REAL BUSINESS CYCLE MODELS

The optimal growth model says that less developed countries should grow relatively fast, converging to a constant growth rate, which we call the balanced growth path. However, the growth rate of developed countries is in fact not constant. Instead we observe fluctuations around the balanced growth path, which are the booms and recessions that characterize the business cycle. There are many questions we have about the business cycle.

- How do we measure the cycle? How large a deviation from balanced growth constitutes a recession?
- How does variable X co-move with the business cycle?

  Definition 17 *A variable X is PROCYCLICAL if X is positively correlated with the business cycle, $Y - Y_{trend}$, where Y is real GDP per capita.*

  Definition 18 *A variable X is COUNTERCYCLICAL if X is negatively correlated with the business cycle, $Y - Y_{trend}$, where Y is real GDP per capita.*

- Is X more or less variable than the cycle?
- What is the effect of policy variables such as monetary variables or fiscal variables on the business cycle?
- What is the duration of the business cycle? What affects its duration?
- What causes business cycles?

Historically, the pioneers of the real business cycle theory were Mitchell, Kuznets, and Mills. Mitchell studied co-movements of the business cycle with various aggregate variables, looking for leading indicators. Mills asked whether prices were driven by demand or supply. Kuznets looked at the time path of growth. Also Frisch (1933) proposed that business cycles where the result of slow propogation of shocks through the economy. Schumpeter noticed that business cycles arose with the onset of the industrial revolution.

At this point, the depression came on the scene and Keynes proposed a demand-driven business cycle. The model was static, and business cycles were driven by changes in investment (animal spirits) or government policy. This continued until the 1970s, with the work of Lucas. Lucas showed that the government policy has less of an effect on the business cycle than previously thought, especially if households anticipate what the government policy is.

In 1982, Kydland and Prescott proposed the first modern real business cycle model. Called real because expectations were deemphasized. Kydland and Prescott emphasized the role of productivity changes, rather than monetary policy, as a cause of business cycle fluctuations.

RBC MODEL

I Assumptions

Maintain all neoclassical optimal growth assumptions. But add the following.

A Preferences

Suppose households get utility from leisure as well as consumption. Let l_t be the fraction of non-personal (personal: eating, sleeping, flossing, etc.) hours spent on leisure. h_t will be the fraction spent working. We assume:

1. $h_t + l_t = 1$. Normalize so that each household has 1 unit of time. h is the faction of time spent working.

2. $u = u(C_t, l_t)$

3. u is C^2 in l, and $U_l > 0$, $U_{ll} < 0$.

4. The Inada conditions hold: $U_l(c{,}0) = \infty$ and $U_l(c, 1) = 0$.

B Production

Two production functions will be used.

1. Continuous shock:

$$Y_t = e^{z_t} F(K_t, h_t L_t) \tag{1.2.1}$$

   Note that $h_t L_t$ is a measure of total time worked by the labor force. z_t is a stochastic shock:

$$z_{t+1} = \rho z_t + \epsilon_{t+1} \tag{1.2.2}$$

$$\epsilon_{t+1} \quad \text{iid} \tag{1.2.3}$$

Here $0 \leq \rho < 1$, and we will consider the case of $\rho = 1$ later. z_0 is given. Current and past values of z_t are known, but future values are unknown. Example: for the case of $\rho = 0$ and $\epsilon \sim N(0, \sigma_\epsilon^2)$ is the set up for Brock and Miriman (1972) JET.

2. Discrete shock:

$$Y_t = z_t F(K_t, h_t L_t) \tag{1.2.4}$$

Here z_t follows a discrete Markov process. Let $z_t \in \{z_1, \ldots, z_m\}$ with $z_1 < \ldots < z_m$. Then we write:

$$\text{Prob}(z_{t+1}|z_t) = \Pi = \begin{bmatrix} \pi_{11} & \ldots & \pi_{1m} \\ \vdots & \ddots & \vdots \\ \pi_{m1} & \ldots & \pi_{mm} \end{bmatrix} \tag{1.2.5}$$

Here COLS are probabilities of z_{t+1} and ROWS are conditional on z_t. So

$$\text{Prob}(z_{t+1} = z_2 | z_t = z_3) = \pi_{32} \tag{1.2.6}$$

Hence the rows of Π sum to one. We usually will assign Π in such a way that if in a high state (above the mean) at t, the probability of remaining in a high state in $t+1$ is greater than the probability of switching to a low state in $t+1$.

II A Recursive Representation

The key trick to writing the problem recursively is the law of iterated expectations:

$$E[z_{t+1}|z_0] = E[\ldots E[E[z_{t+1}|z_t]|z_{t-1}]\ldots|z_0] \tag{2.0.1}$$

So we can write the problem as:

$$\max E_0 \left\{ \sum_{t=0}^{\infty} \beta^t r[S_t, S_{t+1}, z_t] \right\} \tag{2.0.2}$$

Here $E_0 = E(.|z_0)$. The maximization is subject to a distribution for z:

$$z_t \sim \Omega(z_{t-1}) \tag{2.0.3}$$

z_0 is given and known at time 0. Our strategy is to write the problem recursively again, using the law of iterated expectations.

A Heuristic via Working Backwards

Start with an arbitrary V_0, which we interpret as the utility the household receives if the world ends in T. Then:

$$V_1(S_{T-1}, Z_{T-1}) = \max_{S_T \in \Gamma(S_{T-1})} \left\{ r(S_{T-1}, S_T, Z_{T-1}) + \beta E\left[V_0(S_T, Z_T) | Z_{T-1} \right] \right\} \tag{2.1.4}$$

$$= r(S_{T-1}, S_T^*, Z_{T-1}) + \beta E\left[V_0(S_T^*, Z_T) | Z_{T-1} \right] \tag{2.1.5}$$

$$\begin{aligned} V_2(S_{T-2}, Z_{T-2}) \quad = \quad & \max_{S_{T-1} \in \Gamma(S_{T-2})} \Big\{ r(S_{T-2}, S_{T-1}, Z_{T-2}) \\ & + \beta E\left[r(S_{T-1}, S_T^*, Z_{T-1}) + \beta E\left[V_0(S_T^*, Z_T) | Z_{T-1} \right] | Z_{T-2} \right] \Big\} \end{aligned} \tag{2.1.6}$$

So working backwards, given the social planner acts optimally in period T, what is the optimal action in $T-1$?

$$\begin{aligned} = \quad & r\left(S_{T-2}, S_{T-1}^*, Z_{T-2}\right) + \beta E\left[r\left(S_{T-1}^*, S_T^*, Z_{T-1}\right) | Z_{T-2} \right] + \\ & \beta^2 E\left[E\left[V_0(S_T^*, Z_T) | Z_{T-1} \right] | Z_{T-2} \right] \end{aligned} \tag{2.1.7}$$

The law of iterated expectations means that the last term is just the expectation with respect to information from $T-2$. Hence:

$$V_T(S_0, Z_0) = \sum_{t=0}^{T} \beta^t E\left[r\left(S_t^*, S_{t+1}^*, Z_t\right) | Z_0 \right] + \beta^T E\left[V_0(S_T^*, Z_T) | Z_0 \right] \tag{2.1.8}$$

With some regularity conditions (note that these are less trivial than with the optimal growth model, since we need to account for the possibility of arbitrarily large shocks in the continuous case), we get that:

$$\lim_{T \to \infty} V_T(S_0, Z_0) = E\left[\sum_{t=0}^{\infty} \beta^t r\left(S_t^*, S_{t+1}^*, Z_t\right) | Z_0 \right] \equiv V(S_0, Z_0) \tag{2.1.9}$$

Hence the limiting value function and the original optimization problem are equivalent in the sense the that the two problems yield the same solution. The limit is the globally stable fixed point of the value function equation:

$$V(S,Z) = \max_{S' \in \Gamma(S)} \{r(S,S',Z) + \beta E[V(S',Z')\,|Z]\} \tag{2.1.10}$$

We let V, the fixed point, be the **VALUE FUNCTION**.

