraint. All of the other equilibrium variables are solved by the remaining 8 equations and therefore have the same solution.

In other words, deficits do not matter. Indeed, issuing more government debt does not even raise the interest rate on goverment debt! Ricardian equivalence follows less from optimization as it does from the resource constraint and supply equals demand. Extra borrowing does not add resources to the economy and therefore does not change the aggregate resource constraint. Because supply equals demand and all households are identical, each household must loan the government $\hat{B}'$. But then each household gets $\Delta TR = \hat{B}'$ back. So households have identical income and make identical decisions. Note that the household is not richer despite owning a bond and receiving a tax cut: in period $t+1$ (or some subequent period) the government must raise taxes to repay the accumulated debt.

Because Ricardian equivalence does not depend much on optimization, it is robust across models. From the above discussion, it is clear that heterogeneity might matter, and the assumption that government debt is risk free will matter. Finally, the assumption of lump sum taxes matters.

A Risky Government Debt

If government debt has default risk, and if the default risk rises with the quantity of outstanding bonds, then clearly the interest rate will rise with $\hat{B}'$ as lenders demand higher interest payments for taking on more risk. However, one can repeat the above arguments and show that only the interest rate is affected, no other endogenous variables change.

B Heterogeneity

Suppose first we are in an open economy. Then we might borrow from China, but give the transfer to a US household. In this case we might see real effects. However, overall demand and resources will not change. Instead, we will see a decrease in net exports (China would have spent the money on US exports in the absence of US debt), and an increase in domestic consumption.

If we borrow from the rich who would likely have invested the money in the private sector, and give to the poor who consume, then we again are unlikely to see a change in resources or overall demand, but could see a decrease in investment spending and an increase in consumption.

If we borrow and the young and give a tax cut to the old, we will see redistribution across generations.

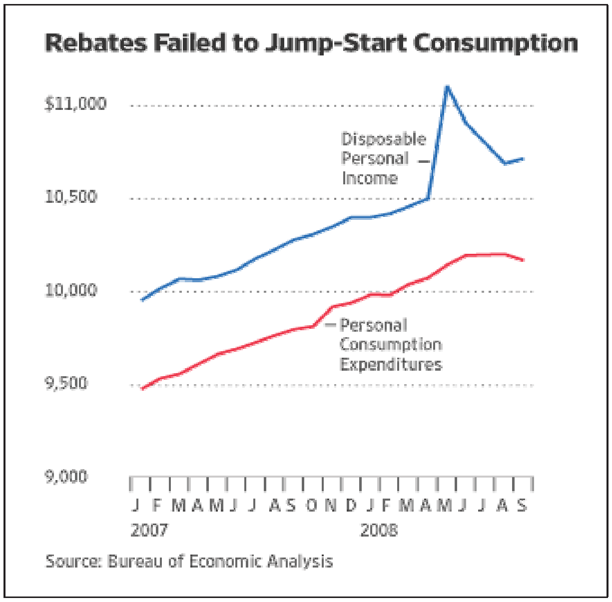
C Lump Sum Taxes

If we, say, decrease the tax rate on wage income rather than the lump sum tax, we will likely see an increase in hours worked. However, the increase in hours worked will be caused by the change in wage income taxes, rather than the deficit. A decrease in wage taxes financed by a lump sum tax increase will have the same effect as a decrease in wage taxes financed by a deficit.

D Actual Results

Overall, Ricardian equivalence is robust across models and seems quantitatively robust across alternative assumptions: we will not see an increase in aggregate demand for reasonable increases in the deficit, although we might see small changes in how resources are distributed. Clearly government debt does not create wealth, and has at best limited power to move wealth around the economy.

Actual experiences with tax rebates (close to lump sum tax cuts) financed by borrowing tend to confirm the Ricardian view. Little change in consumption resulted from such rebates. The following is a typical example:



Consumption changed little, despite the tax rebate. Why? Because households were rebated back money they lent to the government, leaving their after tax income up, but their net income available after loaning money to the government unchanged.

VII Steady State

A Basic properties of the steady state

The certainty equivalence steady state evaluates (5.3.8)-(5.3.16) at their steady state values.

$$u_c\left(C, \frac{1}{\hat{p}}\right) = \beta v_k(S) = \beta u_c\left(C, \frac{1}{\hat{p}}\right)(F_k(K,H) + 1 - \delta)$$

$$\rho = F_k(K,H) - \delta \tag{7.1.1}$$

$$u_c\left(C, \frac{1}{\hat{p}}\right) F_h(K,H) = \psi \tag{7.1.2}$$

$$\frac{1+\mu}{\hat{p}} u_c\left(C, \frac{1}{\hat{p}}\right) = \beta v_{\hat{m}}(S) = \beta \frac{1}{\hat{p}}\left(u_c\left(C, \frac{1}{\hat{p}}\right) + u_m\left(C, \frac{1}{\hat{p}}\right)\right)$$

$$(1+\mu-\beta)\, u_c\left(C, \frac{1}{\hat{p}}\right) = \beta u_m\left(C, \frac{1}{\hat{p}}\right) \tag{7.1.3}$$

$$\frac{1+\mu}{1+R}\frac{1}{\hat{p}}u_c\left(C, \frac{1}{\hat{p}}\right) = \beta v_{\hat{b}}\,(S) = \beta\frac{1}{\hat{p}}u_c\left(C, \frac{1}{\hat{p}}\right)$$

$$1+R = (1+\mu)\,(1+\rho) \tag{7.1.4}$$

$$C = F\,(K{,}H) - \delta K - g \tag{7.1.5}$$

$$g + TR = \frac{\hat{B}}{\hat{p}}\left(\frac{1+\mu}{1+R} - 1\right) + \frac{\mu}{\hat{p}} \tag{7.1.6}$$

Equations (7.1.1)-(7.1.6) solve for unknowns K, TR, H, C, R, and $\hat{p}$.

It is clear from the steady state equations that Ricardian Equivalence holds in the steady state, which is a special case of equilibrium. It is also clear from (7.1.3) that the Friedman rule is optimal and from (7.1.4) that the Friedman rule is equivalent to a nominal rate equal to zero. Equation (7.1.4) shows that the Fisher relation holds approximately: the nominal rate approximately equals the real rate plus the inflation rate for $\mu \cdot \rho$ small. Labor supply is endogenous so the steady state is not neutral.

B Steady state deficits

First, from (7.1.6), the steady state primary deficity is:

$$D_p = g + TR = \frac{\hat{B}}{\hat{p}}\left(\frac{1+\mu}{1+R} - 1\right) + \frac{\mu}{\hat{p}} \tag{7.2.7}$$

Using (7.1.4):

$$= \frac{\hat{B}}{\hat{p}}\left(\frac{1}{1+\rho} - 1\right) + \frac{\mu}{\hat{p}} \tag{7.2.8}$$

$$D_p = g + TR = \frac{\hat{B}}{\hat{p}}\,(\beta - 1) + \frac{\mu}{\hat{p}} \tag{7.2.9}$$

A steady state primary deficity is certainly possible with seniorage. We can spend more than taxes indefinitely regardless of the bond stock if we finance the difference with seniorage.

However, although we call this a deficit in an accounting sense, in reality seniorage is a tax on money holdings. So we are not really running a deficit if it is financed with seniorage.

Suppose instead we assume seniorage equals zero ($\mu = 0$), so any steady state deficit cannot be financed with seniorage. Then:

$$D_p = -\frac{\hat{B}}{\hat{p}} \frac{R}{1+R} \tag{7.2.10}$$

The deficit is negative. Therefore, we can only have a steady state primary surplus. If we have a steady state primary surplus, we can use the extra revenue over spending to finance the interest on the constant bond stock. The total bond stock therefore remains constant. A primary deficit would mean we have increasing interest and principle to pay over time which is inconsistent with a constant bond stock.

Now for the conventional deficit (2.3.11) is at the steady state:

$$D_c = \frac{\hat{B}}{(1+R)\,\hat{p}} \left((1+\mu) - 1\right) + \frac{\mu}{\hat{p}} = \mu \frac{\hat{B}}{(1+R)\,\hat{p}} + \frac{\mu}{\hat{p}} \tag{7.2.11}$$

A steady state conventional deficit is possible as long as we can print money to finance it. But with no seniorage, we have:

$$D_c = 0. \tag{7.2.12}$$

A steady state conventional deficit (or surplus) with a constant bond stock is not possible without seniorage. Any non-zero value of D_c implies a bond stock which is changing, which is inconsistent with our definition of a steady state.

With a constant bond stock, there are no free lunches. We cannot borrow money and never pay it back.

C Steady state with growing bond stock and $\mu = 0$

A *Stationary State* is when all variables are constant. It is a special case of a *Steady State* where all variables grow at a constant rate (zero in the case of a stationary state). Most variables in our steady state grow at rate zero, but aggregate variables grow at the constant rate of the productivity augmented population. Per capita variables grow at the constant rate of the productivity growth rate. Suppose we modify our definition of steady state so that the bond stock grows at a constant rate:

$$\hat{B}' = (1+\zeta)\,\hat{B} \tag{7.3.13}$$

The steady state primary deficit with $\mu = 0$ becomes:

$$D_p = \frac{\hat{B}(1+\zeta)}{\hat{p}} \frac{1}{1+R} - \frac{\hat{B}}{\hat{p}} \tag{7.3.14}$$

$$= \frac{\hat{B}}{\hat{p}} \left(\frac{1+\zeta}{1+R} - 1 \right) \tag{7.3.15}$$

So for a steady state primary deficit we must have $\zeta > R$. From equation (7.1.4) with $\mu = 0$, that is equivalent to: $\zeta > r = \rho$ or $1 + \zeta > \frac{1}{\beta}$. However, this violates the transversality condition. Each period we must borrow to finance the new principle plus interest, so the rate of growth in the bond stock must be greater than the rate of interest. This is a (government run) Ponzi scheme, whereby old investors are paid off with the proceeds from new investors. Thus, no primary deficit is possible in the steady state even if bond stocks are allowed to grow.

