Real Business Cycle Models with Money

For nearly the rest of the course, we are going to focus on the role of money in the economy.

- The supply of money (M1) is positively correlated with the business cycle (procyclical).
- The long run correlation between money and output is weak or zero.
- Friedman and Schwartz (1963) money changes lead the business cycle by about two years (most recent data: only about 2 quarters).
- M1 growth is counter cyclical.
- M1 growth and inflation are negatively correlated (?).
- In the long run, M1 growth and inflation are highly positively correlated.
- The correlation of M1 growth and most real variables is weak.
- The correlation of M1 growth and velocity is weak.

Statistic	Std Dev	Correlation with M1 growth	Correlation with Output
Output	1.72	-0.12	1.0
Consumption	0.86	0.02	0.77
Investment	8.24	-0.18	0.76
Hours	1.59	-0.15	0.86
Nominal Interest Rate	1.29	-0.27	0.10
Price Level	0.88	-0.57	-0.16
Inflation	0.57	-0.29	0.34
M1 Level	1.52	0.25	0.33
Velocity	1.94	0.01	0.37

Table 1: Monetary facts. HP filtered quarterly data. Nominal rate is the 1-month Tbill rate.

The basic issue is that we would like money to be procyclical, but relatively neutral in the long run. But the basic intution says that money should be always neutral. It doesn't matter if shmoos are valued at 10 pesos, 2 dollars, or 50 dollars. Agents only care about real variables. Hence the assignment is a difficult one.

I Incomplete Information/Price Rigidities Models

Most models histroically have assumed some kind of incomplete information or pure irrationality to get around this. If agents do not realize the true wage they receive, they might work more hours, for instance. Alternatively, if agents do not observe the real interest rate correctly, they might invest more in real terms. This gives a role for monetary policy since the money supply is under control of the government. This also provides the short run/long run differece in correlations: Presumably agents are only misinformed in the short run.

Alternatively, we could assume wage or price rigidities. For example, there may be menu-costs to adjusting prices. Perhaps prices are only rigid in the short run. Still, we must not only have a fixed wage/price but also a mechanism by which real variables change. Why would I change my hours when monetary aggregates change yet my wage does not?

II The Value of Money

Money has to have positive value in equilibrium to match any correlations. Cannot just helicopter drop in pieces of paper and expect the real economy to be altered.

Recall the three uses for money:

1. Numeraire good. Quoting prices in terms of money is not, however, a reason to hold money.

2. Storage of Value.

3. Medium of exchange.

A Storage of value

That money provides a short term store of value does not in general imply that money is held in equilibrium. Assuming identical risk characteristics, money is dominated in terms of rate of return by capital assets, since money pays no dividend. We could introduce deflation in order for money to have a return equal to capital, but this would be unrealistic and essentially introduce a bubble. Alternatively, we could assume there is no other asset, or that loan markets are in some sense incomplete. This is the motivation of the OG model. Another essential feature is that the consumption good be non-storable. This forces agents to hold money as opposed to extra consumption goods.

B Medium of Exchange

The medium of exchange motive is a reason to hold money in equilibrium. But why hold money which loses value over time instead of bartering? Large economies are too complicated for barter, but in our model with one good, barter becomes a possibility. If households just consume their own production (or their own firm's production), then money has no value. What we need is some requirement that you can't buy your own production. A DECENTRALIZATION of markets. This leads to shopping time models, where money has value in that it reduces shopping time.

- EXPLICIT SHOPPING TIME. Cash-in-Advance models, which explicitly model shopping time.
- IMPLICIT SHOPPING TIME. Money-in-the-Utility-Function models which use an implicit model whereby increases in money holdings imply increases in leisure time.

The real effects now are that inflation causes a decline in activities which require shopping, or decentralized markets in favor of activities which do not. If consumptin goods require shopping, and the cost of shopping goes up, then households consume less, and work less, in favor of leisure, which requires no shopping.

Money in the Utility Function (Cooley – Hanson Framework)

I Introduction

Basic premise is that holding money provides utility. For example:

- Reduction in shopping time increases leisure time.
- Liquidity service, unmodeled transactions cost to converting capital or bonds to cash. Or some cost to being short on cash.

Possible problems still remain. Does money have typically concave preferences with respect to shopping time? Are non-satiation and the Inada conditions satisfied?

II Assumptions

Consider the RBC model, without labor markets. I'll also do just the continuous distribution for productivity shocks.

A Money

Let the per capita stock of money evolve according to

$$\bar{M}_t = (1+\mu)\,\bar{M}_{t-1} \tag{2.1.1}$$

Money is the numeraire, hence we have a price P_t in terms of dollars/goods or the amount of dollars required to purchase 1 unit of consumption good. Hence the supply of real money balances are $q_t = \frac{\bar{M}_t}{P_t}$, or the number of goods one can buy with $\bar{M}_t$ units of currency. Hence:

$$\frac{P_{t+1}}{P_t} q_{t+1} = (1+\mu)\,q_t \tag{2.1.2}$$

Here μ is a policy variable set by the FED, although a relatively simple one since μ is set once for all time.

B Preferences

$$U = u\left(c_t, \frac{M_t}{P_t}\right) \tag{2.2.3}$$

Note that c_t is per capita consumption.

Our most pressing concern is the derivatives U_m and U_{mm}. It is natural to assume concavity, $U_m > 0$, $U_{mm} < 0$. But this doesn't fit easily with money. Once all consumption goods are purchased, there is no more use for money. Hence it might be more natural to assume some sort of satiation point.

Let us see what the derivatives are given money reduces "shopping time". Let

$$U = u\,(c_t, l_t) \tag{2.2.4}$$

Further let the time constraint be $1 = l_t + \bar{h} + \psi_t$. Here ψ_t is shopping time, Assume time spent working is constant. Further, let

$$\psi_t = \psi\left(c_t, \frac{M_t}{P_t}\right) \tag{2.2.5}$$

After substituting in we get:

$$U = u\left(c_t, 1 - h - \psi\left(c_t, \frac{M_t}{P_t}\right)\right) \tag{2.2.6}$$

This is in the form we need.

The derivatives are:

$$u_m = -u_l \psi_m \tag{2.2.7}$$

Now if $\psi_m < 0$, then $U_m > 0$. first derivative. Hence we need that increases in money reduce shopping time. Second derivative is:

$$u_{mm} = u_{ll} \left(\psi_m\right)^2 - u_l \psi_{mm} \tag{2.2.8}$$

Here it is sufficient to have $\psi_{mm} \geq 0$ for concavity. Marginal shopping time is non-decreasing.