Hence if:

$$V(S_0,z_0) = E\left[\sum_{t=0}^{\infty} \beta^t r\left[S_t^*, S_{t+1}^*, z_t\right] |Z_0\right] \tag{2.1.11}$$

We can write:

$$V(S,z) = \max_{S' \in \Gamma(S,z)} \{r[S,S',z] + \beta E[V(S',z')\,|z]\} \tag{2.1.12}$$

The expectation operator preserves properties such as continuity and concavity. Hence we would expect the theorems to go through, with z acting like a state variable. In fact they do, with the main difficulty being the extra distributional assumptions we need on z. So we need r to be concave in z, bounded in z, Γ to be convex and non-empty in z, etc. Plus we need r to be integrable in z and Γ to be measurable, etc.

B Alternative Heuristic

As an alternative, start with the infinite horizon problem.

$$V(S_0,Z_0) = E\left[\sum_{t=0}^{\infty} \beta^t r\left(S_t^*, S_{t+1}^*, Z_t\right) |Z_0\right] \tag{2.2.13}$$

$$= r(S_0,S_1^*,Z_0) + E\left[\sum_{t=1}^{\infty} \beta^t r\left(S_t^*, S_{t+1}^*, Z_t\right) |Z_0\right] \tag{2.2.14}$$

$$= r(S_0,S_1^*,Z_0) + \mathrm{E}\left[\mathrm{E}\left[\sum_{t=1}^{\infty} \beta^t r\left(S_t^*, S_{t+1}^*, Z_t\right) |Z_1\right] |Z_0\right] \tag{2.2.15}$$

$$= r\left(S_0, S_1^*, Z_0\right) + \mathrm{E}\left[\mathrm{E}\left[\sum_{t=0}^{\infty} \beta^{t+1} r\left(S_{t+1}^*, S_{t+2}^*, Z_{t+2}\right) | Z_1\right] | Z_0\right] \tag{2.2.16}$$

$$= r\left(S_0, S_1^*, Z_0\right) + \beta \mathrm{E}\left[\mathrm{E}\left[\sum_{t=0}^{\infty} \beta^{t} r\left(S_{t+1}^*, S_{t+2}^*, Z_{t+2}\right) | Z_1\right] | Z_0\right] \tag{2.2.17}$$

$$V\left(S_0, Z_0\right) = r\left(S_0, S_1^*, Z_0\right) + \beta \mathrm{E}\left[V\left(S_1^*, Z_1\right) | Z_0\right] \tag{2.2.18}$$

The problem from period 1 onward is the same as the problem at period zero, except for different starting conditions.

III RBC Model: Social Planning Problem

A Problem

The value function for the discrete case is:

$$\begin{aligned} V(k, z) \quad = \quad & \max_{k', h \in \Gamma(k,z)} \left\{ u\left[zF(k,h) + (1-\delta)k - (1+\eta)k', 1-h\right] + \right. \\ & \left. \beta E\left[V\left(k', z'\right) | z\right] \right\} \end{aligned} \tag{3.1.1}$$

$$\Gamma(k, z) = \left\{k', h | h \in [0,1]\ , 0 \leq (1+\eta)k' \leq y + (1-\delta)k\right\} \tag{3.1.2}$$

That is:

$$\begin{aligned} V(k, z_i) \quad = \quad & \max_{k', h \in \Gamma(k,z)} \left\{ u\left[z_i F(k,h) + (1-\delta)k - (1+\eta)k', 1-h\right] + \right. \\ & \left. \beta\left[\pi_{i1}, \ldots, \pi_{in}\right] \cdot \begin{bmatrix} V\left(k', z_1\right) \\ \ldots \\ V\left(k', z_n\right) \end{bmatrix} \right\} \end{aligned} \tag{3.1.3}$$

The value function is for the continuous case:

$$\begin{aligned} V(k, z) \quad = \quad & \max_{k', h \in \Gamma(k,z)} \left\{ u\left[e^z F(k,h) + (1-\delta)k - (1+\eta)k', 1-h\right] + \right. \\ & \left. \beta E\left[V\left(k', \rho z + \epsilon'\right)\right] \right\} \end{aligned} \tag{3.1.4}$$

B First Order conditions

We have 2 focs, one w.r.t. investment and one w.r.t. labor choice. I'll just do the continuous case:

$$(1+\eta)\, u_c \left[e^z F(k,h) + (1-\delta)k - (1+\eta)k', 1-h\right] = \beta E\left[V_k(k', \rho z + \epsilon')\right] \tag{3.2.1}$$

Here we have the usual marginal utility of consumption equal to the marginal utility of investment.

$$\begin{aligned} &-u_l\left[e^z F(k,h) + (1-\delta)k - (1+\eta)k', 1-h\right] + \\ &u_c\left[e^z F(k,h) + (1-\delta)k - (1+\eta)k', 1-h\right] e^z F_h(k,h) = 0 \end{aligned} \tag{3.2.2}$$

Equation (3.2.2) sets the marginal utility of leisure equal to the marginal utility of working. The marginal utility of working is equal to the wage times the marginal utility of consumption. Working gives wealth right now. Hence the decision is essentially static.

These give rise to policy functions: $k' = \phi_1(k,z)$ and $h = \phi_2(k,z)$. So the dynamics are characterized by the two policy functions and k_0 and z_0 given.

C Envelope equations

These are the same as in the growth model. For the continuous case:

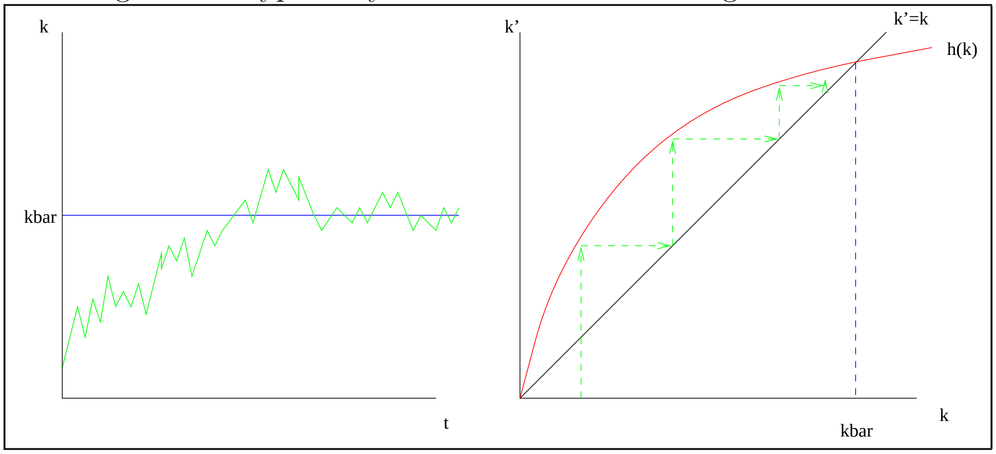
$$v_k(k,z) = u_c[c]\left(e^z F_k(k,h) + 1 - \delta\right). \tag{3.3.3}$$

Note that we could have one with respect to z. However, we do not need to calculate it since it is not present in the focs.

IV Steady States

Note that k and z are random variables which depend on ϵ. So k and z will never converge to a single value. However:

Figure 24: Typical dynamics of the stochastic growth model



k and z might drift around stationary values. Or the distribution of k might converge to a distribution which is indepedent of the state variables.

Definition 19 *INVARIANT DISTRIBUTION: A distribution of the state variables which is constant in time.*

A continuous case-z

The solutions to the first order conditions are functions $k' = \phi_1(k, z)$ and $h = \phi_2(k, z)$ and $z' = \rho z + \epsilon$. First let's find the invariant distribution of z. For the distribution to be constant, the moments must also be constant. That is, the mean, variance, etc. are constants. Hence:

$$\bar{z} = E(z_{t+1}) = \rho E(z_t) + E(\epsilon') \tag{4.1.1}$$

$$\bar{z} = \rho \bar{z} + \bar{\epsilon} \tag{4.1.2}$$

Hence any statinary distribution must have a mean of $\bar{z} = \frac{1}{1-\rho}\bar{\epsilon}$. The distribution of z is just the distribution of a sum of random variables. For example, if all variables are mean zero, their sum is also.