For the conventional deficit with constant growth in bonds and $\mu = 0$, we have:

$$D_c = \frac{\hat{B}(1+\zeta)}{\hat{p}} \frac{1}{1+R} - \frac{\hat{B}}{\hat{p}} \frac{1}{1+R} = \zeta \frac{\hat{B}}{\hat{p}} \frac{1}{1+R} \tag{7.3.16}$$

A steady state conventional deficit is possible for $0 < \zeta < \rho$. Imagine a primary surplus that is not quite large enough to pay the interest on the debt. Then next period, the government must borrow to replace the existing bond stock plus a little bit extra for the remaining interest. The bond stock grows, but slow enough so as to not violate the transversality condition.[1]

From this exercise, we learn that aggregate demand does not change (Ricardian equivalence holds) even with a steady state conventional deficit, whereby some debt is never repaid. It is therefore not anticipation of future taxes that causes households to save rather than spend the increase in lump sum transfers, but rather simply that government bonds do not create any additional resources and therefore can only shift demand from one sector to another.

[1] E.g. let the primary surplus be 15, the bond stock 100, and $R = 0.2$. Then we have 20 in interest payments of which the primary surplus covers 15, leaving 5 remaining. So new borrowing is $100 + 5$. The bond stock grows by five percent which is less than the interest rate of 20 percent.

D Maximum Deficit With Seniorage

Let utility be $\log C_t + \alpha \log \frac{1}{\hat{p}_t}$ so that (7.1.3) becomes:

$$(1 + \mu - \beta) \frac{1}{C} = \frac{\beta\alpha}{\frac{1}{\hat{p}}} \tag{7.4.17}$$

$$\frac{\mu}{\hat{p}} = \frac{\mu}{1 + \mu - \beta} \beta\alpha C \tag{7.4.18}$$

Note that from (7.1.1), since marginal products are homogenous degree zero, the capital to labor ratio is independent of μ. But then from (7.1.2), C is independent of μ.

Equation (7.4.18) implies seniorage revenue equals $-\infty$ at the Friedman rule, indicating the Friedman rule requires infinite lump sum taxes to finance. Seniorage is increasing in μ and no money growth implies no seniorage. Regardless of the level of inflation, some currency is held since otherwise with log utility the household would get $-\infty$ utility.

The maximum seniorage is:

$$\frac{\mu}{\hat{p}} = \beta\alpha C \tag{7.4.19}$$

Now equations (7.2.9) and (7.4.18) imply:

$$D_p = (\beta - 1) \frac{\hat{B}}{\hat{p}} + \frac{\mu}{\hat{p}}, \tag{7.4.20}$$

$$= \beta\alpha C \left(\frac{(\beta - 1) \hat{B} + \mu}{1 + \mu - \beta} \right), \tag{7.4.21}$$

which is increasing in μ. Therefore, in the limit as $\mu \to \infty$, we have:

$$D_p = \beta\alpha C. \tag{7.4.22}$$

The maximum primary deficit equals the steady state seniorage. This is consistent with any finite steady state stock of bonds, since as $\mu \to \infty$ the real value of the stock of bonds goes to zero.

VIII Unpleasant Arithmetic

A longstanding monetarist claim is that inflation is a monetary phenomena, caused by increases in the growth rate of money. This has obvious support from the steady state equation where $\pi = \mu$. Nonetheless, all government policies are linked in the government budget constraint. If we view seniorage as the variable which ensures the government budget constraint holds, rather than TR, then changes in fiscal things like deficits and taxes could affect seniorage and therefore inflation through the government constraint.

Consider the following hypothesis:

Definition 30 *Unpleasant Monetarist Arithmetic: Bond financed primary deficits are eventually more inflationary the deficits financed with seniorage.*

Consider two countries with an equal sized primary deficit and an equal sized initial bond stock, $\hat{B}_t = 0$. The first country finances with seniorage. From the government constraint:

$$D^p = \frac{\mu}{\hat{p}} \tag{8.0.1}$$

$$\mu = D^p \hat{p} \tag{8.0.2}$$

Notice that I am using μ to balance the constraint and am taking g, TR (and therefore D_P), and $\hat{B}'$ as given. The first country incurs inflation as a result of money growth, but in turn the cause of the money growth was the primary deficit.

Suppose a second country opts for borrowing and no seniorage ($\mu = 0$). Then:

$$D^p = \frac{\hat{B}'}{\hat{p}} \frac{1}{1+R} - \frac{\hat{B}}{\hat{p}} \tag{8.0.3}$$

$$\hat{B}' = (1+R)\,\hat{B} + D^p \hat{p}\,(1+R) \tag{8.0.4}$$

Looking at the above equation, borrowing here grows at the rate of interest, which is not sustainable due to the transversality condition. Eventually, we get to a point where no more lending can take place ($\hat{B}' = 0$), as the government cannot even pay the interest on a new bond issue. At this point, assuming a fixed primary deficit, the government must resort to seniorage, at which point:

$$D^p = -\frac{\hat{B}}{\hat{p}} + \frac{\mu}{\hat{p}} \tag{8.0.5}$$

$$\mu = \hat{p}D^p + \hat{B} \tag{8.0.6}$$

And so inflation is eventually higher for a government that finances a primary deficit with bonds versus seniorage.

The Latin American experience of the 1980s tends to confirm this view. Countries like Columbia had stable 20% inflation for years since their debt did not accumulate. Conversely countries like Brazil and Argentina had debts that accumulated to unsustainable levels. Failing to cut government spending or raise taxes, they resorted to massive seniorage, creating hyperinflation.

OG Model with Money

I Introduction

Original OG model was by Samuelson (1958). Main treatment here follows chapter 3,4 of Blanchard and Fisher, and Grandmont Econometrica (1985).

- The main advantage of the OG model is that we can study life cycle issues such as social security, bequests, etc. The OG model provides a compact framework since the agent only has a few decisions to make, not an infinite number of decisions as in the ILA models.

- Role for money: completes markets by allowing agents not yet alive to "trade" with agents who are currently alive. Households desire to save for retirement (sell goods for assets). Old want to buy goods, but cannot sell assets since they will be dead next period and cannot deliver goods next period (cannot pay the young back next period). This is not efficient. A social planner could transfer goods from the young to the old.

- Person may consume their own production. However, purchase of consumption of other generations requires money, as in cash in advance.

- No capital, so money has a store of value role as well.

An important issue in OG models is the relationship between the CE and PO. In general the competitive equilibria is not parato optimal. This is usually due to some kind of incompleteness in the markets. Money often is used to add an extra asset to complete the markets. In general OG models overaccumulate capital, leading to inefficient allocations.

It is also not clear what the social welfare function should be. Does the social planner care only about the current generation? If so, no resources are allocated to the next generation not yet born. The social welfare functions used in the ILA models:

$$W = \sum^{\infty} \beta^t u \left[\frac{C_t}{L_t}\right] \tag{1.0.1}$$

$$W = \sum^{\infty} \beta^t L_t u \left[\frac{C_t}{L_t}\right] \tag{1.0.2}$$

do not have clear analogs in the OG model. However, if there is a bequest motive the current generation cares about all future generations and this kind of social welfare function might be appropriate.

These ideas are best illustrated by the following example. Suppose we have a hotel with n rooms and n guests. It is easy to see that allocating one room per guest is Parato efficient (you cannot change the allocation without making at least one person worse off). It is also welfare maximizing, since allocating two rooms to one guest and zero rooms to another would result in the guest with no room having a much higher marginal utility of consumption than the guest with two rooms.

Now let us suppose n is countably infinite. Start by allocating again one room to everyone. Now give guest two's room to guest one, and repeat, giving guest $t+1$'s room to guest t. We now have higher efficiency and higher welfare. Guest one is better off with two rooms and everyone else is equally well off. The double infinity of goods and households breaks the welfare theorems.

Now apply to the OG model.

II Assumptions

The economy is a pure endowment economy so money plays the store of value role. Consumption goods are perishable, and hence not storable.

Although the setup is simple, the simplicity is deceptive, for we can show that more complicated models behave isomorphically to this model. For example, the two period lived case is isomorphic to the n period lived case.

A Agents

Agents live for 2 periods and are identical. There is a continuum of agents (this is used because one can show that a model with a continuum of agents is identical to a model with one agent).

B Goods

There is a single perishable commodity. Consumption when young is C_{1t}, consumption when old is C_{2t}.

C Money

There exists a stock $\bar{M}$ per capita of fiat money, invented by the old agents alive at period zero. Money is storable. Money is usually constrained to be positive: $m_t \geq 0$. So we cannot go short on money. We wish to avoid old agents going infinitly short on money and then dying before their debts are paid.

D Prices

Money is the numeraire good, hence P_t is in dollars per good. The model is perfect foresight-rational expectations so that $E\left(P_{t+1}\right) = P_{t+1}$.

E Utilty

Additively seperable utility function:

$$u = u\left(C_{1t}\right) + V\left(C_{2,t+1}\right) \tag{2.5.1}$$

We have the usual concavity restrictions: $U_c > 0$, $U_{cc} < 0$, $V_c > 0$, $V_{cc} < 0$. The Inada conditions also apply: $U_c\left(0\right) = \infty$ and $U_c\left(\infty\right) = 0$ and the same for V.

F Endowment Economy

Labor is supplied inelastically to a fixed, CRS production technology. Hence production is also fixed. Call this production an endowment w. We have $w_1 \geq 0$ and $w_2 \geq 0$.

G Budget Constraints

$$w_1 - \frac{M_t}{P_t} = C_{1t} \tag{2.7.2}$$

$$w_2 + \frac{M_t}{P_{t+1}} = C_{2,t+1} \tag{2.7.3}$$

Hence there is really only one decision. Once we know the savings decision, we implicitly know the consumption in each period.