Liquidity services model is similar, we need liquidity costs to be convex (Fenstra, JME, 1986).

C Government

Government controls the growth rate of money. Government prints money to cover its exogenous expenditures g per capita units of the consumption good. The government also gives out transfer payments, TR_t per capita (one can think of TR_t as a helicopter drop of cash from the government). Hence the government budget constraint is:

$$g + TR_t = \mu \frac{\bar{M}_t}{P_t} \tag{2.3.9}$$

Note that the right hand side are seniorage revenues:

$$\text{seniorage } = \text{value of printed money } - \text{ printing costs } = \mu \frac{\bar{M}_t}{P_t}, \tag{2.3.10}$$

assuming printing costs are zero. Note the constraint is in terms of the consumption good. So the idea is that any excess seniorage will be returned to the consumers as transfers.

D Firms

Money is a public good, so the social planning problem and competitive equilibrium will generally yield different outcomes (the social welfare theorems no longer hold). Hence we must work directly with the competitive equilibrium. First, we consider the firm's problem.

$$\Pi = \max_{k_t, h_t} e^{z_t} F\left(k_t, h_t\right) - r_t k_t - w_t h_t \tag{2.4.11}$$

$$r_t = e^{z_t} F_k\left(k_t, h_t\right) \tag{2.4.12}$$

$$w_t = e^{z_t} F_h\left(k_t, h_t\right) \tag{2.4.13}$$

In equilibrium, firms have zero profits, and $h_t = 1$ since households have no preference for leisure. Hence:

$$r_t = e^{z_t} F_k\left(K_t{,}1\right) \equiv e^{z_t} f_k\left(K_t\right) \tag{2.4.14}$$

$$w_t = e^{z_t} F_h\left(K_t{,}1\right) = e^{z_t} f\left(K_t\right) - e^{z_t} f_k\left(K_t\right) K_t \tag{2.4.15}$$

The last equality follows from Euler's theorem (see page 51 of my notes on growth).

E budget constraint

$$r_t k_t + w_t + (1-\delta)\, k_t + \frac{M_t}{P_t} + TR_t = (1+\eta)\, k_{t+1} + c_t + \frac{M_{t+1}}{P_t} \tag{2.5.16}$$

The budget constraint units are consumption goods per capita. Simplifying:

$$\begin{aligned} e^{z_t} f_k\left(K_t\right)\left(k_t - K_t\right) + e^{z_t} f\left(K_t\right) \quad + \quad & (1-\delta)\, k_t + \frac{M_t}{P_t} + TR_t = \\ & (1+\eta)\, k_{t+1} + c_t + \frac{M_{t+1}}{P_t} \end{aligned} \tag{2.5.17}$$

Note I am carefully keeping track of the difference between household capital k, and aggregate capital K. The household controls it's own capital over time. The household does not control the aggregate capital stock, which determines the factor prices.

III Problem

A Optimization Problem

The problem is to maximize utility subject to the budget constraint.

$$W = \max_{k_{t+1}, m_{t+1}} E\left[\sum_{t=0}^{\infty} \beta^t u\left(c_t, \frac{M_t}{P_t}\right)\right] \tag{3.1.1}$$

Subject to:

$$e^{z_t} f_k(K_t)(k_t - K_t) \quad + \quad e^{z_t} f(K_t) + (1-\delta) k_t + \frac{M_t}{P_t} + TR_t =$$
$$(1+\eta) k_{t+1} + c_t + \frac{M_{t+1}}{P_t} \tag{3.1.2}$$

$$z_{t+1} = \rho z_t + \epsilon_t \tag{3.1.3}$$

$$\frac{\bar{M}_{t+1}}{P_t} = (1+\mu) \frac{\bar{M}_t}{P_t} \tag{3.1.4}$$

$$g + TR_t = \mu \frac{\bar{M}_t}{P_t} \tag{3.1.5}$$

We also have general equilibrium restrictions. These are that money supply equal money demand and that identical households make identical decisions.

$$k_t = K_t, \quad M_t = \bar{M}_t \quad \rightarrow \quad M_{t+1} = \bar{M}_{t+1}, \quad k_{t+1} = K_{t+1}. \tag{3.1.6}$$

B Normalization

Aggregate money and prices are not stationary (increasing to infinity), so we need to make them stationary before writing the value function equation. There is obvious way and a not so obvious way. The obvious way is to use the stationary inflation rate and real balances:

$$q_{t+1}(1+\pi_t) = (1+\mu) q_t \qquad \frac{M_t}{P_t} = q_t \tag{3.2.7}$$

From the point of view of the household, or analytically, there is nothing wrong with the above framework. In fact the household does not need to know P_{t+1}, the term $(1+\pi_t) m_{t+1}$ is independent of P_{t+1} (thus the control variable would be $m_{t+1}(1+\pi_t)$). However, from the point of view of the economist trying to solve the model on computer, P_{t+1} is more difficult. In effect, we have to guess a pricing function $p(s_t)$, where s_t is the states. Then we have to show that the equilibrium pricing function P_t is the same (a rational expectations equilibrium). Most likely, we have to iterate on computer, guessing pricing function then updating. This normalization also does not eliminate any variables.

A second strategy is to rely on the economy wide versus individual variables. Let $\hat{m}_t = \frac{M_t}{\bar{M}_t}$ be the fraction of the per-capital money stock that an individual holds. Let $\hat{p} = \frac{P_t}{\bar{M}_t}$ be the inverse of the supply of real balances. Then we have:

$$\frac{M_t}{P_t} = \frac{M_t}{\bar{M}_t}\frac{\bar{M}_t}{P_t} = \frac{\hat{m}_t}{\hat{p}_t} \tag{3.2.8}$$

$$\frac{M_{t+1}}{P_t} = \frac{M_{t+1}}{\bar{M}_{t+1}}\frac{\bar{M}_{t+1}}{\bar{M}_t}\frac{\bar{M}_t}{P_t} = \frac{\hat{m}_{t+1}\left(1+\mu\right)}{\hat{p}_t} \tag{3.2.9}$$

This is easier computationally and eliminates $\bar{M}$.