The variance must also be constant, assuming ϵ is mean zero and independent of z:

$$E\left(z^2\right) = E(\rho z + \epsilon)^2 \tag{4.1.3}$$

$$E\left(z^2\right) = \rho^2 E\left(z^2\right) + \sigma_\epsilon^2 \tag{4.1.4}$$

$$E\left(z^2\right) = \frac{1}{1-\rho^2}\sigma_\epsilon^2 \tag{4.1.5}$$

And so on.

B discrete case-z: Convergence of Discrete Markov Processes

We are interested in the "long run" behavior of z. As the stationary state is a long run prediction of the deterministic model, there is a long run prediction of the stochastic model. This is the invariant distribution. That is, over a long period of time, we expect the technology shocks to behave as the invariant distribution.

This is in Lucas and Stokey (chapter 11). For discrete markov processes, the distribution is given by Π. Consider the distribution two steps forward:

$$\text{Prob}\left(z_{t+2}|z_t\right) =? \tag{4.2.6}$$

Well, consider the first element:

$$\text{Prob}\left(z_{t+2} = z_1|z_t = z_1\right) \tag{4.2.7}$$

We have:

$$\text{Prob}\left(z_{t+1}|z_t = z_1\right) = \begin{bmatrix} \pi_{11} & \dots & \pi_{1m} \end{bmatrix} \tag{4.2.8}$$

Further:

$$\text{Prob}\left(z_{t+2} = z_1|z_{t+1}\right) = \begin{bmatrix} \pi_{11} & \dots & \pi_{m1} \end{bmatrix}' \tag{4.2.9}$$

So the probability is just the dot product of the two vectors. Each mutiplication represents a possible way of getting to z_1.

More generally, we have:

$$\text{Prob}\left(z_t|z_0\right) = \Pi^t \tag{4.2.10}$$

Or we can write this recursively:

$$\Pi_{t+1} = \Pi \cdot \Pi_t \tag{4.2.11}$$

Here Π_0 is the identity matrix. So the question is, what is the stationary distribution function for the above equation? That is, what is $\bar{\Pi}$ in

$$\lim_{t->\infty} \Pi^t = \bar{\Pi} \tag{4.2.12}$$

$\bar{\Pi}$ also satisfies:

$$\bar{\Pi} = \Pi \cdot \bar{\Pi} \tag{4.2.13}$$

We are interested in the long run behavior of the technology shocks. What does the model converge to?

Definition 20 *STATIONARY TRANSITION FUNCTION: a transition function which is independent of time or state variables.*

The stationary transition function fully characterizes the invariant distribution for discrete Markov processes.

1 existence

Convergence of discrete Markov processes is not automatic. Consider the following example:

$$\Pi = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.2.14}$$

With probability one we enter state 2 or 3 and then stay there forever. What we need to rule out is non-ergodic sets:

Definition 21 *A NON-ERGODIC set is a strict subset of z's where the probability of staying in the subset is one, once reached.*

A similar problem arises when we constantly switch states, as in the following example:

$$\Pi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4.2.15}$$

The above distribution function has a cyclically moving subset:

Definition 22 *CYCLICALLY MOVING SUBSET. A system that remains in one subset for a fixed period of time and then switches to another subset.*

In each case, the limiting distribution always depends on the current state, the initial conditions do not wash out.

A useful condition:

$$\sum_{j=1}^{m} \min_i \pi_{ij} > 0 \tag{4.2.16}$$

If the above condition is satisfied, Π is ergodic and has no cyclically moving subsets and converges to a unique stationary transition function geometrically. The condition requires at least one column to have no zeros.

2 finding the stationary transition function

1. Note that convergence is quick. Hence I recommend iterating on computer as an easy way to check the transition function.

2. There is a method of direct calculation if Π is symmetric. Here are the steps.

   (a) First we break down Π by diagonalization. For any symmetric matrix there exists an $m \times m$ matrix B such that:

   $$\Pi = B \cdot \Delta B^{-1} \tag{4.2.17}$$

   where:

   $$\Delta = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_m \end{bmatrix} \tag{4.2.18}$$

   Here λ_i's are the eigenvalues of Π.

   (b) Compute the eigenvalues by solving the characteristic polynomial:

   $$|\Pi - \lambda I| = 0 \tag{4.2.19}$$

   Since Π is symmetric, we should get m real eigenvalues.

   (c) Then compute B as the solution to

   $$\Pi = B \cdot \Delta B^{-1}. \tag{4.2.20}$$

   The columns of B equal the eigenvectors of of Π (note that the solution for B

may not be unique, in which case you can use any non-zero solution). That is:

$$B = [B_1, \ldots, B_m], \tag{4.2.21}$$

where B_i solves:

$$\Pi B_i = \lambda_i B_i, \tag{4.2.22}$$

$$(\Pi - \lambda_i I) B_i = 0. \tag{4.2.23}$$

(d) Now note that:

$$\Pi^t = B \cdot \Delta^t B^{-1} \tag{4.2.24}$$

Where:

$$\Delta^t = \begin{bmatrix} \lambda_1^t & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_m^t \end{bmatrix} \tag{4.2.25}$$

(e) Hence:

$$\bar{\Pi} = \lim_{t \to \infty} B \cdot \Delta^t B^{-1} = B \cdot \bar{\Delta} B^{-1}. \tag{4.2.26}$$

EXERCISE: Compute the stationary transition matrix for the transition matrix:

$$\Pi = \begin{bmatrix} n & 1-n \\ 1-n & n \end{bmatrix} \tag{4.2.27}$$

Here $0 < n < 1$. Also give the intuiton behind the result.

3 Invariant distribution for k: continuous case

Recall that the first order conditon and envelope equations are:

$$(1+\eta) u_c \left[e^z F(k,h) + (1-\delta) k - (1+\eta) k', 1-h\right] = \beta E \left[V_k (k', \rho z + \epsilon')\right], \tag{4.2.28}$$

$$V_k (k,z) = u_c \left[e^z F(k,h) + (1-\delta) k - (1+\eta) k', 1-h\right] \left[e^z F_k (k,h) + (1-\delta)\right], \tag{4.2.29}$$

$$\mathrm{E}[k_{t+1}] = \mathrm{E}[k_t] = \bar{k}. \tag{4.2.30}$$

It is difficult to go much further, although Lucas and Stokey establish that invariant distributions exist for special cases. Indeed there is now a large literature just establishing the existence of invariant distributions. This is beyond our scope. However, if we can solve for the policy function, we can often also solve for the moments of the invariant distribution.

Suppose the policy function is linear or log-linear.

$$k_{t+1} = a + bk_t + cz_t \tag{4.2.31}$$

$$k_{t+1} = \beta\gamma A e^{z_t} k_t^{\gamma} \tag{4.2.32}$$

Then we can use the same procedure as with z to get the moments of the invariant distribution for k. For example the invariant distribution for $\log k$ in the log linear case has a constant mean. Take expectations of both sides:

$$E\left[\log k_{t+1}\right] = \log\left(\beta\gamma A\right) + E\left(z_t\right) + \gamma \log k_t. \tag{4.2.33}$$

$$\bar{k} \equiv E\left[\log k\right] = \frac{\log\left(\beta\gamma A\right)}{1-\gamma}. \tag{4.2.34}$$

The linear model has a constant mean of the invariant distribution:

$$E\left[k_{t+1}\right] = a + bE\left[k_t\right] + cE\left[z_t\right] \tag{4.2.35}$$

Recall that z_t has zero stationary mean and let $E\left[k_{t+1}\right] = E\left[k_t\right] = \bar{k}$. Then:

$$\bar{k} = \frac{a}{1-b}, \tag{4.2.36}$$

and similarily for the variance.

Note that since ϵ is normally distributed, z is also normally distributed, as it is the sum of normal distributions. Thus the invariant distribution for z is normal. But then since k (linear case) and $\log k$ (log linear case) are just sums of z's, k or $\log k$ will have an invariant distribution which is normal. However, in general solutions are not available.