III Problem

A Competitive Problem

$$\max_{M_t} U\left[w_1 - \frac{M_t}{P_t}\right] + V\left[w_2 + \frac{M_t}{P_{t+1}}\right] \tag{3.1.1}$$

Subject to the general equilibrium restriction that money supply equals money demand:

$$M_t = \bar{M} \tag{3.1.2}$$

This problem is nice and easy because it is already a two period problem. There is no need to establish a value function, it would just be equal to the second period utility function. No need to normalize, $\bar{M}$ is constant.

B FOCs

$$-\frac{1}{P_t}U_c\left[w_1 - \frac{M_t}{P_t}\right] + \frac{1}{P_{t+1}}V_c\left[w_2 + \frac{M_t}{P_{t+1}}\right] = 0. \tag{3.2.3}$$

Substitute in the general equilibrium restriction:

$$-\frac{1}{P_t}U_c\left[w_1 - \frac{\bar{M}}{P_t}\right] + \frac{1}{P_{t+1}}V_c\left[w_2 + \frac{\bar{M}}{P_{t+1}}\right] = 0. \tag{3.2.4}$$

Equation (3.2.3) is the demand curve $M\left(M_t, P_t; P_{t+1}\right) = 0$. Prices must follow a first order difference equation $M\left(\bar{M}, P_t; P_{t+1}\right) = 0$. It is important to recognize that that P_t and not P_{t+1} is determined here. In turn P_{t+1} is determined from an updated equation and so on.

IV The Value of Money

Manipulating the FOC gives:

$$\frac{P_t}{P_{t+1}} = \frac{U_c}{V_c} \tag{4.0.1}$$

Hence the marginal rate of substitution between old and young consumption must equal the return on money.

A Social Planning Problem

Let's compare (4.0.1) to the social planning problem, which is:

$$\max_{c_{1t},c_{2t}} U\left(C_{1t}\right) + V\left(C_{2t}\right) \tag{4.1.2}$$

Subject to the resource constraint:

$$C_{1t} + C_{2t} = w_1 + w_2 \tag{4.1.3}$$

Substitute in to get:

$$\max U\left(w_1 + w_2 - C_{2t}\right) + V\left(C_{2t}\right) \tag{4.1.4}$$

The first order condition is:

$$-U_c\left(w_1 + w_2 - C_{2t}\right) + V_c\left(C_{2t}\right) = 0 \tag{4.1.5}$$

$$\frac{U_c\left(C_{1t}\right)}{V_c\left(C_{2t}\right)} = 1. \tag{4.1.6}$$

We desire to equalize the marginal utility of consumption when old and young. To see what kind of savings behavior is implied, we look at what kind of savings behavior moves the economy from the endowment point to the point which maximizes social welfare.

B Classical case

In this case we have:

$$\frac{V_c\left(w_2\right)}{U_c\left(w_1\right)} < 1 \tag{4.2.7}$$

At the endowment we are consuming too much when old, would like to consume more when young (w_2 is relatively large).

$$\frac{V_c\left(w_2 \downarrow\right)\uparrow}{U_c\left(w_1 \uparrow\right)\downarrow} < 1 \tag{4.2.8}$$

POSSIBILITY ONE. In this case the young would like to exchange money for goods, to consume more now. They would 'promise' to exchange goods for money when old to clear the balance sheets. Hence the young would like to go short money and hold negative balances. Remember that the young generation is born with zero balances. But remember that the old will never buy money, since they cannot consume it.[2] Hence money has no value and we just consume the endowment.

POSSIBILITY TWO: We can compensate the young for holding money by giving the young a high return.

$$\frac{P_{t+1}}{P_t} = \frac{V_c}{U_c} \tag{4.2.9}$$

We must then have $P_{t+1} < P_t$ since prices are positive. the right and left hand sides are both less than one in this case. But then prices go to zero and money has infinite value.

The classical case therefore results in money having no value and an endowment economy.

C Samuelson Case (Monetary case)

In this case we have:

$$\frac{V_c\,(w_2)}{U_c\,(w_1)} > 1 \tag{4.3.10}$$

In thise case the young would like to consume more when old. This case seems more logical, we usually think of having less income when retired, so we must save. Perhaps w_2 is near zero. Hence money has positive value and the young exchange goods to the old for money to store value. The generations are linked through the money asset.

D equality case

Note that if

$$\frac{V_c\,(w_2)}{U_c\,(w_1)} = 1 \tag{4.4.11}$$

Then the endowment is efficient and there is no need for money.

In summary money has value if and only if MRS at the endowment is less than one. Money has value because it provides a store of value and because it completes the market by providing an asset that allows agents to trade across time.

[2]One way around this is to constrain old to hold zero balances through legal means (ie no bankruptcy) and then assume time goes backwards to $-\infty$.

V Price Dynamics of the OG model

Recall that the FOC was:

$$-\frac{1}{P_t}U_c\left[w_1-\frac{\bar{M}}{P_t}\right]+\frac{1}{P_{t+1}}V_c\left[w_2+\frac{\bar{M}}{P_{t+1}}\right]=0 \tag{5.0.1}$$

A Steady State

Transform the choice variable into real balances:

$$\theta_t=\frac{\bar{M}}{P_t} \tag{5.1.2}$$

Then the FOC is:

$$-\theta_t U_c\left[w_1-\theta_t\right]+\theta_{t+1}V_c\left[w_2+\theta_{t+1}\right]=0 \tag{5.1.3}$$

The steady state is $\theta_{t+1}=\theta_t=\theta$. Clearly $\theta=0$ is one steady state where we consume the endowment. Another satisifies:

$$-U_c\left[w_1-\theta\right]+V_c\left[w_2+\theta\right]=g\left(\theta\right)=0 \tag{5.1.4}$$

Let $\theta\to w_1$ then $g\left(\theta\right)\to-\infty$ by the Inada conditions. Let $\theta\to 0$ then

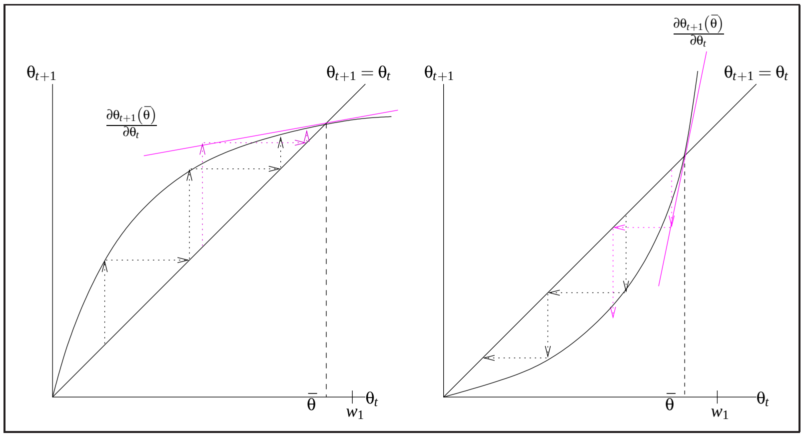
$$g\left(\theta\right)\to -U_c\left(w_1\right)+V_c\left(w_2\right)>0 \tag{5.1.5}$$

Because of the Samuelson condition. Hence there exists a steady state somewhere between 0 and w_1.

B Dynamics

We use our usual trick of linearizing around the steady state to get the dynamics.

$$\left.\frac{\partial\theta_{t+1}}{\partial\theta_t}\right|_{\theta}=-\left.\frac{\frac{\partial h}{\partial\theta_t}}{\frac{\partial h}{\partial\theta_{t+1}}}\right|_{\theta}=\frac{U_c-\theta U_{cc}}{V_c+\theta V_{cc}} \tag{5.2.6}$$



Recall that $c_1 = w_1 - \theta$ and $c_2 = w_2 + \theta$:

$$\left.\frac{\partial\theta_{t+1}}{\partial\theta_t}\right|_{\theta} = \frac{U_c\left(1 - \frac{\theta}{w_1-\theta}\frac{(w_1-\theta)U_{cc}}{U_c}\right)}{V_c\left(1 - -\frac{\theta}{w_2+\theta}\frac{(w_2+\theta)V_{cc}}{V_c}\right)} \tag{5.2.7}$$

$$\left.\frac{\partial\theta_{t+1}}{\partial\theta_t}\right|_{\theta} = \frac{U_c\left(1 + \frac{\theta}{w_1-\theta}R_u(\theta)\right)}{V_c\left(1 - \frac{\theta}{w_2+\theta}R_v(\theta)\right)} \tag{5.2.8}$$

Here R is the coefficient of relative risk aversion. Now recall that the first order condition says that $U_c = V_c$ at the monetary steady state. Hence:

$$\left.\frac{\partial\theta_{t+1}}{\partial\theta_t}\right|_{\theta} = \frac{\left(1 + \frac{\theta}{w_1-\theta}R_u(\theta)\right)}{\left(1 - \frac{\theta}{w_2+\theta}R_v(\theta)\right)} \tag{5.2.9}$$

So for stability we need:

$$\left|\frac{\left(1 + \frac{\theta}{w_1-\theta}R_u(\theta)\right)}{\left(1 - \frac{\theta}{w_2+\theta}R_v(\theta)\right)}\right| < 1 \tag{5.2.10}$$

Assuming agents are risk averse, the numerator is positive and greater than one. Hence for stability, we need the denominator to be very negative, by making R_v very big. We need

$$1 + \frac{\theta}{w_1 - \theta}R_u < -1 + \frac{\theta}{w_2 + \theta}R_v \tag{5.2.11}$$

$$R_v > \frac{2\left(w_2+\theta\right)}{\theta} + \frac{w_2+\theta}{w_1-\theta} R_u \tag{5.2.12}$$

Thus the steady state is stable if old agents are sufficiently risk averse relative to young agents.