IV Recursive Problem

Rewriting the government budget constraint and household budget constraint gives:

$$g + TR_t = \mu\frac{1}{\hat{p}_t} \tag{4.0.1}$$

$$\begin{aligned} &e^{z_t} f_k\left(K_t\right)\left(k_t - K_t\right) + e^{z_t} f\left(K_t\right) + \left(1-\delta\right) k_t + \frac{\hat{m}_t}{\hat{p}_t} + TR_t = \\ &\quad \left(1+\eta\right) k_{t+1} + c_t + \frac{\hat{m}_{t+1}\left(1+\mu\right)}{\hat{p}_t} \end{aligned} \tag{4.0.2}$$

Notice that transfers in the government budget, which the household takes as given, is a function of other variables the household takes as given. Thus we can substitute one set of variables the household takes as given for another:

$$\begin{aligned} &e^{z_t} f_k\left(K_t\right)\left(k_t - K_t\right) + e^{z_t} f\left(K_t\right) + \left(1-\delta\right) k_t + \frac{\hat{m}_t + \mu}{\hat{p}_t} - g = \\ &\quad \left(1+\eta\right) k_{t+1} + c_t + \frac{\hat{m}_{t+1}\left(1+\mu\right)}{\hat{p}_t} \end{aligned} \tag{4.0.3}$$

We are now in a position to write the value function.

$$\begin{aligned} V\left(k, K, \hat{m}, z\right) &= \max_{k', \hat{m}'}\left\{u\left[e^{z} f_k\left(K\right)\left(k-K\right) + e^{z} f\left(K\right) + \left(1-\delta\right) k + \frac{\hat{m}+\mu}{\hat{p}} - g\right.\right. \\ &\qquad \left.\left. - \left(1+\eta\right) k' - \frac{\left(1+\mu\right)\hat{m}'}{\hat{p}}, \frac{\hat{m}}{\hat{p}}\right] + \beta \mathrm{E}\left[V\left(k', K', \hat{m}', \rho z + \epsilon\right)\right]\right\} \end{aligned} \tag{4.0.4}$$

Again we have the general equilibrium restrictions that if the household is the same as the other households, everyone must make the same decisions:

$$k = K, \quad \hat{m} = 1 \quad \rightarrow \quad \hat{m}' = 1, \quad k' = K' \tag{4.0.5}$$

A Equilibrium

Let us define the equilibrium.

Definition 23 *A* Recursive Competitive Equilibrium *given individual states* $s = (k, K, \hat{m}, z)$ *and aggregate states* $S = (K, K, 1, z)$ *consists of individual decisions:* $c(s)$*,* $\hat{m}'(s)$*,* $k'(s)$*, aggregate decisions:* $C(S)$*,* $K'(S)$*, prices:* $\hat{p}(S)$*,* $r(S)$*,* $w(S)$*, and a value function* $v(s)$*, such that: the value function holds (1 equation), households optimize (2 equations), the first order conditions hold when equilibrium conditions,* $s = S$ $(k = K,\ \hat{m} = 1)$ *imply* $s' = S'$ $(k' = K',\ \hat{m}' = 1)$*, hold, (2 eqns), firms optimize (2 eqns), the budget constraint holds, and the resource constraint holds.*

I count 9 equations and 9 unknowns.

B First order conditions

$$\frac{(1+\mu)}{\hat{p}} u_c\left(c, \frac{\hat{m}}{\hat{p}}\right) = \beta \mathrm{E}\left[V_m\left(k', K', \hat{m}', \rho z + \epsilon\right)\right] \tag{4.2.6}$$

$$(1+\eta)\, u_c\left(c, \frac{\hat{m}}{\hat{p}}\right) = \beta \mathrm{E}\left[V_k\left(k', K', \hat{m}', \rho z + \epsilon\right)\right] \tag{4.2.7}$$

Note that in equation (4.2.6), the left hand side is the MU_c, foregone by investing in money, and the right hand side is the value of holding real money balances. The first order conditions determine $\hat{m}'(k, K, \hat{m}, z)$ and $k'(k, K, \hat{m}, z)$.

Now substitute in the general equilibrium requirements. Substitute out for for $\hat{m}'$, $\hat{m}$, k, and k'. What is left will determine prices and aggregate decisions.

$$\frac{(1+\mu)}{\hat{p}} u_c\left[C, \frac{1}{\hat{p}}\right] = \beta \mathrm{E}\left[V_m\left(K', K', 1, \rho z + \epsilon\right)\right], \tag{4.2.8}$$

$$(1+\eta)\, u_c\left[C, \frac{1}{\hat{p}}\right] = \beta \mathrm{E}\left[V_k\left(K', K', 1, \rho z + \epsilon\right)\right], \tag{4.2.9}$$

$$C = e^z f(K) + (1-\delta) K - (1+\eta) K' - g. \tag{4.2.10}$$

These two equations determine $\hat{p}(K,z)$ and $K'(K,z)$. Since these variables are aggregate decisions/prices, they are a function of only aggregate states (observe no individual states are in equations 4.2.8 and 4.2.9). Effectively, we are equating supply and demand to get the price.

C Envelope equations

We have:

$$v_m(k, K, \hat{m}, z) = \frac{1}{\hat{p}}\left[u_c\left(c, \frac{\hat{m}}{\hat{p}}\right) + u_m\left(c, \frac{\hat{m}}{\hat{p}}\right)\right] \tag{4.3.11}$$

Money provides utility directly, and can be traded in for consumption at rate $\hat{p}$.

$$v_k(k, K, \hat{m}, z) = u_c\left(c, \frac{\hat{m}}{\hat{p}}\right)\left(e^z f_k(K) + 1 - \delta\right) \tag{4.3.12}$$

In equilibrium:

$$v_m(K, K, 1, z) = \frac{1}{\hat{p}}\left[u_c\left(C, \frac{1}{\hat{p}}\right) + u_m\left(C, \frac{\hat{m}}{\hat{p}}\right)\right] \tag{4.3.13}$$

$$v_k(K, K, 1, z) = u_c\left(C, \frac{1}{\hat{p}}\right)\left(e^z f_k(K) + 1 - \delta\right) \tag{4.3.14}$$