4 Certainty Equivalence

One way around lack of knowledge of the steady state is to assume certainty equivalence:

$$E\left(f\left(x\right)\right) = f\left(E\left(x\right)\right) \tag{4.2.37}$$

This reduces the problem down to finding the deterministic steady state.

$$-\left(1+\eta\right)u_c\left[e^zF\left(k,h\right)+\left(1-\delta\right)k-\left(1+\eta\right)k',1-h\right]+\beta E\left[V_k\left(k',\rho z+\epsilon'\right)\right]=0 \quad (4.2.38)$$

$$-\left(1+\eta\right)u_c\left[e^zF\left(k,h\right)+\left(1-\delta\right)k-\left(1+\eta\right)k',1-h\right]+\beta V_k\left(E\left[k'\right],0\right)=0 \quad (4.2.39)$$

The problem is now deterministic, so we need to find $\bar{k}=E\left[k_{t+1}\right]=E\left[k_t\right]=k_t$ (since k_t is known at time t). Taking expectations of both sides:

$$-\mathrm{E}\left[\left(1+\eta\right)u_c\left[c,1-h\right]+\beta V_k\left(\bar{k},0\right)\right]=0, \quad (4.2.40)$$

$$-\left(1+\eta\right)u_c\left[\bar{c},1-\bar{h}\right]+\beta V_k\left(\bar{k},0\right)=0 \quad (4.2.41)$$

$$\mathrm{E}\left[V_k\left(k,z\right)\right]=V_k\left(\bar{k},0\right)=u_c\left[\bar{c},1-\bar{h}\right]\left[F_k\left(\bar{k},\bar{h}\right)+\left(1-\delta\right)\right] \quad (4.2.42)$$

Hence we get:

$$\left(1+\eta\right)=\beta\left[F_k\left(\bar{k},\bar{h}\right)+\left(1-\delta\right)\right] \quad (4.2.43)$$

$$\rho=F_k\left(\bar{k},\bar{h}\right)-\delta. \quad (4.2.44)$$

This is the same as the deterministic model. Note that certainty equivalence is most often used in Linear-Quadratic approximations where the return function is quadratic and the first order conditions are linear, where the approximation makes some sense. In general we would expect the certainty equivalence steady state capital to be higher than the true mean of the invariant distribution, since the risk aversion biases the planner towards current consumption.

C Certainty Equivalence steady state for h

The first order condition for h is:

$$-u_l\left[c,1-h\right]+u_c\left[c,1-h\right]e^zF_h\left(k,h\right)=0 \quad (4.3.1)$$

Under certainty equivalence we have:

$$u_l\left[\bar{c}, 1-\bar{h}\right] = u_c\left[\bar{c}, 1-\bar{h}\right] F_h\left(\bar{k}, \bar{h}\right) \tag{4.3.2}$$

Hence the stationary fraction of hours worked is not in general independent of the utility function.

V Calibration

We are interested in two questions: Does a model driven by shocks to productivity replicate key moments of the economy (for example, the variance of investment spending is about four times the variance of the business cycle)? Second, how much of business cycle variation is explained by productivity shocks alone (not government policy).

Unfortunately, we cannot solve for $\mathrm{E}\left[x^2\right]$, to get the variance of investment spending. Hence we must solve the model on computer. We need to solve computationally for v and the optimal decision ϕ so that we can analyze the moments of the state variables (we cannot analyze most of the moments analytically without ϕ). To solve computationally, we must:

1. Assign functional forms and parameter values consistent with some long run features of the data. This process is called calibration.
2. Solve the model on computer to get v and ϕ.
3. Simulate the model to generate random economies. Find moments of the simulated data such as $\mathrm{E}\left[x^2\right]$, and compare to the moments in the real data.

We begin with the calibration process.

A Estimation vs Calibration

Computational questions (use calibration):

- Big Potatoes or small potatoes.
- Do productivity shocks cause large or small fluctuations in investment spending?
- Do productivity shocks account for most or little of business cycles?

Analytical Questions (use theory):

- Is the comparative static +/- or ambiguous?
- What causes growth?

Empirical Questions (use econometrics).

- What is the magnitude of a particular parameter?
- Given the sign is ambiguous, what sign is observed in the data?
- Can we reject or accept the model?

To estimate the parameters, we would have to know the functional form for v. Then we could use GMM by setting a moment restriction. Typically the expected value of the FOC would be equal to zero. Work by Hansen and Gallant and others exists on solving for v and estimating simultaneously. What you do is start with a given v_0 and given parameters, solve for v_1, re-estimate the parameters, then find v_2 and iterate. Calibration, estimates parameters seperately from steady state equations or using other studies.

B Steps for Calibration

We need to assign parameters to the model so it may be solved on computer. The process:

1. Restrict functional forms of the model to a parametric class which is consistent with assumptions.
2. Construct a data set which is consistent with the model.
3. Assign parameter values such that the behavior of the economy matches features of the data in as many dimensions as there are parameters.

C Step 1: Restrict Functional Forms

A stylized fact is that the economy displays balanced growth: consumption, capital, investment, and output all grow at the same rate. As we saw in the optimal growth model, CRR utility with Cobb-Douglass production has this property and satisfies all assumptions.

$$Y_t = e^{z_t} F\left(K_t, L_t h_t\right) = e^{z_t} K_t^{\gamma} \left(L_t h_t\right)^{1-\gamma} \tag{5.3.1}$$

$$u = \frac{\left[C_t^{1-\alpha} l_t^{\alpha}\right]^{1-\sigma} - 1}{1-\sigma} \tag{5.3.2}$$

Here σ again will be the coefficient of relative risk aversion and α is the share of leisure in the composite commodity (ie the utility function has constant elasticity of substitution, with the elasticity being $\frac{1-\alpha}{\alpha}$). As α gets small, a lot of leisure is needed to give the same utility as a unit of consumption. The utility function implies hours worked is constant in the long run. This is debatable, hours seem to be falling. Also γ is the capital share, we will see that the fraction of income which goes to the owners of capital is γ.

D Constructing the Data Set

Problems:

- There is no government sector. How then, do we treat government output? Is it consumption or investment?
- Housing is measured as investment, but other consumer durables are not. Our concept of a consumption item is that it gives utility for only 1 period.

Hence we must add government investment and consumer durables to investment, and the rest of government spending to consumption.

1 Construct Private Capital Data

We will need data on private, government, and durable capital K_i $i = p, G, D$. The National Income and Product Accounts (NIPA) has two sets of tables: "GDP and Personal Income" and "Fixed Assets." Private capital data is available from the fixed asset tables, although this data is annual. This estimate does not include land or inventories. Land values can be added from the Flow of Funds Account, *Balance Sheet of the United States*, a FED publication. Inventories and structures are available as either an annual series from the fixed asset tables or as a quarterly series in the Balance Sheet of the US. Inventories are like capital in that they are used to produce consumption goods in subsequent periods, with a one to one production technology. Thus:

$$K_p = K_{\text{NIPA}} + \text{ LAND } + \text{ INVENTORIES} \tag{5.4.1}$$

2 Construct Private Investment Data

We also need data on investment X_i. The NIPA has data on private investment, which does not include the change in inventories (also available from the NIPA). In addition, we have no foreign sector. Imagine instead a technology where US produced exports are “inputs” into a production function which produces imports (“corn” is an input for “Toyotas”). If so, we should add imports to domestic production. In the data we have:

$$Y = C + X + G + EX - M. \tag{5.4.2}$$

Now in the data, consumption is composed of domestic consumption C_D and imports so:

$$Y = C_D + M + X + G + EX - M = C_D + I + G + EX. \tag{5.4.3}$$

Assuming balanced trade $X = M$, then:

$$Y = C_D + X + G + M. \tag{5.4.4}$$

So the intermediate good X vanishes, and we have the final good imports.