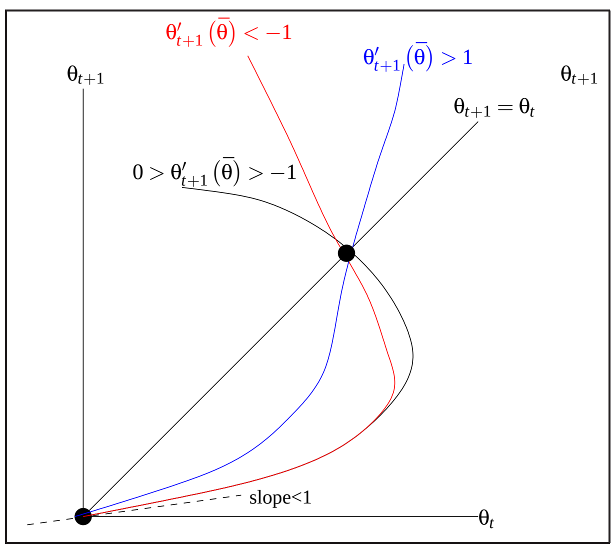
We know the sign of the derivative at the monetary steady state is either positive and greater than one or negative and that there exists a steady state at $\theta = 0$.

Let us also compute the derivative at 0. Using Equation (5.2.6), the derivative at the point $\theta = 0$ is:

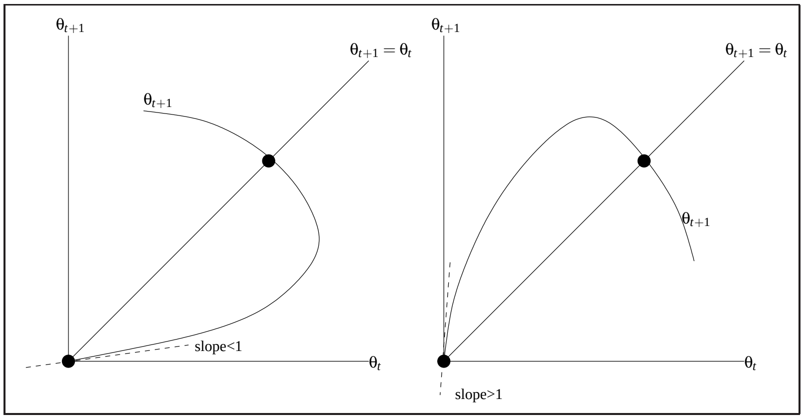
$$\left.\frac{\partial \theta_{t+1}}{\partial \theta_t}\right|_0 = -\left.\frac{\frac{\partial h}{\partial \theta_t}}{\frac{\partial h}{\partial \theta_{t+1}}}\right|_0 = \frac{U_c\left(w_1\right)}{V_c\left(w_2\right)} < 1 \tag{5.2.13}$$

Here the last inequality follows from the Samuelson condition.

Therefore, the offer curve is either upward sloping or backward bending. We have 3 possible cases:



Notice that for the backward bending cases, I have used that the derivative at the non-monetary steady state is less than 1.



Notice further that in the backward bending case, the $\theta_{t+1}(\theta_t)$ is not unique and is therefore not a function. What we have done is not technically correct, since in fact future beliefs about θ_{t+1} determine θ_t, that is, we should have $\theta_t(\theta_{t+1})$.

VI Uncertainty

Suppose agents are uncertain about FEDs monetary policy, or what the level of money supply is. We had previously assumed perfect foresight,

$$E(\theta_{t+1}) = \theta_{t+1} \tag{6.0.1}$$

But suppose there was an unobserved change in FED policy. Rational expectations theory supposes that rationality is an equilibrium condition. Over time agents will make mistakes but slowly converge to perfect foresight again as mistakes are corrected. We would like to operationalize that theory by supposing how agents react out of the REE. We will give a learning model in which agents make mistakes, but eventually improve forecasts until no more errors are made. And yet the dynamics will 'flip' and the monetary steady state may become unstable.

A An Adaptive model

Suppose now that we replace θ_{t+1} with $E(\theta_{t+1}) = \theta^e_{t+1}$. The first order condition is $h\left(\theta^e_{t+1}, \theta_t\right) = 0$. Current prices are determined by expectations of future prices.

We are assuming certainty equivalence here, or possibly putting all weight of expectations on one value. We now need a forecasting model. Start with a simple one:

$$\theta_{t+1}^{e} = \theta_{t-1} \tag{6.1.2}$$

Hence the model is "adaptive" or backward looking.

B Dynamics: Steady State

Hence the actual dynamics follow:

$$h\left(\theta_{t-1}, \theta_t\right) = 0 \tag{6.2.3}$$

Note that the steady state is unchanged in this model: the condition is still $h\left(\theta, \theta\right) = 0$, just like the last model. Forecasts are correct at the steady state:

$$\theta_{t+1}^{e} = \theta_{t-1} = \theta_{t+1} = \theta \tag{6.2.4}$$

We say that the adaptive rule "detects" the steady state. As the economy moves closer to the steady state, forecasts become more and more accurate.

Note that the dynamics reverse:

$$\left.\frac{\partial \theta_{t+1}}{\partial \theta_t}\right|_{\theta} = -\left.\frac{\frac{\partial h}{\partial \theta_t}}{\frac{\partial h}{\partial \theta_{t+1}}}\right|_{\theta} \tag{6.2.5}$$

Now becomes:

$$\left.\frac{\partial \theta_t}{\partial \theta_{t-1}}\right|_{\theta} = -\left.\frac{\frac{\partial h}{\partial \theta_{t-1}}}{\frac{\partial h}{\partial \theta_t}}\right|_{\theta} \tag{6.2.6}$$

Hence stability under perfect foresight implies instability under adaptive expectations. Similar results obtain in more complicated models where the agent is able to learn the new FED policy.

Thus we obtain these possible cases:

Rational expectations/perfect foresight:

1. This condition implies a stable cobweb (derivative greater than -1).

$$R_v > \frac{2\left(w_2 + \theta\right)}{\theta} + \frac{w_2 + \theta}{w_1 - \theta} R_u \tag{6.2.7}$$

2. This condition implies an unstable cobweb (derivative less than -1).

$$\frac{w_2 + \theta}{\theta} < R_v < \frac{2\,(w_2 + \theta)}{\theta} + \frac{w_2 + \theta}{w_1 - \theta} R_u \tag{6.2.8}$$

3. This condition implies explosive growth or stable non-monetary steady state (depending on the initial condition), (derivative is greater than 1).

$$\frac{w_2 + \theta}{\theta} > R_v \tag{6.2.9}$$

Thus the reverse dynamics under learning are:

1. This condition implies a stable monetary steady state (derivative is positive, less than 1).

$$R_v < \frac{w_2 + \theta}{\theta} \tag{6.2.10}$$

2. This condition implies a stable cobweb (derivative negative, greater than -1).

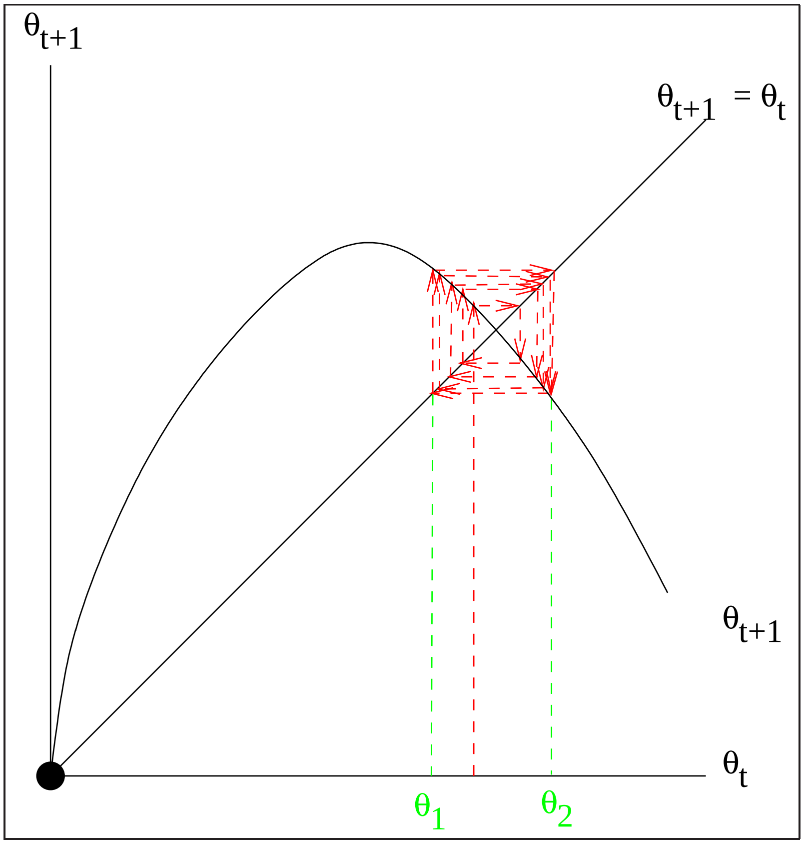
$$\frac{w_2 + \theta}{\theta} < R_v < \frac{2\,(w_2 + \theta)}{\theta} + \frac{w_2 + \theta}{w_1 - \theta} R_u \tag{6.2.11}$$

3. This condition implies an unstable cobweb, see below for what happens. (derivative greater than -1).

$$R_v > \frac{2\,(w_2 + \theta)}{\theta} + \frac{w_2 + \theta}{w_1 - \theta} R_u \tag{6.2.12}$$

C Endogenous Business Cycles

Notice that in the third case, we have a maximum and the relationship is an inverted-U. As the steady state just becomes unstable (draw with steady state derivative just less than -1), then it is easy to show that a two state cycle is stable.