V Super Nuetrality of the Steady State

A Certainty Equivalence Steady state

In the steady state $\hat{p}, k, \hat{m}$, and $z = 0$ are constant. Using the envelope for K to eliminate the derivative of v gives:

$$(1+\eta) u_c\left(c, \frac{1}{\hat{p}}\right) = \beta u_c\left(c, \frac{1}{\hat{p}}\right)\left(f_k\left(\bar{K}\right) + 1 - \delta\right) \tag{5.1.1}$$

$$\rho = f_k\left(\bar{K}\right) - \delta \tag{5.1.2}$$

Using the envelope for $\hat{m}$ to eliminate the derivative of v in the first order condition results in:

$$\frac{(1+\mu)}{\bar{\hat{p}}} u_c\left(C, \frac{1}{\hat{p}}\right) = \beta \frac{1}{\bar{\hat{p}}}\left(u_c\left(C, \frac{1}{\hat{p}}\right) + u_m\left(c, \frac{1}{\hat{p}}\right)\right), \tag{5.1.3}$$

$$(1+\mu-\beta)\, u_c\left(C, \frac{1}{\hat{p}}\right) = \beta u_m\left(C, \frac{1}{\hat{p}}\right) \tag{5.1.4}$$

The steady state capital is not a function of μ, m, or $\hat{p}$. Equation (5.1.2) is the modified golden rule. Steady state consumption is:

$$C = f\left(\bar{K}\right) - (\delta+\eta)\,\bar{K} - g, \tag{5.1.5}$$

which is also independent of money. We have finally that $\bar{\hat{m}} = 1$, of couse. Equation (5.1.4) determines the steady state real money balances $\hat{p}$. Note that money balances are not independent of the utility function. The steady state is then equations (5.1.2), (5.1.4) and (5.1.5), and $z = 0$, which solve for C, K, $\hat{p}$, and z.

B Super Neutrality and Neutrality

Suppose we have an increase in the growth rate of money. Does the steady state real money balances, change? If so, how? From equation (5.1.4), we see that:

$$(1+\mu-\beta) = \frac{\beta u_m\left(C, \frac{1}{\hat{p}}\right)}{u_c\left(C, \frac{1}{\hat{p}}\right)} \tag{5.2.6}$$

Increasing μ clearly raises the left hand side. The right hand side must therefore also rise. Assuming $u_{mm} < 0$, a decrease in $\frac{1}{\hat{p}}$ increases the numerator. Assuming $u_{cm} \geq 0$, the denominator falls. So a sufficient condition is $u_{cm} \geq 0$ for steady state real balances to fall when the money growth rate increases.

Note that this is where one gets into trouble with the money in the utility function model. In the liquidity model, u_{cm} is generally positive (Fenstra, JME 1986). Also generally found to be positive when calibrated.

EXERCISE: Under what conditions is u_{cm} positive in the shopping time model?

For neutrality we have:

Definition 24 *The economy is super neutral if all real variables are independent of the* *money growth rate.*

Definition 25 *The economy is neutral if all real variables except real money balances are independent of the money growth rate.*

Here the model does not have the super-neutrality property in the steady state, because real balances change with μ. However, neutrality holds since the remaining steady state real variables (K, C) are independent of the steady state nominal sector ($\hat{p}$ and $\hat{m}$). Essentially the same result as Sidrauski (1967). Model then matches the long run features of the data. However, the result is not very robust.

EXERCISE: Show the result fails in the model with labor markets.

However, Danthine, Donaldson, and Smith JME 1987 show that deviations are generally small (a good example of a computational question). For example, $\mu = 0$ gives a $\bar{K} = .18485$ and $\mu = 4$ gives $\bar{K} = .18629$.

VI Optimal Monetary Policy

A natural question is what value of μ or π maximizes social welfare? To answer this question, we find the efficient allocations using the social planning problem. Then we find the value of μ which makes the social welfare problem identical to the competitive model. Since the social planning problem maximizes welfare, the competitive economy will as well.

A Social Planning Problem

First we translate the budget constraint into an overall resource constraint. The social planner does not use a price system or a government, but instead simply allocates resources. Recall that the budget constraints are:

$$\begin{aligned} & e^{z_t} f_k\left(K_t\right)\left(k_t - K_t\right) + f\left(K_t\right) + (1-\delta)\, k_t + \frac{M_t}{P_t} + TR_t = \\ & \quad \left(1+\eta\right) k_{t+1} + c_t + \frac{M_{t+1}}{P_t} \end{aligned} \tag{6.1.1}$$

$$TR_t = \mu \frac{M_t}{P_t} + g \tag{6.1.2}$$

$$\frac{M_{t+1}}{P_t} = (1+\mu)\,\frac{M_t}{P_t} \tag{6.1.3}$$

Substituting in the government budget constraint, equilibrium, and the law of motion for money into the budget constraint we obtain the overall resource constraint:

$$e^{z_t} f(k_t) + (1-\delta) k_t = (1+\eta) k_{t+1} + c_t + g \tag{6.1.4}$$

This is just the national income identity.

Hence the social planner's problem is to maximize utility subject to the resource constraint.

$$\max_{k_{t+1}, c_t, m_{t+1}} \sum_{t=0}^{\infty} \beta^t u\left(c_t, \frac{M_t}{P_t}\right) \tag{6.1.5}$$

Subject to the resource constraint:

$$e^{z_t} f(k_t) + (1-\delta) k_t = (1+\eta) k_{t+1} + c_t \tag{6.1.6}$$

B Value Function

So prices vanish, except real money balances are still in the utility function. Our normalization here is more straightforward: we can just use $m_t = \frac{M_t}{P_t}$. The value function is similar to before:

$$\begin{aligned} V(k, m, z) &= \max_{k', m'} \left\{ u\left[e^z f(k) + (1-\delta) k - (1+\eta) k' - g, m_t \right] + \right. \\ &\quad \left. \beta \mathrm{E}\left[V(k', m', \rho z + \epsilon) \right] \right\} \end{aligned} \tag{6.2.7}$$

C FOC's

The first order conditions are:

$$\beta \mathrm{E}\left[V_m(k', m', \rho z + \epsilon) \right] = 0 \tag{6.3.8}$$

$$(1+\eta) u_c(c, m) = \beta \mathrm{E}\left[V_k(k', m', \rho z + \epsilon) \right] \tag{6.3.9}$$

Real money balances are costless for the social planner to make. Hence it is optimal to satiate the economy with real money balances, or to produce as much as possible.