But what if $X > M$? Then we have some net foreign investment, $NFI = NX = EX - M$. Again, thinking in terms of GNP, this is no different than domestic investment, since GNP measures US owned production. Some of our production EX was exchanged for a bond or other investment, which we can convert to consumption later. So we need only add net exports to investment. Thus:

$$X_p = X_{NIPA} + \Delta\text{INVENTORIES } + EX - M \tag{5.4.5}$$

Then in the GNP data, let total investment X_{all} equal domestic investment X plus net exports. Then:

$$Y = C + X + G + EX - M = C + X_{\text{all}} + G, \tag{5.4.6}$$

$$= C + X_D + G + EX - M. \tag{5.4.7}$$

Thus we compute consumption and GNP as in the data (in that consumption includes imports and GNP identical except net exports are added to investment spending). Note: we must still remove durable goods from consumption, and add durable goods service flows to GNP, see 8.

3 Construction Private Capital Income Data

We will need capital income to calculate γ_p, which we will in turn use to calculate private capital income. Recall from the competitive model:

$$r = F_k\left(K, H\right). \tag{5.4.8}$$

Interest here is gross of depreciation expenses. The firm pays the household r, and the household pays depreciation expenses, δ. In reality, for interest income such as corporate profits, the firm pays the depreciation expenses and gives an interest dividend (or capital gains) that is net of deprecation, R:

$$R = F_k\left(K,H\right) - \delta. \tag{5.4.9}$$

Rearranging:

$$\left(R + \delta\right) K = K F_K\left(K, H\right). \tag{5.4.10}$$

Plugging in for F gives:

$$K\left(R + \delta\right) = \gamma K^{\gamma} H^{1-\gamma}. \tag{5.4.11}$$

The left hand side is total gross capital income, I, and the right hand side is γ times national income.

$$I = K\left(R + \delta\right) = \gamma K^{\gamma} H^{1-\gamma} = \gamma Y. \tag{5.4.12}$$

This equation is called the *CAPITAL SHARE EQUATION* since it shows the fraction of income I that capital owners receive. That fraction, γ, is called the CAPITAL SHARE or the fraction of output that goes to the owners of capital. Here $R + \delta$ is the "cost of capital" or the gross rental price of capital. Hence the right hand side is the total income derived from capital rental (I). Our strategy is to calculate γ via data on I and Y.

$$\gamma = \frac{I}{Y} \tag{5.4.13}$$

Notice the idea of a ratio, I/Y is unit free and thus we can use nominal instead of real data and don't have to use per-productivity unit data ($i/y = I/Y$ in the steady state). The problem is that data for I is unavailable for durables and government, and inconsistently collected for the private sector.

We expect the capital share equation to hold regardless of the type of capital, with R constant to prevent arbitrage:

$$K_i\left(R+\delta_i\right)=\gamma_i Y_i \equiv I_i \quad i=P,G,D \tag{5.4.14}$$

Here P is private, G is government, and D is consumer durables. The interest rate is constant.

For private capital, we have a measure of I_p from the National Income and Product Accounts (NIPA) as the sum of rental income, corporate profits, and net interest. Firms subtract depeciation as an expense, so we must add it back in. In addition, proprietors income (PI) and indirect business taxes (IBT), are not broken down by capital versus wage income. Indirect business taxes are final goods taxes. These are part of the value of the good but are government income not assigned to either capital or labor. Indirect business taxes can be recovered from the NIPA tables as the difference between National Income (NI) and Net National Factor Income ($NNFI$). Assume these incomes have the same capital share, then we have:

$$\gamma_p Y_p=I_p=I_{p,NIPA}+\gamma_p\left(IBT+PI\right)+DEP. \tag{5.4.15}$$

Total depreciation is available from the NIPA. Hence:

$$\gamma_p=\frac{I_{p,NIPA}+DEP}{Y_p-PI-IBT} \tag{5.4.16}$$

Let $Y_p=GNP$, since services flows from durables and government are not already included in the GNP. Then we can compute $I_p=\gamma_p Y_p$ using γ_p from (5.4.16).

4 Construct R

We also need durable and government capital income data. This we will get by assuming the interest rate is the same for all three types of capital. We can therefore use R and Y_i, $i=G,D$ to calculate I_i. To estimate the interest rate, which we assume by is the same for all investments, we have:

$$I_p=RK_p+\delta_p K_p \tag{5.4.17}$$

The second term is total depreciation expenses (DEP) so:

$$R = \frac{I_p - DEP}{K_p} \tag{5.4.18}$$

This gives an average interest rate of 6.3%.

5 Construct δ_i

For government and durables, we also need some measure of the depreciation rate, since there is no DEP-like statistic. We know:

$$X_t = K_{t+1} - (1 - \delta) K_t \tag{5.4.19}$$

Here X_t is investment, which is available for both consumer durables and and government purchases from the NIPA. Since on the balanced growth path investment and capital and output grow at the same rate:

$$\frac{X_{it}}{K_{it}} = (1 + g_{yi}) - (1 - \delta_i), \tag{5.4.20}$$

where g_y is the growth rate of total capital. In the balanced growth path, all variables grow at the same rate which is the growth rate of total output, $g_y = (1 + \theta)(1 + \eta) - 1$. Therefore, in the steady state:

$$\frac{X_i}{K_i} = g_y + \delta_i \tag{5.4.21}$$

$$\delta_i = \frac{X_i}{K_i} - g_y \tag{5.4.22}$$

We have X_G from the NIPA and X_D is "consumption" of consumer durables in the NIPA. In the balanced growth path, everything grows at the same rate, so g_y is the growth rate of total output. For the average growth rate of real output, we need to use real GDP from the NIPA. The NIPA fixed asset tables have annual data for the stock of government capital and consmer durables. The capital series are annual, and most other data is quarterly. We can either use annual data throughout, or use linear interpolation to estimate the quarterly capital data. Using linear interpolation, we can get $\delta_G = .05$ and $\delta_D = .26$. Consumer durables depreciate more quickly than private capital, and government capital more slowly than private.

6 Construct Durable and Government I

Since we have K_i, $i = D,G$, we can use

$$K_i (R + \delta_i) = I_i, \tag{5.4.23}$$

to get I_i for consumer durables and government investment.

7 Construct Hours Data

Average weekly worked data comes from either the establishment survey (CES) or the household survey (BLS). The establishment survey reports less hours worked, because self-employed are not surveyed. In addition, households may exaggerate their employment status ('I'm a consultant'). To convert to a fraction of time spent working we need the employment to population ratio (also available from the BLS), and a measure of total weekly time available.

One way to to determine the average fraction of time spent working is from micro studies. If we sleep 8 hours, then we have 16 hours of time per day, 7 days a week yields 16x7 = 112 hours of time a week. Including personal duties reduces the time to between 100 (Prescott) and 94.5 hours per week (Hill, 1985).

Assuming a 40 hour work week plus two weeks vacation we have $40 \cdot 50/52 = 38.5$ hours per week at work. But the average work week has actually dropped from 38.40 in 1964 to about 33.7 in 2004.

Finally, some work zero hours. The employment to population ratio is about 0.62 in 2004. Using these data:

$$h = \frac{\text{CES HOURS} \cdot \frac{\text{EMP}}{\text{POP}}}{94.5}. \tag{5.4.24}$$

This gets an average of about 0.22. Cooley and Prescott instead use $h = 0.31$, which comes from micro studies about time spent working (Ghez and Becker, 1975 and Juster and Stafford, 1991), but these tend to ignore retirees and others not working.

8 Construct Aggregate Data

Now add across sectors to get $I = I_p + I_g + I_d$. We also have $X = X_p + X_g + X_d$ and $K = K_p + K_g + K_d$. For output, we do not have a separate estimate of the labor income from the flows from government assets and consumer durables. Therefore, we will just add in the capital income: $Y = GNP + I_D + I_G$. Consumption is non-durable consumption plus government consumption: $C = C - X_D + G$. For population, we can use the NIPA data or the BLS data. We have now constructed data for I, X, K, Y, C, L, and h which is consistent with the model.

E Assign parameter values

Note: results are yearly and data set is 1954-2002.

1 Capital Share (γ)

This is easy:

$$\gamma = \frac{I}{Y} \approx .42 \tag{5.5.1}$$

The capital share varies across countries. Without government we get about .36. Its about .25 without consumer durables or government.