Condition for stability of a two state cycle:

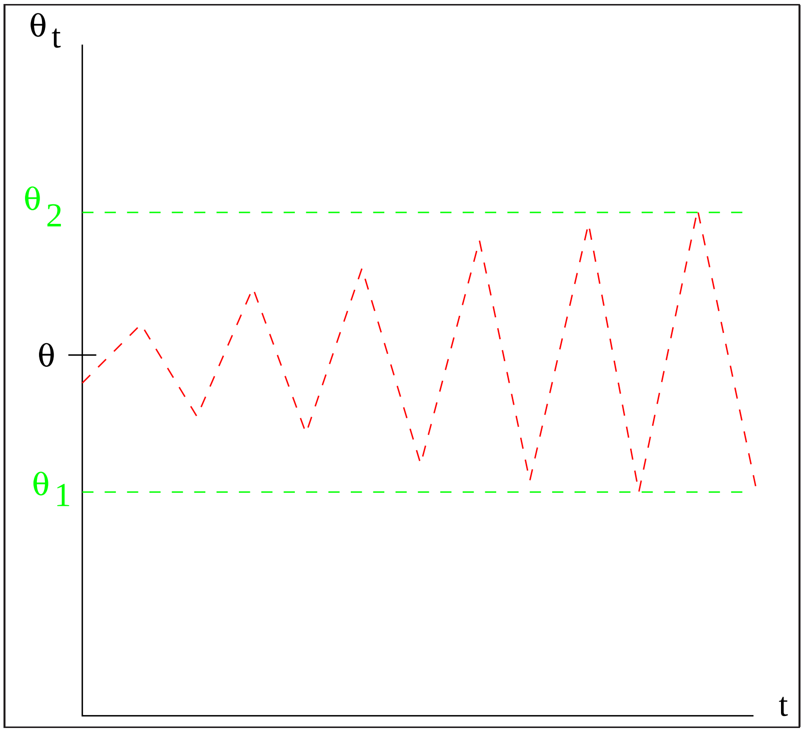
$$\left| \left.\frac{\partial\theta_t}{\partial\theta_{t-1}}\right|_{\theta_1} \left.\frac{\partial\theta_t}{\partial\theta_{t-1}}\right|_{\theta_2} \right| < 1 \tag{6.3.13}$$

Here θ_1 and θ_2 are the rest points in the above graph. These satisfy a system of two equations and two unknowns:

$$h\left(\theta_1, \theta_2\right) = 0 \tag{6.3.14}$$

$$h\left(\theta_2, \theta_1\right) = 0 \tag{6.3.15}$$

The dynamics look like this:



This is an "endogenous business cycle" (Grandmont, 1985). The economy cycles for purely endogenous reasons (generated through learning). No exogenous productivity shock is required for cyclic behavior.

D Complex Dynamics

As we increase R_v still further, the two state cycle becomes unstable and a 4 state cycle becomes stable. Then an 8 state cycle and so on. Stability of an n state cycle:

$$\left| \prod_{i=1}^{n} \frac{\partial \theta_t}{\partial \theta_{t-1}} \bigg|_{\theta_i} \right| < 1 \tag{6.4.16}$$

Now a cycle of infinite length in which θ appears random (and in fact is how random numbers are generated on computer) and never repeats is known as *Chaos*. Chaos is stable for R_v around 8-9. This is viewed as somewhat unrealistic, and so the prevailing wisdom is that business cycles are not chaos. However, Kelly, et. al. 1998 show that under imperfect competition, chaos may occur regardless of risk aversion.

E Other results of interest

Kelly (1992) shows that stability under adaptive expectations is equivalent to stability for more complex learning models. For example:

$$\theta^e_{t+1} = \alpha\theta_{t-1} \tag{6.5.17}$$

Here the parameter α can be updated as new information comes in, perhaps through a OLS regression.

Sunspot equilibrium: add uncertainty to forecast:

$$\theta^e_{t+1} = \alpha\theta_{t-1} + (1-\alpha)\,\epsilon_t \tag{6.5.18}$$

Here ϵ_t is extrinsic uncertainty or a sunspot. It only matters because people believe it does. Stability of perfect foresight implies existence of sunspots, according to Azariadis (1981). Woodford (1989) establishes stability under learning.

DISTORTIONARY TAXES

I Introduction

A Optimal tax package

The most obvious question to ask is actually a public finance question: what is the optimal tax package? The government needs to finance government spending on public goods and redistribution. To finance these expenditures, should the government tax capital income, wage income, total income, consumption, the inflation tax, or some combination? All non-lump sum taxes distort decisions away from those that maximize welfare. Which result in the least distortion and therefore the most welfare?

Put more simply, consider the case where someone changes behavior strictly to avoid paying a tax. For example, the person works less and leisures more since wages are being taxed. The resulting change in behavior lowers the welfare of the individual, since they are making a decision they would not choose in absence of the tax. Conversely, the government is also worse off since it gains no revenue when households change behavior to avoid the tax. Thus an efficient tax minimizes such distortions to decision making.

This is both a qualitative and a quantitative issue. It may be that one tax is better than another, but it is an open question as to whether or not there is much difference quantitatively. Many who argue in government for higher taxes, argue that the magnitude of distortions is small. Many will work 40 hours per week regardless of the tax rate.

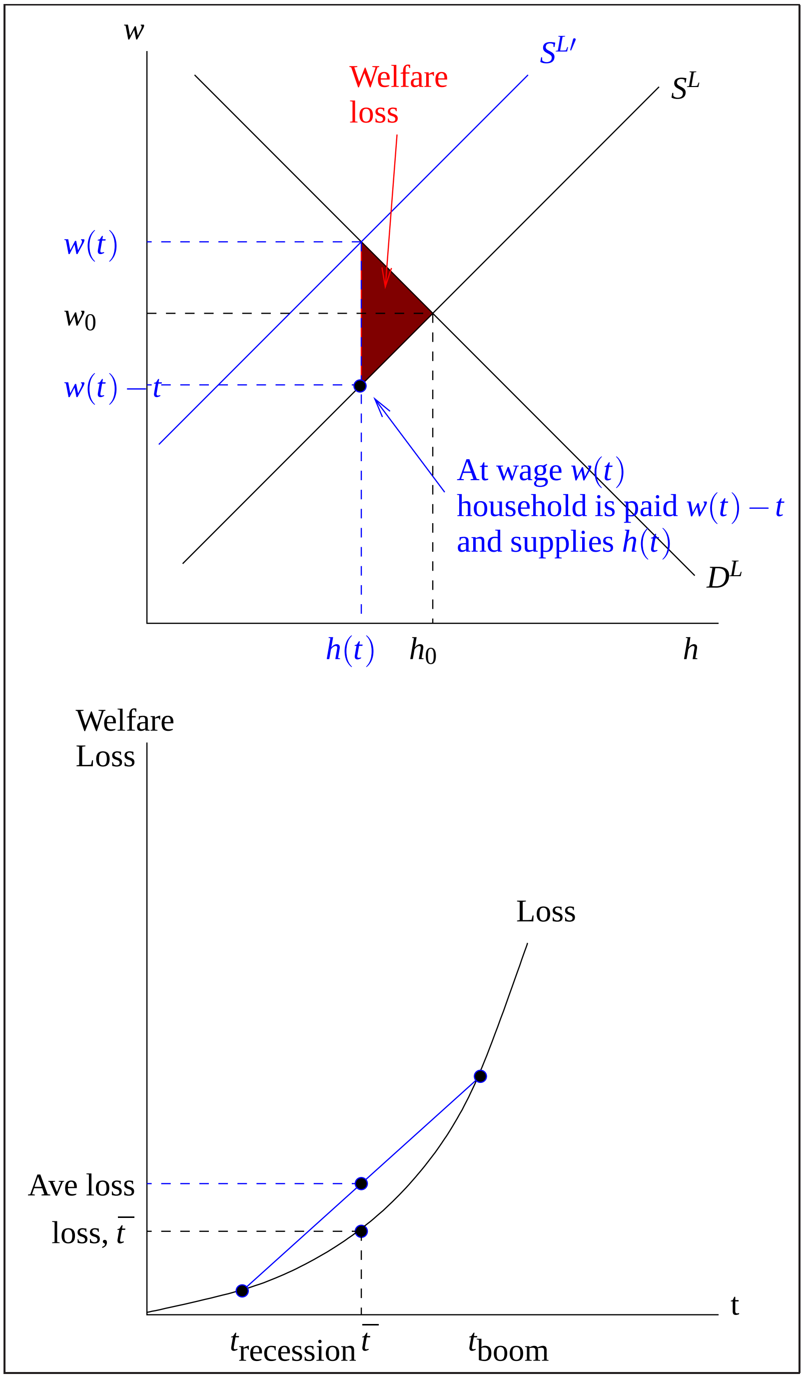
We will persue here largely from the quantitative side. In 603, we will look more closely at optimal tax packages, although we can derive some results here as well. We will compare alternative plans with something related to the current tax system and see if the welfare effects are big potatoes or small potatoes.

B Business Cycle Effects

Second, we would like to know how taxes affect the business cycle. How does changing the investment income tax affect the volatility of investment? Wage income taxes are subject to frequent changes. Could these be a source of variation in hours and GDP which the RBC model does not account for?

Optimal tax policy can vary over the business cycle as well. We like smooth consumption, so the optimal tax policy would presumably act in this manner. In a boom, we could increase taxes and run a surplus, while in a recession, reduce taxes and run a deficit to reduce business cycle variation. A problem with this idea is that distortions are typically convex. Meaning if we double the tax rate we more than double the welfare loss. This argues for constant taxes, which would have higher welfare than taxes which are higher and then lower. Thus the optimal tax presumably trades off these two effects and eliminates some but not all of the business cycle variation.

The following graph shows the argument for constant taxes.



We see that in a simple partial equilibrium framework from undergrad, that the dead-weight welfare loss from the tax is the area of the triangle. The area is:

$$\text{welfare loss } = \frac{1}{2} \cdot \text{ base } \cdot \text{ height} = \frac{1}{2} t \left(h_0 - h\left(t\right)\right). \tag{1.2.1}$$

The loss is convex in t as long as h_{tt} is not to positive.