D Envelope equations

The envelope equations are:

$$V_m(k, m, z) = U_m \tag{6.4.10}$$

Money provides utility directly through reduced shopping.

$$v_k(k, m, z) = u_c(c, m)\left(e^z f_k(k) + 1 - \delta\right) \tag{6.4.11}$$

E Steady state and optimal policy

In the steady state:

$$\beta u_m[c, m] = 0 \tag{6.5.12}$$

Now if real money balances do not satiate, we want $m \to \infty$. If money balances do satiate, u_m eventually goes to zero as m approaches some m^*. So we wish to choose m^*, or let m grow without bound. How can this be achieved in the competitive model?

Recall that the steady state condition for m was:

$$(1 + \mu - \beta)\, u_c(c, m) = \beta u_m(c, m) \tag{6.5.13}$$

Hence we need:

$$1 + \mu - \beta = 0 \tag{6.5.14}$$

So we need:

$$\pi = \mu = -(1 - \beta) \tag{6.5.15}$$

This is the famous CHICAGO RULE, deflate at $1 - \beta$.

We can formulate the rule in terms of the interest rate and inflation rate. The steady state equation for k is:

$$1 + \eta = \hat{\beta}\left(f_k(k) + 1 - \delta\right) \tag{6.5.16}$$

If we assume the population welfare function, and let $r = f_k - \delta$ be the net rate of interest, then:

$$1 = \beta (r + 1) \tag{6.5.17}$$

Hence:

$$\beta = \frac{1}{r + 1} \tag{6.5.18}$$

Combining (6.5.15) and (6.5.18):

$$\mu = \pi = \frac{P' - P}{P} = -1 + \frac{1}{r + 1} \tag{6.5.19}$$

$$\frac{P'}{P} = \frac{1}{r + 1} \tag{6.5.20}$$

$$(1 + \pi)(1 + r) - 1 = 0. \tag{6.5.21}$$

Notice that the left hand side is the nominal interest rate. So the optimal monetary policy is to set the nominal rate $R = 0$. Now when writing the nominal rate it is often typical to assume that $\pi r \approx 0$. In this case (6.5.21) reduces to:

$$\pi + r + \pi r = 0, \tag{6.5.22}$$

$$r + \pi = R = 0, \tag{6.5.23}$$

$$\pi = -r. \tag{6.5.24}$$

Hence the optimal monetary policy is to deflate at the real rate of interest (Friedman, 1969). The Friedman rule can also be written as setting the nominal interest rate equal to zero. The interpretation if money and financial assets earn the same rate of return, there is no disincentive to holding money and households hold enough real balances until satiation occurs. Since money is costless to produce, we should produce as much real balances as is needed.

This result is very robust.

- The result is robust to adding labor markets, holds for Cash-in-Advance models, etc.

- Gomis and Peralta have a model where it is not robust. Sellers have monopoly power and overcharge buyers. Inflation allows the buyers to buy with dollars that are worth less, thus potentially canceling this advantage.

- If there are factor or income taxes only (no lump sum taxes), the government minimizes marginal deadweight loss across all taxes. Note: In our model, a negative inflation tax is OK because the government can always finance expenditures by lump sum taxes, (which are negative lump sum transfers). A positive inflation tax may be required if the government must finance with income or factor taxes (Phelps, 1972). But Barro and Fisher (1976) argue that the tax still turns out to be negative quantitatively.

- Lucas and Stokey (1988) and Kimbrough (1986) argue that money is an intermediate good (shopping is an intermediate good for consumption) and therefore should not be taxed.

- We will see in the cash-in-advance models, that although money is an intermediate good, there is no alternative input to production (ie no alternative method of payment). Therefore, taxing money has low welfare loss (Cooley-Hanson, JET, 1992). Money thus acts like a tax on consumption. This has led to some modifications of CIA models to allow alternative methods of payment. The welfare loss is larger in this case.

- The Friedman rule may not be implementable. Recall that $\beta u_m\left(\bar{c}, \frac{1}{\bar{\hat{p}}}\right) = 0$. If u has no local satiation, then $\frac{1}{\bar{\hat{p}}} = \infty$. From the equation for government transfers:

$$TR = \mu \frac{1}{\bar{\hat{p}}} - g = -(1-\beta)\frac{1}{\bar{\hat{p}}} - g \;\rightarrow\; -\infty \tag{6.5.25}$$

  Thus if no local satiation exists, the government must eventually have infinite lump sum taxes.

Cash-In-Advance

I Introduction

The set up separates production and consumption. The idea is, rather than add heterogeneity to get trade, simply assume you cannot consume your own production.

A Decentralized Markets

If everyone can get together in a big room and submit demand curves to the auctioneer, money is not necessary and will at best be neutral. Similarly, we need a rule that you cannot consume your own production. Otherwise, no one would hold expensive cash to buy goods from others.

Each household will now be divided into workers and shoppers. One member of the household sells production goods, and the other buys, but not at the same place.

B Time Line

At the start of the period an agent has currency holdings, which the shopper takes to the market, while the producer stays home, produces, and sells the resulting output to another household shopper.

An important consideration is when the transfer occurs. We will consider two models. One in which the transfer occurs prior to the shopper leaving (ie the shopper gets the transfer) and the other where the transfer occurs after the shopper leaves (ie the worker gets the transfer).

Consumption goods must then be purchased with cash, while output and non-depreciated capital are sold for money. The worker sells while the shopper buys.