2 Population growth (η)

Note that world-wide, a constant growth rate doesn't work too well. Also, population is projected to level off in a few decades. For the US, though, the population growth rate has been roughly constant. I get an average growth rate of $\eta = .012$ per year:

$$L_t = (1 + \eta)\, L_{t-1}, \tag{5.5.2}$$

$$\eta = \frac{L_t - L_{t-1}}{L_{t-1}}, \tag{5.5.3}$$

$$\eta = \frac{1}{T} \sum_{t=1}^{T} \frac{L_t - L_{t-1}}{L_{t-1}}. \tag{5.5.4}$$

3 Growth of per capita output (θ)

This is just the average growth rate of real per capita output, about 0.02.

$$\frac{\frac{Y_{t+1}}{L_{t+1}}}{\frac{Y_t}{L_t}} = 1 + \theta. \tag{5.5.5}$$

4 Depreciation (δ)

We have:

$$\delta = \frac{X}{K} - g_y = \frac{X}{K} + 1 - (1+\theta)(1+\eta) \tag{5.5.6}$$

I get $\delta = .057$, well below what is allowed for taxes, close to .1.

5 Rate of time preference and Discount rate (β)

Using the modified golden rule (the certainty equivalence steady state):

$$\rho = F_k(K, H) - \delta, \tag{5.5.7}$$

$$\rho K = K F_k(K, H) - \delta K. \tag{5.5.8}$$

Since production is Cobb-Douglas,

$$\rho K = \gamma Y - \delta K, \tag{5.5.9}$$

$$\rho = \gamma \frac{Y}{K} - \delta. \tag{5.5.10}$$

I get $\rho = 0.04$, which implies $\beta = \frac{1+\eta}{1+\rho} = .96$.

6 Decay in technology shocks (ρ)

We use the Solow procedure to generate data on technology, then estimate ρ. We have:

$$Y_t = e^{z_t} K_t^{\gamma} (L_t h_t)^{1-\gamma} \tag{5.5.11}$$

$$\log(Y_t) = \gamma \log(K_t) + (1-\gamma)\log(A_t L_t h_t) + z_t \tag{5.5.12}$$

We have data on output, capital, and total hours worked, and we have γ. We can construct $A_t = (1+\theta)^t$. We will need to use real output and capital. So we can compute z_t. Then estimate ρ using

$$z_t = \rho z_{t-1} + \epsilon_t \tag{5.5.13}$$

Using the quarterly data we get $\rho = .975$ (annual is 0.9) and a unit root is difficult to reject here. In this case, OLS estimate is still consistent, but standard errors are higher. One may also just assume $\rho = 1$ and use procedure in the homework. The standard deviation of the residuals is $\sigma_\epsilon = .04$ (annualized is 0.08).

7 Preferences (α and σ)

Only the steady state for hours depends on the functional form of the utility function. Recall that the utility function does not matter for the steady state capital. With our utility specification, we then get two unknowns and only one equation, the steady state for hours worked. Hence we usually just assume a value for σ and try to compute α. Most studies of risk aversion get values between 1-4 (1 is log). Hurd Econometrica 1989 gets 1.12 using a life cycle model with panel data. Prescott 1986 FED-MINN Quarterly Review gets about 1 as well.

The first order condition for hours worked is:

$$u_l = u_c e^z F_h(k, h) \tag{5.5.14}$$

At the steady state given log utility we have:

$$\alpha \frac{1}{1-\bar{h}} = (1-\alpha) \frac{(1-\gamma)\bar{y}}{\bar{c}\bar{h}} \tag{5.5.15}$$

$$\frac{\alpha}{1-\alpha} \frac{\bar{h}}{1-\bar{h}} = \frac{(1-\gamma)\bar{y}}{\bar{c}} \tag{5.5.16}$$

Using the estimate of $\bar{h}$ above, we solve the above equation and get $\alpha = 0.76$.

VI Computational Solution: Quadratic Approximation (QA)

We wish to find if the moments match the moments of the data. To do this, we must solve for the value function and the policy functions, and then iterate forward a large number of times to generate simulated data. We have calibrated the model. The next step is to solve the model.

There are many ways to solve RBC models. We have already seen two ways: the analytical solution via either guess-and-verify or direct value function iteration. Many algorithms solve the model on computer. The easiest and most robust of these is the QA method (although the method is also usually the least accurate). The idea is to form a quadratic approximation of the return function, and to linearize the constraints. The first order conditions will then be linear, and easily solved on computer for a given value function. Then the optimal values are plugged into the Bellman's equation to give the updated value function. We will see that the updated value function is also quadratic, hence the procedure can be repeated.

Recall the generalized problem:

$$\max_{c_t} E\left[\sum^{\infty} \beta^t r\left(s_t, c_t\right)\right] \tag{6.1}$$

subject to:

$$S_{t+1} = B\left(s_t, c_t, \epsilon_{t+1}\right) \tag{6.2}$$

$$V\left(s\right) = \max_{c}\left\{r\left(s,c\right) + \beta E\left[V\left(B\left(s,c,\epsilon\right)\right)\right]\right\} \tag{6.3}$$

So I have left in the control variables and instead substituted the constraint in for next period's value function. We suppose m states exist and n controls. It follows that B is an $m \times 1$ vector of transition equations.

The procedure:

1. Find the deterministic steady state, $\bar{x} = \left(\bar{S}, \bar{C}\right)$.
2. Approximate r with a 2nd order Taylor expansion around $\bar{x}$. Approximate B with a first order Taylor expansion around $\bar{x}$. The first and second derivatives are approximated with finite differences.
3. Convert the problem to a single quadratic Bellman's equation, given an initial V_0.
4. Calculate the optimal decisions.
5. Plug the decisions into the Bellman's equation, and iterate.

A Find the steady state

1 Calculate env and foc

First calculate the first order conditions and envelopes:

$$r_c\left(s,c\right) + \beta E\left[v_s\left(B\left(s,c,\epsilon\right)\right)B_c\left(s,c,\epsilon\right)\right] = 0 \tag{6.4}$$

At the certainty equivalence steady state:

$$r_c(\bar{s},\bar{c}) + \beta v_s(\bar{s}) B_c(\bar{s},\bar{c},\bar{\epsilon}) = 0 \tag{6.5}$$

Given that utility is scaler, r_c is $1 \times n$ vector of foc's and $v_s \cdot B_c$ is $1 \times m \cdot m \times n$. So we have n equations.

The envelopes are:

$$v_s(s) = r_s(s,c) + \beta E\left[v_s(B(s,c,\epsilon)) B_s(s,c,\epsilon)\right] \tag{6.6}$$

At the certainty equivalence steady state.

$$v_s(\bar{s}) = r_s(\bar{s},\bar{c}) + \beta v_s(\bar{s}) B_s(\bar{s},\bar{c},\bar{\epsilon}) \tag{6.7}$$

Here we have v_s and r_s as $1 \times m$, and $v_s \cdot B_s$ is $1 \times m \cdot m \times m$. So we have m envelopes.

2 solve foc and env

Solve the envelopes for v_s to get:

$$v_s(\bar{s})\left[I_m - \beta B_s(\bar{s},\bar{c},\bar{\epsilon})\right] = r_s(\bar{s},\bar{c}) \tag{6.8}$$

$$v_s(\bar{s}) = r_s(\bar{s},\bar{c})\left[I_m - \beta B_s(\bar{s},\bar{c},\bar{\epsilon})\right]^{-1} \tag{6.9}$$

Here I_m is the $m \times m$ identity matrix.

Substitute into the first order condition to get:

$$\begin{bmatrix} r_c(\bar{s},\bar{c}) + \beta r_s(\bar{s},\bar{c})\left[I_m - \beta B_s(\bar{s},\bar{c},\bar{\epsilon})\right]^{-1} B_c(\bar{s},\bar{c},\bar{\epsilon}) \\ \bar{s} - B(\bar{s},\bar{c},\bar{\epsilon}) \end{bmatrix} = 0 \tag{6.10}$$

So we have n foc's and m transitions for a total of $m + n$ equations for unknowns s and c.