II Assumptions

Start with the RBC model.

A Taxes

We model taxes as autoregressive processes, similar to money growth and productivity. A known tax change in the future affects decisions today and throughout all periods, making it difficult to write a recursive problem. However, here taxes are random and thus cannot be completely anticipated. We thus have "tax shocks," which are simply unanticipated changes in the tax code.

With that in mind, consider first a tax on wage income $\tau_{h,t}$ and a tax $\tau_{k,t}$ on capital income. These taxes may be correlated with the productivity shocks, each other, or government spending and follow a first order Markov process:

$$\begin{bmatrix} z_{t+1} \\ g_{t+1} \\ \tau_{h,t+1} \\ \tau_{k,t+1} \end{bmatrix} \equiv T_{t+1} = (I - \rho_T)\,\hat{T} + \rho_T T_t + \epsilon_{t+1} \tag{2.1.1}$$

It is easy to see that the certainty equivalence steady state is:

$$\bar{T} = \hat{T}, \tag{2.1.2}$$

with $\hat{z} = 0$. The shock is distributed as:

$$\epsilon_{t+1} \sim N\left(0, \Sigma\right) \tag{2.1.3}$$

Here we have to set the variance low enough so that we don't see negative taxes. An alternative that is probably better is to assume the taxes are distributed log-normally, while the productivity shock is distributed normally as before. More realism would require that tax rates tend to be constant for short periods of time (less than a year, say).

The assumption here is that all taxes are expected to revert towards $\hat{T}$. Any deviation constitutes an unanticpated tax change, or shock. But in reality some tax shocks are essentially permanent, and are therefore changes in $\hat{T}$.

Clearly $\rho_T = [\rho_z, \rho_g, \rho_h, \rho_k]$. Policy variables here include ρ_h, ρ_g, and ρ_k, $\bar{g}$, $\bar{\tau}_h$, and $\bar{\tau}_k$, and Σ.

B Government Budget

We add in distortionary taxes so that:

$$g_t + TR_t = \tau_{h,t} w_t H_t + \tau_{k,t} \left(r_t - \delta\right) K_t \tag{2.2.4}$$

Notice that we still have a government transfer which is set to balance the government constraint. The actual tax rates are not policies, only the parameters of the distribution are policies. Note that we can alternatively use bonds rather than a lump sum transfer to balance the government constraint. By Ricardian equivalence, the equilibrium will not change.

We can view the set up as the government having unpredictable spending needs and/or tax cuts each period. It then computes it's budget balance, and borrows whatever is needed so that the government constraint holds.

C Preferences

I will add a preference for public goods to the model:

$$U = u\left(c_t, g_t, 1 - h_t\right) \tag{2.3.5}$$

McGratten (1994) uses $u\left(c_t + \pi g_t, 1 - h_t\right)$ and argues that the indivisible labor assumption will add too much variance to aggregate hours when combined with variance induced by random tax changes.

D household budget constraint

The household budget constraint is now:

$$r_t k_t - \tau_{k,t} \left(r_t - \delta\right) k_t + w_t h_t - \tau_{h,t} w_t h_t + TR_t = c_t + k_{t+1} - \left(1 - \delta\right) k_t \tag{2.4.6}$$

Substituting in for the government transfer and firm first order conditions and simplifying gives:

$$\left[e^{z_t} F_k\left(K_t, H_t\right)-\delta\right] \cdot\left[k_t-\tau_{k,t}\left(k_t-K_t\right)\right]+e^{z_t} F_h\left(K_t, H_t\right)\left(h_t-\tau_{h,t}\left(h_t-H_t\right)\right)$$
$$=c_t+k_{t+1}-k_t+g_t \tag{2.4.7}$$

III Value Function and Equilibrium

The value function is:

$$\begin{aligned} v(T, k, K) &= \max_{k', h}\left\{u\left[\left(e^z F_k(K, H)-\delta\right)\left(k-\tau_k(k-K)\right)+e^z F_h(K, H)\left(h-\tau_h(h-H)\right)\right.\right. \\ &- \left.\left. k'+k-g, g, 1-h\right]+\beta \mathrm{E}\left[v\left(\left(I-\rho_T\right) \hat{T}+\rho_T T+\epsilon', k', K'\right)\right]\right\} \end{aligned} \tag{3.0.8}$$

Here we have an equilibrium restriction that $k=K$ implies $k'=K'$ and $h=H$.

Definition 31 *An* Equilibrium *given a set of individual state variables $s=[T, k, K]$, a set of aggregate state variables $S=[T, K, K]$, and government policies $\left[\rho_T, \bar{T}, \Sigma\right]$ is a set of individual decisions $[k', h, c]$, a set of aggregate decisions $[C, K', H]$, a set of prices $[r, w]$, a government decision TR, and a value function v, such that the household first order conditions hold (2 equations), the household budget constraint holds, the resource constraint holds, the household first order conditions hold at equilibrium (2), the firm first order conditions hold (2), the government budget constraint holds, and the Bellman's equation holds.*

So we have 10 equations for 10 unknowns.

IV First order conditions and envelope equations

A First order conditions

We have:

$$u_c(c, g, 1-h)=\beta \mathrm{E}\left[v_k\left(s'\right)\right] \tag{4.1.1}$$

$$\left(1-\tau_h\right) e^z F_h(K, H)\, u_c(c, g, 1-h)=u_l(c, g, 1-h) \tag{4.1.2}$$

In equilibrium, these become:

$$u_c(C, g, 1-H) = \beta \mathrm{E}\left[v_k(S')\right] \tag{4.1.3}$$

The investment tax does not affect the price of consumption, so the left hand side is unchanged. We will see the effect of the investment tax on the right hand side through the envelope equation.

$$(1-\tau_h) e^z F_h(K,H) u_c(C, g, 1-H) = u_l(C, g, 1-H) \tag{4.1.4}$$

The value of working falls with the tax, but leisure is not taxed and is therefore unaffected.

$$C = e^z F(K,H) + (1-\delta) K - K' \tag{4.1.5}$$

No change to the aggregate resources, except through changes in hours.

B Envelope equation

We have:

$$v_k(s) = u_c(c, g, 1-h)\left[(1-\tau_k)\left(e^z F_k(K,H) - \delta\right) + 1\right] \tag{4.2.6}$$

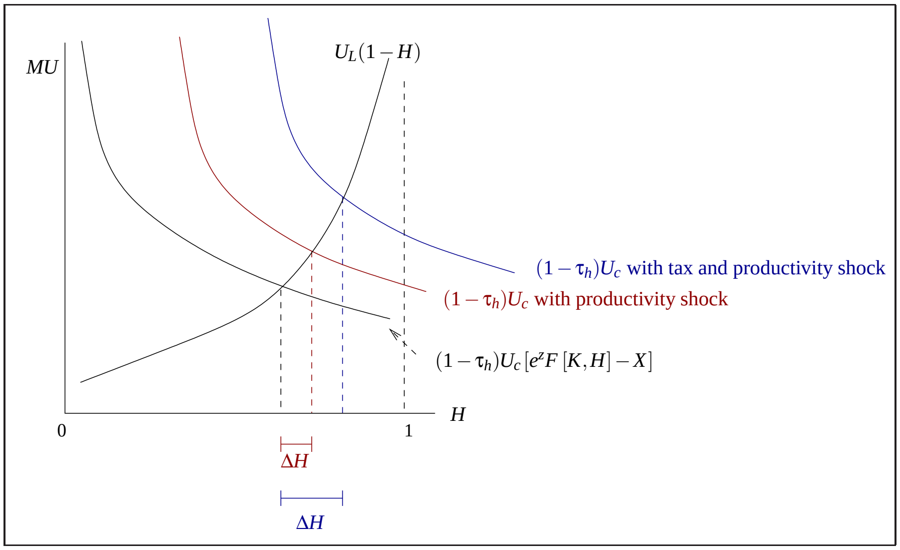
Then net return on capital falls with the investment tax. In equilibrium:

$$v_k(S) = u_c(C, g, 1-H)\left[(1-\tau_k)\left(e^z F_k(K,H) - \delta\right) + 1\right] \tag{4.2.7}$$

C Hours worked volatility and wage income taxes

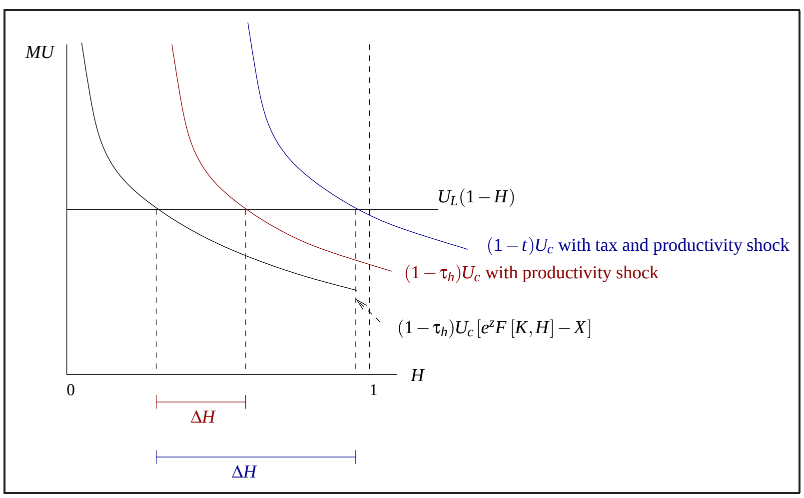
Here is an analysis of the effect of the wage income tax on hours. Suppose for simplicity that utility is separable in leisure and that K' is unaffected by τ_h. Then we have:

$$(1-\tau_h) e^z F_h(K,H) u_c\left(e^z F(K,H) - X, g\right) = u_l(1-H) \tag{4.3.8}$$



It is immediately clear that the right hand side is independent of z, while the left hand side is increasing in z. Therefore H is procyclical and variation in productivity causes variation in hours. However, calibrated values of u_l using log utility result in a fairly steep utility of leisure curve, generating little variation in H relative to the data (about half the variation in output, the red line in the above graph). Now let us add tax variation. If the correlation between shocks to productivity and tax rates are not too positive, we see the variation in the tax shock adding to the variation in H, making for a more realistic variation in H.