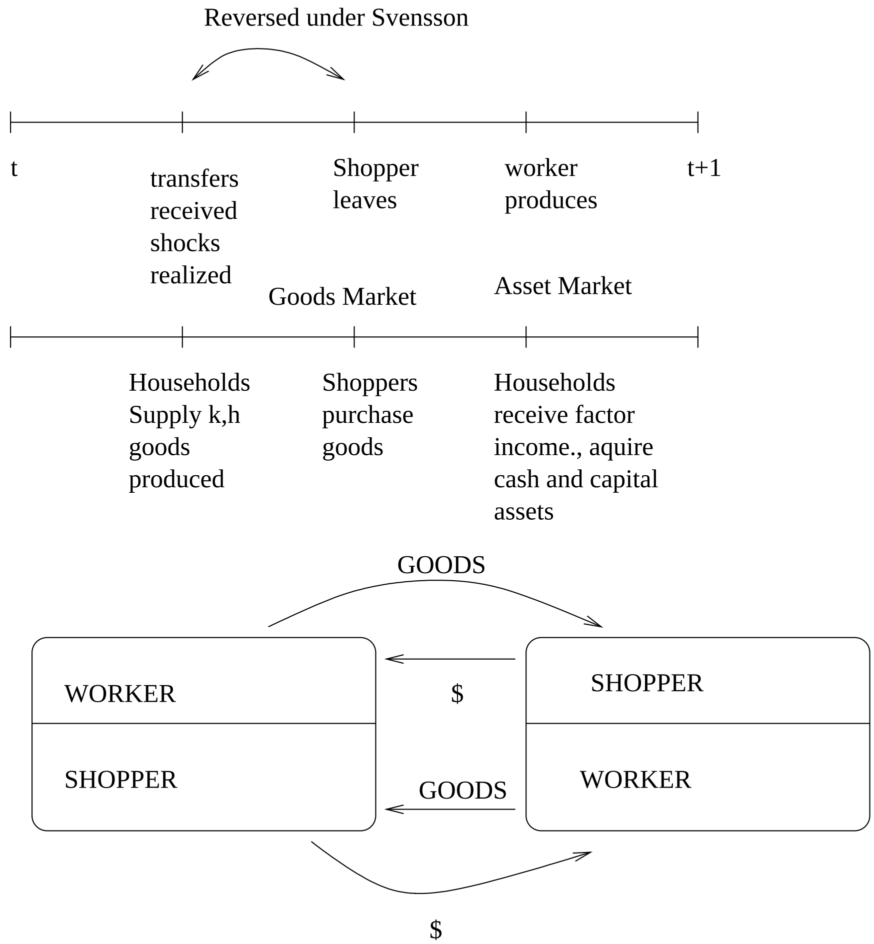

Figure 26: Decentralization of Markets.

Investment is purchased normally, allocate some of output and capital to investment. The key is that households may not consume their own shmoos. Cash revenues from the sale of production goods are carried over to the next period.

An alternative interpretation is that asset markets are incomplete. In the first sub-period, the asset market is closed but the goods market is open. Thus trading for goods involves only money. Then after the goods market closes, the asset market opens for trading of capital goods for capital goods.

II Assumptions

Start with the RBC model and labor markets, with no population growth.

A Cash in advance constraint

The requirement that households have cash to make purchases creates a cash-in-advance constraint. Two main assumptions are needed for the cash in advance constraint. The first is what good require cash and the second is when the transfer occurrs.

1 What goods require cash?

1. Cash good/credit good. Add a new consumption good which is purchased only with cash. This good has the same price as the shmoo/capital good, since we assume it is produced with an identical technology. Think of it as a red shmoo and a white shmoo. One can think of credit goods as goods purchased with a credit card.

   Let c_{1t} denote consumption of the cash good, while c_{2t} is consumption of the credit good. The "cash in advance constraint" is then:

   $$\frac{M_t + TR_t}{P_t} \geq c_{1t} \tag{2.1.1}$$

   Here I have written the cash in advance constraint assuming the transfer goes to the shopper.

   Since shopping is done in advance, before money is obtained through the sale of production, we are constrained in how much of the cash good we can buy, because we need the cash in advance, which comes from the sale of goods last period. This is the set up of Lucas and Stokey (1987) Econometrica and chapter 7 of Cooley (Cooley-Hansen). Real balances in period t are $\frac{M+TR}{P}$.

2. All consumption goods. The model we will consider in class is Svennsson article, which requires cash for all purchases (the original CIA constraint from Clower, 1956 also required cash for all consumption goods). The cash-in-advance (CIA) constraint, assuming the shopper does not get the transfer, is:

   $$\frac{M_t}{P_t} \geq c_t \tag{2.1.2}$$

3. Other versions:

   - CIA for consumption and investment goods (Stockman 1981).

- Cash or costly credit for all consumption goods but credit is more costly for some goods than others (Dotsey and Ireland).
- Cash is required for firms to make wage and interest payments.

2 Transfer to the worker or shopper?

1. To the shopper. Suppose the money shock is realized at the beginning of the period. This is essentially equivalent to giving the transfer payment to the shopper. Hence there is no uncertainty about how much money the producer needs to carry over. If the transfer is unusually high, so are prices, but then the shopper also has more cash. If the transfer is unusually low, so are prices, but then the shopper has less cash. So the CIA constraint binds in this case, as the household will only carry enough cash to finance purchases and no more, since money has a lower return than capital.

2. Percautionary demand/Liquidity model. Suppose that money shocks are announced at the end of the period, i.e. the random transfer payment is given to the producer, not the shopper. Then there is uncertainty about how much money the producer needs to carry forward. If the transfer is high, then so are prices. The worker already has a large transfer and does not want to accept cash. But then the shopper might be short cash. To prevent this, the household will carry extra cash and so sometimes the constraint does not bind. This is known as precautionary or liquidity demand.

   - The essential trade-off is that money earns a lower return (actually negative, assuming positive money growth) than capital. Hence the consumer holds as little money as possible.
   - However, uncertainty may exist over what real balances are next period. If there is a big money growth shock, we have inflation and quantity effects. A possible cash crunch of not being able to purchase as much goods as desired due to a high inflation and low cash balances. Or there may be percautionary money holdings, hold extra cash to avoid this possibility.
   - If the transfer is at the beginning of the period, the CIA constraint always binds. Money has real effects due to the distortion that the inflation tax puts on consumption of cash vs. credit goods. This model is harder to solve, but has certain advantages of realism:

- Velocity can vary. Consider first the CIA constraint that $m_t = c_t$. Then the consumption velocity is given by $MV = Pc$ and hence $V = 1$ (here capital V is velocity and v is the value function). Similarly, in Stockman the velocity of private spending $(c + I)$ is also one. The Lucas and Stokey framework has consumption velocity which varies. The data suggests increasing inflation results in increased velocity. A positive money shock brings the CIA constraint closer to binding, hence velocity increases.
- Money is not perfectly correlated with consumption, which better reflects the data.

3 Realism of the CIA constraint

Is this realistic? Clearly we are paid at the end of $t-1$ or beginning of t for work done in period $t-1$. Thus we do need cash in advance: we must keep some of our wages in cash since we shop in period t before receiving period t wages. Most people keep some of $t-1$ wages in cash for precautionary reasons: in case unforseen spending needs arise. Finally, credit cards are clearly a credit technology: they allow you to use period t wages to pay for period t spending.