This can be solved by the computer using a nonlinear equation solver, such as Matlab's fsolve.

B Approximate r

Note that:

$$\begin{aligned} r(S,C) &\approx r(\bar{x}) + \begin{bmatrix} r_s(\bar{x}) & r_c(\bar{x}) \end{bmatrix} \begin{bmatrix} S-\bar{S} \\ C-\bar{C} \end{bmatrix} + \\ & \frac{1}{2} \begin{bmatrix} S-\bar{S} & C-\bar{C} \end{bmatrix} \begin{bmatrix} r_{ss}(\bar{x}) & r_{sc}(\bar{x}) \\ r_{cs}(\bar{x}) & r_{cc}(\bar{x}) \end{bmatrix} \begin{bmatrix} S-\bar{S} \\ C-\bar{C} \end{bmatrix} \end{aligned} \tag{6.2.1}$$

$$B(S,C,\epsilon) \approx B(\bar{x},\bar{\epsilon}) + \begin{bmatrix} B_s(\bar{x},\bar{\epsilon}) & B_c(\bar{x},\bar{\epsilon}) & B_\epsilon(\bar{x},\bar{\epsilon}) \end{bmatrix} \begin{bmatrix} S-\bar{S} \\ C-\bar{C} \\ \epsilon-\bar{\epsilon} \end{bmatrix} \tag{6.2.2}$$

We can approximate the derivatives via finite differencing. Let $\tilde{h}$ be a small positive number, and h^i be a vector of zeros except the ith element which is $\tilde{h}$.

$$r_i(\bar{x}) \approx \frac{r(\bar{x}+h^i) - r(\bar{x}-h^i)}{2\tilde{h}} \tag{6.2.3}$$

$$r_{ij}(\bar{x}) \approx \frac{r(\bar{x}+h^i+h^j) - r(\bar{x}+h^i-h^j) - r(\bar{x}-h^i+h^j) + r(\bar{x}-h^i-h^j)}{4\tilde{h}^2} \tag{6.2.4}$$

C Rewrite the value function quadratically

Let $y = [1,x]^T$ and $w = [1,s]$. Then we can write the approximation as:

$$r(x) \approx r(\bar{x}) + r_x(\bar{x})(x-\bar{x}) + \frac{1}{2}(x-\bar{x})^T r_{xx}(\bar{x})(x-\bar{x}) \tag{6.3.1}$$

$$\begin{aligned} = & \left[r(\bar{x}) + r_x(\bar{x})\bar{x} + \frac{1}{2}\bar{x}^T r_{xx}(\bar{x})\bar{x} \right] + \\ & r_x(\bar{x})x + \frac{1}{2}\bar{x}^T r_{xx}(\bar{x})x + \frac{1}{2}x^T r_{xx}(\bar{x})\bar{x} + \frac{1}{2}x^T r_{xx}(\bar{x})x \end{aligned} \tag{6.3.2}$$

$$= \left[r(\bar{x}) + r_x(\bar{x})\bar{x} + \frac{1}{2}\bar{x}^T r_{xx}(\bar{x})\bar{x}\right] + \left[r_x(\bar{x}) + \bar{x}^T r_{xx}(\bar{x})\right] x + \frac{1}{2} x^T r_{xx}(\bar{x}) x \tag{6.3.3}$$

$$\equiv A + 2Dx + x^T C x \tag{6.3.4}$$

$$= \begin{bmatrix} 1 & x^T \end{bmatrix} \begin{bmatrix} A & D \\ D^T & C \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} \tag{6.3.5}$$

$$\equiv y^T Q y \tag{6.3.6}$$

Notice that I have exploited that for square symmetric matricies $a^T = a$.

Similarly, one can write:

$$B(x) = B(\bar{x}) + B_x(\bar{x})(x - \bar{x}) \tag{6.3.7}$$

$$= \begin{bmatrix} 1 & x^T \end{bmatrix} \begin{bmatrix} B(\bar{x}) - B_x(\bar{x})\bar{x} \\ B_x(\bar{x}) \end{bmatrix} \equiv y^T B \tag{6.3.8}$$

Note that B is thus an $m + n + 1$ by m matrix.

It follows that we can write the problem as:

$$v(s) = \max_c \left\{ y^T Q y + \beta v\left(y^T B\right) \right\} \tag{6.3.9}$$

D Iteration

1. The first step is to guess a V_0 and condense the problem into one matrix of the form $Y^T R_0 Y$.

2. Then we calculate the optimal decisions by solving the first order conditions.

3. We then plug in the solutions to get V_1.

4. Repeat steps 1-3 until $||V_i - V_{i-1}|| < 1e - 8$.

5. Find optimal policy using solution to V.

1 Choose v_0 and condense

Let $w = [1,s]^T$. Choose v_0 as a matrix of dimension $m+1 \times m+1$. We are essentially treating the constant as an extra state. Then:

$$w^T v_1 w = \max_c \left\{ y^T Q y + \beta w'^T v_0 w \right\} \tag{6.4.1}$$

Here $w' = B \cdot y$.

$$= \max_c \left\{ y^T Q y + \beta y^T B^T v_0 B y \right\} \tag{6.4.2}$$

$$= \max_c \left\{ y^T \left[Q + \beta B^T v_0 B \right] y \right\} \tag{6.4.3}$$

$$\equiv \max_c \left\{ y^T R y \right\} \tag{6.4.4}$$

2 Optimize

Notice that we can write the objective function as:

$$w^T v_1 w = \max_c \left\{ y^T R y \right\} \tag{6.4.5}$$

$$w^T v_1 w = \max_c \left\{ \begin{bmatrix} w^T & c^T \end{bmatrix} \begin{bmatrix} R_w & R_{wc}^T \\ R_{wc} & R_c \end{bmatrix} \begin{bmatrix} w & c \end{bmatrix} \right\} \tag{6.4.6}$$

First check the second order conditions for a maximum. The minimum requirement is that the E matrix is negative definite, I would use $\max\left(eig\left(R_c\right)\right) < 0$. I would also check the entire R matrix, although that matrix can be negative along the way.

Time to whip out some vector calculus. The first order condition is:

$$2R_{wc}w + 2R_c c = 0 \tag{6.4.7}$$

$$c = -inv\left(R_c\right) R_{wc} w \equiv Fw \tag{6.4.8}$$

Plug in to get:

$$w^T v_1 w = w^T R_w w + 2w^T F^T R_{wc} w + w^T F^T R_c F w \tag{6.4.9}$$

$$= w^T \left(R_w + 2F^T R_{wc} + F^T R_c F\right) w \tag{6.4.10}$$

And hence:

$$v_1 = R_w + 2F^T R_{wc} + F^T R_c F \tag{6.4.11}$$

The convergence criteria is:

$$\max |v_1 - v_0| < \tilde{h} \tag{6.4.12}$$

VII Simulation

We now have characterized the optimal policy:

$$C = F \cdot [1, s], \tag{7.0.1}$$

$$S' = B \cdot [1, S, \epsilon]^T. \tag{7.0.2}$$

Here B is the $\hat{B}$ matrix with an additional column for the random variable.

For the optimal growth model, we have:

$$\begin{bmatrix} k' \\ h \end{bmatrix} = \begin{bmatrix} F_{11} + F_{12}k + F_{13}z \\ F_{21} + F_{22}k + F_{23}z \end{bmatrix} \tag{7.0.3}$$

$$z' = \rho z + \epsilon \tag{7.0.4}$$

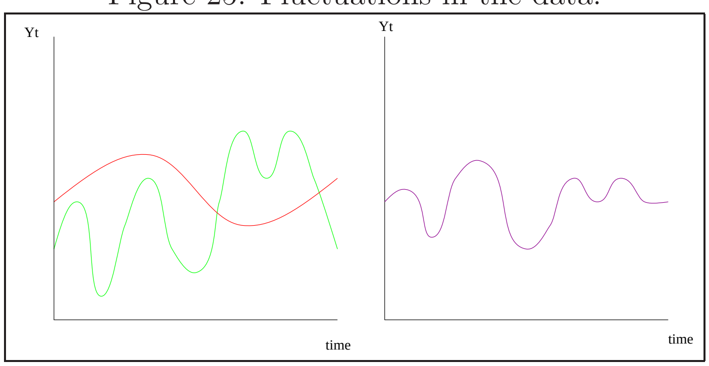
Now we can also find the other variables, output via the production function, consumption via the budget constraint, investment via the law of motion for capital. So we might simulate the model 100 times and get values for h,y,c,x,k. Once this is done, there is one more thing remaining.