Compare with the Hansen indivisible labor assumption, where we have variation along the extensive margin and therefore much bigger changes in hours for a productivity shock.



Here the tax shock adds too much variation unless the tax and productivity shock are very highly correlated. We will see after calibrating the model and looking at the moments that the tax model generally does better than the indivisble labor model (see Mcgratten 1994).

V Optimality

A Non-Optimality of the tax-distorted economy

Consider the equilibrium equations of the model with distortionary taxes versus the socially optimal RBC model with exogenous government spending:

$$u_c\left(C, g, 1-H\right) = \beta \mathrm{E}\left[v_k\left(T', K', K'\right)\right] \tag{5.1.1}$$

$$\left(1-\tau_h\right) e^z F_h\left(K,H\right) u_c\left(C, g, 1-H\right) = u_l\left(C, g, 1-H\right) \tag{5.1.2}$$

$$v_k\left(T,K,K\right) = u_c\left(C, g, 1-H\right)\left[\left(1-\tau_k\right)\left(e^z F_k\left(K, H\right) - \delta\right) + 1\right] \tag{5.1.3}$$

$$C = e^z F\left(K,H\right) + \left(1-\delta\right) K - K' - g \tag{5.1.4}$$

The allocations for the social planning problem are determined via:

$$u_c\left(C, g, 1-H\right) = \beta \mathrm{E}\left[w_k\left(K'\right)\right] \tag{5.1.5}$$

$$e^z F_h\left(K,H\right) u_c\left(C, g, 1-H\right) = u_l\left(C, g, 1-H\right) \tag{5.1.6}$$

$$w_k\left(K\right) = u_c\left(C, g, 1-H\right)\left[\left(e^z F_k\left(K, H\right) - \delta\right) + 1\right] \tag{5.1.7}$$

$$C = e^z F\left(K,H\right) + \left(1-\delta\right) K - K' - g \tag{5.1.8}$$

Inspection of equations (5.1.1)-(5.1.4) versus (5.1.5)-(5.1.8) reveals that $\tau_h = \tau_k = 0$ for all t is optimal (this could be achieved by for example setting $\hat{T} = 0$ and $\Sigma = 0$). The taxes are distortionary in that they distort decisions away from the social optimum.[3]

B Psuedo Planning Problem

Can we find an alternative set of preferences for which the allocations of the competitive equilibrium are optimal? Let $\tau_k = 0$ and $u = u(c,g) + J(l)$ be separable in leisure and consider the following psuedo planning problem (PSP):

$$\hat{w}(T,K) = \max_{K',H} \left\{ u\left[e^z F(K,H) + (1-\delta)K - K' - g, g \right] + \right. \tag{5.2.9}$$

$$\left. \frac{1}{1-\tau_h} J(1-H) + \beta \mathrm{E}\left[\hat{w}(T',K') \right] \right\} \tag{5.2.10}$$

Now the first order conditions and envelopes for the competitive economy match the PSP when we define $v_k = \hat{w}_k$. We can solve the PSP, which has no prices and is therefore easier to work with and get the same allocations as the competitive economy. We learn from this excercise that a labor tax is equivalent to increased preferences for leisure. I invite you to find a PSP for the case where $\tau_k > 0$ and the capital tax is on the gross, rather than the net return $(\tau_k (r_t + (1-\delta)) k_t)$.

C Digression on Supermodularity

Here is a comparative static theorem developed by Topkis (1978) and presented in Milgrom and Shannon (1992). By using supermodularity, we avoid a lot of extra assumptions needed by the implicit function theorem.

Definition 32 *A twice differentiable function is supermodular if and only if all of the cross partials are positive.*

That is:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} > 0 \ , \ \frac{\partial^2 f}{\partial x_i' \partial x_j'} > 0 \ , \ \frac{\partial^2 f}{\partial x_i' \partial x_j} > 0 \tag{5.3.11}$$

What is nice about this theorem is that it is easily reversed, as submodular functions are non-increasing.

[3] I am assuming the planner cannot affect the exogenous process for g. If we suppose the planner could, we would have an extra first order condition that $u_c = u_g$, which is solved at g^*. The competitive equilibrium allocations would then be equivalent to the planning problem on this extra dimension if for example $\bar{g} = g^*$ and $\Sigma = 0$.

Definition 33 *Let x,$y \in \Gamma$, where Γ is a vector space. If $x \vee y \in \Gamma$ and $x \wedge y \in \Gamma$ then Γ is a lattice.*

Here $\vee$ is the component wise maximum and $\wedge$ is the component wise minimum. That is, let $x = [1{,}2]$ and $y = [0{,}3]$, then $x \vee y = [1{,}3]$ and $x \wedge y = [0{,}2]$.

THEOREM 12 *Let $f(x, x')$ be supermodular in $[x{,}x']$ and $\Gamma(x)$ be a lattice. Then:*

$$\max_{x' \in \Gamma(x)} \{f(x, x')\} \tag{5.3.12}$$

has $x'^{*}(x)$ non decreasing in x.

Most of the examples we will consider are unconstrained so we won't have to worry about the lattice part. This theorem can be applied directly if we can show super modularity of the objective without proving supermodularity of v. Consider for example, the PSP (5.2.10). Define $x = 1 - \tau_h$ and define f via:

$$\begin{aligned} f &= u\left[e^{z}F(K{,}H) + (1-\delta)K - K' - g, g\right] + \frac{1}{x}J(1-H) + \\ &\quad \beta \mathrm{E}\left[\hat{w}\left((1-\rho_T)\bar{\tau}_h + \rho_T(1-x) + \epsilon', K'\right)\right] \end{aligned} \tag{5.3.13}$$

Note that for simplicity I have ignored that T includes other variables besides τ_h. Now to apply supermodularity, let $x = 1 - \tau$ so that that:

$$f_{K',H} = -u_{cc}(C{,}g)\, e^{z}F_h(K{,}H) > 0 \tag{5.3.14}$$

$$f_{H,x} = \frac{1}{(x)^2}J_l(1-H) > 0 \tag{5.3.15}$$

$$f_{K',x} = -\beta\rho_T \mathrm{E}\left[\hat{w}_{k,\tau}(T', K')\right] \tag{5.3.16}$$

Notice that supermodularity is immediate for $\rho_T = 0$. Hence we can say that for $\rho_T = 0$, both H and K' are non-decreasing in x and therefore both H and K' are non-increasing in τ_h.

Now the case where $0 < \rho_T < 1$ is more complicated, since we must show that w has a negative cross partial derivative (*sub modular*). Assume w_0 is submodular in k, x. Then we know that $K'(K, z, w_0)$ and $H(K, z, w_0)$ are increasing in x, by applying the supermodularity theorem by using equations (5.3.14)-(5.3.16).

Now from the envelope:

$$w_{K,x} = -u_{cc}\left(C,g\right)\left(e^{z}F_{k}\left(K,H\right)+1-\delta\right)\frac{\partial K'\left(w_0\right)}{\partial x}+\frac{\partial H\left(w_0\right)}{\partial x}\cdot$$
$$\left[u_{cc}\left(C,g\right)\left(e^{z}F_{k}\left(K,H\right)+1-\delta\right)e^{z}F_{h}\left(K,H\right)+u_{c}\left(C,g\right)e^{z}F_{kh}\left(K,H\right)\right] \tag{5.3.17}$$

The first term is clearly positive. The second term is positive if and only if:

$$u_{cc}\left(C,g\right)\left(e^{z}F_{k}\left(K,H\right)+1-\delta\right)e^{z}F_{h}\left(K,H\right)+u_{c}\left(C,g\right)e^{z}F_{kh}\left(K,H\right)>0 \tag{5.3.18}$$

$$R<\frac{CF_{kh}\left(K,H\right)}{F_{h}\left(K,H\right)\left(e^{z}F_{k}\left(K,H\right)+1-\delta\right)} \tag{5.3.19}$$

Here R is the coefficient of risk aversion, with a large R corresponding to more desire for smooth consumption. So we see that current labor taxes rise causing a disincentive to work, but also causing an increase in future taxes. The household is poorer in the future and so may want to work harder to smooth consumption if preferences for smooth consumption are sufficiently strong. Still this condition is on the primitives of the model and is easy to verify in practice. Thus given the above condition w_1 is sub modular and so by induction w is sub modular and therefore K' and H are non-increasing in τ_h.

Here is a supermodularity theorem for dynamic programming. Let:

$$v\left(s\right)=\max_{s'\in\Gamma(s)}\left\{u\left(s,s'\right)+\beta v\left(s'\right)\right\} \tag{5.3.20}$$

Then:

THEOREM 13 *Assume:*

1. $u\left(s,s'\right)$ *is supermodular as a function of* $[s,s']$.

2. $\Gamma\left(s\right)$ *is a lattice.*

Then v is supermodular in s and therefore $s'\left(s\right)$ is non-decreasing in s.

VI Steady State

Holding all variables constant at their certainty equivalence steady state values, we see that:

$$u_{c}\left(C,g,1-H\right)=\beta v_{k}\left(S\right)=u_{c}\left(C,g,1-H\right)\left[\left(1-\tau_{k}\right)\left(F_{k}\left(K,H\right)-\delta\right)+1\right] \tag{6.0.1}$$

$$\rho = (1 - \tau_k)\left(F_k(K, H) - \delta\right) \tag{6.0.2}$$

$$(1 - \tau_h) F_h(K,H)\, u_c(C, g, 1 - H) = u_l(C, g, 1 - H) \tag{6.0.3}$$

$$C = F(K, H) - \delta K - g \tag{6.0.4}$$

It is straightforward to show via the implicit function theorem (for separable utility) that:

$$\frac{\partial K}{\partial \tau_k} < 0\ ,\ \frac{\partial H}{\partial \tau_k} < 0\ \Leftrightarrow\ (5.3.19)\ ,\ \frac{\partial Y}{\partial \tau_k} < 0 \tag{6.0.5}$$

The tax decreases the steady state capital stock, making the country poorer in the long run.