B Stochastic Money Growth

The money growth equation is given by:

$$\bar{M}_{t+1} = e^{\mu_t} \bar{M}_t \tag{2.2.3}$$

Real balances follow:

$$\frac{P_{t+1}}{P_t} \bar{m}_{t+1} = e^{\mu_t} \bar{m}_t \tag{2.2.4}$$

The money shock follows an autoregressive process:

$$\mu_{t+1} = \theta \mu_t + \zeta_{t+1} \tag{2.2.5}$$

Let ζ be log-normally distributed, uncorrelated with ϵ, and:

$$E\left[\zeta_{t+1}\right] = (1-\theta)\,\bar{\mu} \tag{2.2.6}$$

This keeps money growth rates bounded away from zero. There is no possibility of negative inflation. Think of $\bar{\mu}$ as the policy variable, or target growth rate.

The FED's actual target is a short term nominal interest rate, not the money growth rate. But in the past it has targeted a money growth rate. The FED also occasionally changes the target, which we don't allow here.

Seniorage revenues are:

$$\bar{M_{t+1}} - \bar{M}_t = (e^{\mu_t} - 1)\,\bar{M}_t = \text{ seniorage} \tag{2.2.7}$$

We assume seniorage revenues are returned to the household in the form of a cash transfer. Later we will allow for open market operations. Therefore, the government budget constraint is:

$$TR_t = (e^{\mu_t} - 1)\,\frac{\bar{M}_t}{P_t} \tag{2.2.8}$$

III Problem: Precautionary Model

A Budget Constraint

$$r_t k_t + w_t h_t + \frac{M_t}{P_t} + TR_t = k_{t+1} - (1-\delta_k)\,k_t + c_t + \frac{M_{t+1}}{P_t} \tag{3.1.1}$$

B Normalized Budget constraint

As before, let $\hat{m}_t = \frac{M_t}{\bar{M}_t}$ and $\hat{p}_t = \frac{P_t}{\bar{M}_t}$ so that:

$$r_t k_t + w_t h_t + \frac{\hat{m}_t}{\hat{p}_t} + TR_t = k_{t+1} - (1-\delta_k)\,k_t + c_t + e^{\mu_t}\frac{\hat{m}_{t+1}}{\hat{p}_t} \tag{3.2.2}$$

The worker produces shmoos, and sells all shmoos for cash except those used for investment x_t. Meanwhile, shopper spends some or all of the cash on the consumption good. Notice the implicit decentralization, although at the end, the worker and shopper pool their resources into one budget constraint.

Normalized transfers are:

$$TR_t = (e^{\mu_t} - 1)\,\frac{1}{\hat{p}_t} \tag{3.2.3}$$

The normalized CIA constraint is:

$$c_t \leq \frac{M_t}{P_t} = \frac{\hat{m}_t}{\hat{p}_t} \tag{3.2.4}$$

C Household Problem: competitive model

$$\max_{c_t, \hat{m}_{t+1}, k_{t+1}, h_t} E\left[\sum_{t=0}^{\infty} \beta^t u\left(c_t, 1 - h_t\right)\right] \tag{3.3.5}$$

Subject to household constraints:

$$c_t \leq \frac{\hat{m}_t}{\hat{p}_t} \tag{3.3.6}$$

$$\begin{aligned} e^{z_t} F_k\left(K_t, H_t\right) k_t + e^{z_t} F_h\left(K_t, H_t\right) h_t + \frac{\hat{m}_t + e^{\mu_t} - 1}{\hat{p}_t} = \\ k_{t+1} - \left(1 - \delta_k\right) k_t + c_t + e^{\mu_t} \frac{\hat{m}_{t+1}}{\hat{p}_t}, \end{aligned} \tag{3.3.7}$$

and subject to general equilibrium and the law of motion for the shocks.

$$\mu_{t+1} = \theta \mu_t + \zeta_{t+1} \tag{3.3.8}$$

$$z_{t+1} = \rho z_t + \epsilon_{t+1} \tag{3.3.9}$$

$$k_t = K_t,\ \hat{m}_t = 1 \ \Rightarrow\ k_{t+1} = K_{t+1} \ \ \hat{m}_{t+1} = 1 \ \ h_t = H_t \tag{3.3.10}$$

D Value function

The budget constraint holds with equality and can thus be substituted into both the utility function and the CIA constraint. The resulting constraint is the constraint of the worker: next period's real balances must be larger than factor income less investment spending. The worker keeps some income for investment spending, and sells the rest of the production for cash. However, the shopper may come back with unused cash in which case $t+1$ real balances will be greater than the value of the goods sold to another household. Further, we need only keep track of 2 of the 3 constraints. The budget and workers constraint holds implies the CIA constraint holds. The worker's constraint does not in general hold with equality. Hence we must keep track of it with a multiplier.

Hence we have:

$$\begin{aligned} v(k,K,\hat{m},z,\mu) &= \max_{k',\hat{m}',h} \Bigg\{ u\Bigg[e^z F_k(K,H)\,k + e^z F_h(K,H)\,h + \frac{\hat{m}+e^{\mu}-1}{\hat{p}} - k' + (1-\delta_k)\,k - \\ & e^{\mu}\frac{\hat{m}'}{\hat{p}}, 1-h \Bigg] + \lambda\Bigg[k' - (1-\delta_k)\,k + \frac{e^{\mu}\hat{m}'}{\hat{p}} - \frac{e^{\mu}-1}{\hat{p}} - e^z F_k(K,H)\,k - \\ & -e^z F_h(K,H)\,h \Bigg] + \beta E\Bigg[V(k',K',\hat{m}',\rho z+\epsilon',\theta\mu+\zeta') \Bigg] \Bigg\} \end{aligned} \tag{3.4.11}$$

Subject to the general equilibrium:

$$k = K,\ \hat{m} = 1\ \Rightarrow\ \hat{m}' = 1\ \ k' = K'\ \ h = H\ \ \lambda = \Lambda \tag{3.4.12}$$

E Equilibrium

Definition 26 *A* Recursive Competitive Equilibrium *given individual states* $s = (k,K,\hat{m},z,\mu)$ *and aggregate states* $S = (K,K,1,z,\mu)$ *consists of individual decisions:* $c(s)$, $\hat{m}'(s)$, $k'(s)$, $h(s)$, $\lambda(s)$, *aggregate decisions:* $C(S)$, $K'(S)$, $H(S)$, $\Lambda(S)$, *prices:* $\hat{p}(S)$, $r(S)$, $w(S)$, *and a value function* $v(s)$, *such that: the value function holds (1 equation), households optimize (3 equations and 1 Kuhn-Tucker equation), the first order conditions hold when equilibrium conditions,* $k = K$, $\hat{m} = 1$ *imply* $k' = K'$, $h = H$, $\lambda = \Lambda$, $\hat{m}' = 1$, *hold, (4 eqns including KT), firms optimize (2 eqns), the budget constraint holds, and the resource constraint holds.*

I count 13 equations and 13 unknowns.