VIII Detrending

We wish to distinguish between business cycles and long run growth. Business cycles are the short run fluctuations in Y, while long run growth is the trend in Y. In the model this is easy, z represents the short run changes, and A represents the trend. But in the actual data, there is no way to distinguish between a short run technology shock, z and a long run technology shock A. Consider the 4% growth this year. Was that due to a short or long run fluctuation? One would expect that a certain amount is trend and the rest is a positive shock.

Typically, we assume a certain portion of growth is trend, and the rest is cycle. Removing the trend is called filtering. Filtering also uses jargon "signal" for trend and "noise" for fluctuations. The key assumption is the frequency. If there is a low and a high frequency cycle in the data:

Figure 25: Fluctuations in the data.



A filter calls the low frequency cycles trend and the high frequency cycles fluctuations.

A Log-Linear

The easiest way is log linear detrending. In the model with exogenous growth:

$$Y_t = e^{z_t} F\left(K_t, L_t A_t h_t\right), \tag{8.1.1}$$

$$\frac{Y_t}{A_t L_t} = e^{z_t} F\left(k_t, h_t\right). \tag{8.1.2}$$

Let y_t be output per capita. Then, along the balanced growth path:

$$\frac{\frac{y_t}{A_t}}{\frac{y_{t-1}}{A_{t-1}}} = \frac{e^{z_t}}{e^{z_{t-1}}}, \tag{8.1.3}$$

$$\log\left[\frac{y_t}{y_{t-1}}\right] = \log\left[\frac{A_t}{A_{t-1}}\right] + z_t - z_{t-1}. \tag{8.1.4}$$

Using that $\log(x+1) \approx x$:

$$\log\left[\frac{y_t}{y_{t-1}}\right] = \frac{A_t}{A_{t-1}} - 1 + z_t - z_{t-1}, \tag{8.1.5}$$

$$= \theta + z_t - z_{t-1}. \tag{8.1.6}$$

We would like to denote θ and the growth component and $z_t - z_{t-1}$ as the cycle component. We have an estimate of θ from the calibration as the growth rate of per capita GNP. Thus from (8.1.6):

$$\log\left[\frac{y_t^G}{y_{t-1}^G}\right] = \theta, \tag{8.1.7}$$

$$\log y_t^G = \log y_{t-1}^G + \theta, \tag{8.1.8}$$

$$\log y_t^G = \log y_0^G + \theta t. \tag{8.1.9}$$

$$y_t^G = y_0^G \exp(\theta t). \tag{8.1.10}$$

The cycle component is the remainder. Notice that the growth component is constant. We assume a constant trend. But it may be that we wish to consider a 10 year cycle as a change in trend rather than a long business cycle. For that we need a procedure that has a potentially non-constant growth component.

B HP Filter

A more general version is the HP filter. First detrend the data using logs to remove the constant trend. Let the cycle component be $\hat{y}_t$. We wish to divide the fluctuations into long and short run cycles. Let the short run cycles (business cycles) be $\hat{y}_t^c$ and $\hat{y}_t^g$ be the long run cycles (part of the growth component). Then $\hat{y}_t = \hat{y}_t^g + \hat{y}_t^c$, and we have:

$$\min_{\hat{y}_t^g \in [0,\hat{y}_t]} \sum_{t=1}^{T} \left[(\hat{y}_t^c)^2 + \lambda \left[\left(\hat{y}_{t+1}^g - \hat{y}_t^g\right) - \left(\hat{y}_t^g - \hat{y}_{t-1}^g\right) \right]^2 \right] \tag{8.2.11}$$

Substitute in the constraint to get:

$$\min_{\hat{y}_t^g} \sum_{t=1}^{T} \left[(\hat{y}_t - \hat{y}_t^g)^2 + \lambda \left[\left(\hat{y}_{t+1}^g - \hat{y}_t^g\right) - \left(\hat{y}_t^g - \hat{y}_{t-1}^g\right) \right]^2 \right] \tag{8.2.12}$$

So the first term says we choose the growth component to capture as much as the signal as possible, but the second component is a smoothness penalty. If the growth component is not smooth, there is a penalty.

For example, clearly if $\lambda = 0$ there is no smoothness penalty and all of $\hat{y}_t$ is attributed to long run growth. As λ gets infinitly large we ignore the first term and choose:

$$\hat{y}_{t+1}^g - \hat{y}_t^g = \hat{y}_t^g - \hat{y}_{t-1}^g = \bar{y} \tag{8.2.13}$$

Hence we have

$$\hat{y}_{t+1}^g = \hat{y}_t^g + \bar{y} \tag{8.2.14}$$

This is equivalent to a time trend:

$$\hat{y}_{t+1}^g = \hat{y}_0^g + t\bar{y} \tag{8.2.15}$$

For $\lambda = 1600$, we remove any frequencies greater than or equal to 8 years.

Why use a second difference? Well it turns out that if per capital output is integrated order 2, that is has a unit root in the first difference, but not in the second difference, then the above formula is the optimal signal extractor. However, most macro time series are integrated of order 1, they have a unit root in the series, but not the first difference. Variations tend to make little difference however. Even the linear trend behaves very similar.

Note that we detrend both the simulated and actual data to get a fair comparison.

IX Results of Stochastic Growth Model

After detrending, we can compute the moments of both the actual and simulated data. SEE GRAPHS AND TABLES.

Features:

1. Productivity shocks explain 78% of the variation in output, without government policies of any kind!

2. x fluctuates 4.4 times as much as y. Close to the data, where x fluctuates 4.8 times as much as y.

3. c, x, h are very procyclical as in the data.

4. all variables highly correlated with y. May need another shock?

5. Model hours fluctuate only half as much as in the real economy. In fact output and h fluctuate nearly identically in the real data. Also, productivity and hours worked are very positively correlated in the model, but negatively correlated in the data. Some feature of the labor market is not well captured here (see below).

X Conclusions

Overall a sucess. Could fix the labor mkaret via household production, indivisible labor, or taxes.

XI Labor Market Modifications

The problem is that the according to the calibrated parameters, in response to a boom, a positive shock, labor productivity rises, so wages rise, so hours rises. But this effect is small in the model, hours are too smooth.

Explanations:

1. σ is too high. Recall the utility function:

$$u = \frac{\left(c^{1-\alpha} l^{\alpha}\right)^{1-\sigma} - 1}{1-\sigma} \tag{11.0.1}$$

Note that if σ is large, we prefer smooth leisure as well as smooth consumption. If smooth leisure is preferred, we work the same in both booms and recessions. So reduce σ to increase the volatility of hours.

2. α is too small. In the model the intertemporal elasticity of substitution for leisure also depends on α. For α large, the leisure is much more substitutable over time and households are ok with leisure that varies a lot as in the data. For α small, households prefer smooth leisure, and are not attracted by higher wages in a boom to work more. Estimates of α from the micro literature are that α is quite small, however. Most people work their 40 hours regardless of wages.

   However, this neglects movement in and out of the labor force. Individuals may decide to work or not work, and then once they do work a fixed number of hours. In this case, the elasticity may be small among those working, as in the data, but nonetheless, the economy may behave as if the elasticity is infinite.

3. Frictions exist in the labor market. One possibility is that tax rates on wages are varying. Since this extra shock is missing from the model, the data will have more variance than the model. We will examine this hypothesis later in the semester when we do fiscal policy.

4. Indivisibilities may exist in the labor market. If we work either 0, 20, or 40 hours per week, then we can have bigger fluctuations as people move from say 20 to 40 hours.

Invite you to read the Hansen paper, which is quite easy.