For the labor tax, we can show via the implicit function theorem (for separable utility) that:

$$\frac{\partial K}{\partial \tau_h} < 0\ ,\ \frac{\partial H}{\partial \tau_h} < 0\ ,\ \frac{\partial Y}{\partial \tau_h} < 0 \tag{6.0.6}$$

The labor tax also makes the country poorer in the long run.

VII Calibration

A Consistent Measurements

1 Data on Taxes

One may simply assemble data on average tax rates via:[4]

$$\tau_{k,t} = \frac{\text{capital taxes paid}_t}{r_t K_t} \tag{7.1.1}$$

$$\tau_{h,t} = \frac{\text{wage taxes paid}_t}{w_t H_t L_t} \tag{7.1.2}$$

Cooley and Hansen (1989) report several sources for the tax data. Note the importance of the double taxation of capital income. We have first a tax on firm profits, and then a tax

[4] Of course, what we would really like is marginal tax rates, not average. This is difficult since marginal rates vary by person.

on dividends and capital gains to the household. Household income is thus:

$$\text{household income } = (1 - \tau_{h,t})\, w_t h_t + (1 - \tau_{d,t})\,(1 - \tau_{p,t})\, r_t k_t + \ldots \tag{7.1.3}$$

Here $\tau_{p,t}$ is the corporate profits tax. The goverment first removes $\tau_{p,t}$ portion of the profits earned at the firm, and then taxes $\tau_{d,t}$ fraction of the remainder as a dividend tax.

Note that capital income tax rates have been falling largely due to the increase in 401Ks, IRAs, etc. which postpone the dividend/capital gains tax. One problem here is that we are using average tax rates rather than marginal. For example, the marginal corporate rate is 35% and yet the average tax rate is only about 22%. When deciding whether or not to invest the next dollar, households and firms focus on the marginal rate.

2 Alternative capital measures

Consumer durable investment is capital tax free. The property tax rate on housing differs from the capital tax on other investments. Government investment is also tax free. Therefore we could either (a) exclude government and consumer durables and use fixed non-residential investment for x and GDP of coporate business for y or (b) include these items and use the lower average tax rate on capital.

B Calibrated Values

1 Tax Rates

The tax rate of course varies each period, but with a coporate rate of 0.31 and a capital gains rate of 0.28, we might see something around:

$$1 - \tau_k = (1 - 0.31)\,(1 - 0.28) \;\; \rightarrow \tau_k = 0.5 \tag{7.2.4}$$

The US has one of the highest capital tax rates in the world, compare to Ireland's 10-15%.

For wage taxes, the average is about $\tau_{h,t} = 0.23$. Welfare loss increases with the square of the tax rate, so we should also see large welfare losses from τ_k simply because it is so big.

2 Calibrate capital share

Note that the tax rates change the steady state and so we need to re-calibrate all the variables. For example, the steady state for K is now:

$$\rho = (1 - \tau_k)\,(F_k\,(K, H) - \delta) \tag{7.2.5}$$

Reported capital income is before taxes, so reported capital income is still equal to:

$$I = rK \tag{7.2.6}$$

$$I = ((r - \delta) + \delta) K \tag{7.2.7}$$

$$r - \delta = \frac{I_p - DEP}{K_p} \tag{7.2.8}$$

Here DEP is reported total depreciation. This gives us the pre-tax interest rate. We can then calculate private capital income as before, and government and durable capital income is $r - \delta K_i$, $i = G{,}D$. With Cobb-Douglas production, we have:

$$I = \theta Y \ , \theta = I/Y \tag{7.2.9}$$

So the capital share is unchanged. The rate of time preference will differ, however:

$$\rho K = (1 - \tau_k)(\theta Y - \delta K) \tag{7.2.10}$$

$$\rho = (1 - \tau_k)\left(\theta \frac{Y}{K} - \delta\right) \tag{7.2.11}$$

So the rate of time preference is smaller than the model without taxes. We care more about the future than we previously thought, as the low savings is the result of taxes, not heavy discounting of the future.

3 Consumption share

The first order condition for hours at the steady state is now:

$$\alpha \frac{1}{C} F_h(K{,}H)(1 - \tau_h) = (1 - \alpha)\frac{1}{1 - H} \tag{7.2.12}$$

$$\alpha \frac{Y}{C}(1 - \gamma)(1 - \tau_h) = (1 - \alpha)\frac{H}{1 - H} \tag{7.2.13}$$

$$\alpha = \frac{1}{1 + \frac{Y}{C}\frac{1-H}{H}(1 - \gamma)(1 - \tau_h)} \tag{7.2.14}$$

So the consumption share is a little larger.

VIII Results

A Volatility

Tax shocks increase the model's volatility of all variables.

Statistic	Data	RBC	Indivisible Labor	Taxes	both
output	1.81	1.49	1.65	1.83	2.23
Consumption	0.91	1.01	1.05	0.98	1.03
Investment	5.11	4.86	5.65	5.53	7.24
Capital	0.45	0.39	0.45	0.44	0.57
Hours	1.52	0.37	0.63	1.31	2.03
Productivity	1.32	1.14	1.06	1.11	1.10
Correlation Hours and Productivity	-0.195	0.93	0.91	0.14	-.08

Table 2: Data from McGratten (1994), tables 2-3. Note that parameters are calibrated to the economy with taxes, and thus differ from the standard RBC results.

So clearly the model with taxes and indivisible labor has too much volatility on all dimensions. The assumption of indivisible labor, that all variation in hours is on the external margin (versus the data where 2/3 of the variation is external), is probably too strong. Conversely, the model with variable taxes only does extremely well across many dimensions.

Notice too that the model with taxes explains about 100% of the cycle, whereas the model without taxes explains about 80% of the variation in the cycle. Thus we can view tax shocks as explaining the remaining 20%. Recall that performing the same procedure with monetary shocks gave no extra variation.

Here are some similar results from Greenwood and Huffman (1991):

Statistic	Data	Constant taxes	zero taxes
output	3.5	3.5	2.9
Consumption	2.2	1.7	1.2
Investment	10.5	20.0	NA
Hours	2.1	2.0	1.4
Productivity	2.2	2.0	NA

Table 3: Data from Greenwood and Huffman (1991), table 1. Model includes endogenous capacity utilization. Note that parameters are calibrated to the economy with taxes, and thus differ from the standard RBC results.

The model and the detrending methods are different but the results that taxes (even constant taxes) add to business cycle variation remains true.

B Welfare losses

Here are some results from Cooley and Hansen (1991).

Policy	τ_h	τ_k	$-TR$	Welfare cost
Current US Policy	0.23	0.5	0.0	13.3%
Replace all taxes with lump sum	0.0	0.0	0.263	0%
Replace labor tax with lump sum	0.0	0.5	0.142	8.1%
Replace capital tax with lump sum	0.23	0.0	0.065	4.07%
Replace capital tax with labor tax	0.34	0.0	0.0	7.77%

Table 4: Steady state welfare consequences of alternative policies. Welfare cost is a percent of GNP. Data from Cooley and Hansen (1991), table I.

So we see that capital taxes cause the highest welfare loss, both because they are high to begin with, and because they involve a dynamic distortion of a declining tax base.

For Greenwood and Huffman (1991), we have:

Policy	Welfare gain
Cut capital income tax rate from 35% to 25%	4.05%
Cut labor income tax rate from 35% to 25%	3.10%
Increase investment tax credit from 7% to 14%	1.91%

Table 5: Steady state welfare consequences of alternative policies. Welfare gain is as a percent of GNP. Data from Greenwood and Huffman (1991), table 2.

For reference, a 4% gain in GNP is about half a trillion dollars.

C Variation in tax rates across the cycle

What is the welfare gain from stabilizing the cycle? For Greenwood and Huffman (1991), we have:

Study	Reduction in variance GNP	Welfare gain
Greenwood and Huffman (1991)	50%	0.67%
Lucas (1987)	100%	0.38%

Table 6: Steady state welfare consequences of business cycle stabilization. Welfare gain is as a percent of GNP. Data from Greenwood and Huffman (1991), table 1.

So we see that business cycle stabilization is much more of a second order concern relative to determining the optimal tax package. Nonetheless, undergraduates and the popular press focus exclusively on stabilization.

Note that Greenwood and Huffman make the unrealistic assumption of $\sigma = 15$, which makes their consumption volatility way too low (table 2). By assuming very strong preferences for smooth consumption, the value of stabilization increases. But even with $\sigma = 15$, it is quite low.

The number by Lucas seems most accurate. This led Lucas to proclaim that a basic problem of macroeconomics had been solved. Notice that the business cycle is considerably more stable than prior to WWII. So we are more saying that further stabilization has little returns.

D Using the actual productivity shocks

Consider the following exercise. For the time series of US GDP, compute the implied productivity shocks if the model was exactly correct, as calibrated.

$$Y_t = e^{z_t} K_t^{\gamma} h_t^{1-\gamma}, \tag{8.4.1}$$

$$z_t = \log Y_t - \gamma \log K_t - (1-\gamma) \log h_t. \tag{8.4.2}$$

We can do a similar procedure with the other shocks.

The model will now predict GDP perfectly, by construction. But we can look at other statistics over time, and see how the model economy with specific productivity shocks performs.

Chen Imrohoroglu, and Imrohoroglu (2008) perform such an analysis. The model predicts the savings rate quite well, except during the recessions of 1981 and 2001. The after tax interest rate is near perfect, except the late 1990s where we can see a clear bubble. One big puzzle in the literature is the high hours in the data in the late 1990s internet bubble years.