F First Order Conditions

$$u_c(c,1-h) - \lambda = \beta E\left[v_k(k',K',\hat{m}',\rho z+\epsilon',\theta\mu+\zeta') \right] \tag{3.6.13}$$

$$\frac{e^{\mu}}{\hat{p}}\left(u_c(c,1-h) - \lambda\right) = \beta E\left[v_m(k',K',\hat{m}',\rho z+\epsilon',\theta\mu+\zeta') \right] \tag{3.6.14}$$

$$\left(u_c(c,1-h) - \lambda\right) e^z F_h(K,H) = u_l(c,1-h) \tag{3.6.15}$$

$$\lambda\left[k' - (1-\delta_k)\,k + \frac{e^{\mu}\hat{m}'}{\hat{p}} - \frac{e^{\mu}-1}{\hat{p}} - e^z F_k(K,H)\,k - e^z F_h(K,H)\,h \right] = 0 \tag{3.6.16}$$

I have left in the Kuhn-Tucker conditions since we don't know if the constraint binds. λ here is the marginal utility of money. Look at the conditions at equilibrium.

$$u_c\left(C{,}1-H\right)-\Lambda=\beta E\left[v_k\left(K',K',1,\rho z+\epsilon',\theta\mu+\zeta'\right)\right] \tag{3.6.17}$$

$$\frac{e^{\mu}}{\hat{p}}\left[u_c\left(C{,}1-H\right)-\Lambda\right]=\beta E\left[v_m\left(K',K',1,\rho z+\epsilon',\theta\mu+\zeta'\right)\right] \tag{3.6.18}$$

$$\left[u_c\left(C{,}1-H\right)-\Lambda\right]e^zF_h\left(K,H\right)=u_l\left(C{,}1-H\right) \tag{3.6.19}$$

$$\Lambda\left[K'-\left(1-\delta_k\right)K-e^zF\left(K,H\right)+\frac{1}{\hat{p}}\right]=0 \tag{3.6.20}$$

$$C=e^zF\left(K,H\right)+\left(1-\delta_k\right)K-K' \tag{3.6.21}$$

$$\Lambda\left[\frac{1}{\hat{p}}-C\right]=0 \tag{3.6.22}$$

So the constraint potentially drives a wedge between the marginal utility of consumption and the marginal value of investment. If the constraint does not bind and $\Lambda=0$, then we have marginal utility of investment must equal marginal utility of consumption, etc. If the constraint does bind, however, we are unable to consume as much as we would like because we are short of cash. Since all goods end up as either consumption or investment, there is too much investment. Alternatively, some goods go unsold, which the producer must take back and use as investment. So the marginal utility of consumption and investment may not be equal here.

G Envelope equations

$$v_k\left(k,K,\hat{m},z,\mu\right)=\left[u_c\left(c{,}1-h\right)-\lambda\right]\left[e^zF_k\left(K{,}H\right)+1-\delta\right] \tag{3.7.23}$$

$$v_m\left(k,K,\hat{m},z,\mu\right)=\frac{1}{\hat{p}}u_c\left(c{,}1-h\right) \tag{3.7.24}$$

In equilibrium:

$$v_k\left(K,K,1,z,\mu\right)=\left[u_c\left(C,1-H\right)-\Lambda\right]\left[e^zF_k\left(K,H\right)+1-\delta\right] \tag{3.7.25}$$

$$v_m\left(K,K,1,z,\mu\right)=\frac{1}{\hat{p}}u_c\left(C,1-H\right) \tag{3.7.26}$$

Again the marginal value of capital is reduced if the constraint is binding. This is because we cannot convert the capital into immediate consumption. Money, however, can always be converted into consumption.

IV Precautionary Money Holdings

We now consider how the size of the shock μ' affects consumption and money holdings.

A Non-binding constraint

Suppose first the constraint is non-binding. We will show that the real economy is independent of the money shock, and that the constraint must become binding for sufficiently high money growth rates. Evaluating the first order conditions and envelopes with $\Lambda=0$ gives:

$$u_c\left(C,1-H\right)=\beta E\left[V_k\left(K',K',1,\rho z+\epsilon',\theta\mu+\zeta'\right)\right] \tag{4.1.1}$$

$$\frac{e^{\mu}}{\hat{p}}u_c\left(C,1-H\right)=\beta E\left[V_m\left(K',K',1,\rho z+\epsilon',\theta\mu+\zeta'\right)\right] \tag{4.1.2}$$

$$u_c\left(C,1-H\right)e^zF_h\left(K,H\right)=u_l\left(C,1-H\right) \tag{4.1.3}$$

$$C=e^zF\left(K,H\right)+\left(1-\delta_k\right)K-K' \tag{4.1.4}$$

$$\Lambda\left[\frac{1}{\hat{p}}-C\right]=0\ ,\ \Lambda=0\ ,\ \frac{1}{\hat{p}}>C \tag{4.1.5}$$

$$V_k\left(K,K,1,z,\mu\right)=u_c\left(C,1-H\right)\left[e^zF_k\left(K,H\right)+1-\delta\right] \tag{4.1.6}$$

$$V_m\left(K,K,1,z,\mu\right)=\frac{1}{\hat{p}}u_c\left(C,1-H\right) \tag{4.1.7}$$

It is easy to see that the economy in this case is identical to the RBC model without money, and in fact the competitive equilibrium maximizes welfare. Equations (4.1.1), (4.1.3), and (4.1.4), along with (4.1.6), solve for K', H, V_k, and C independent of μ and $\hat{p}$ (since V_k is independent of μ, so is $V_k\left(S'\right)$.

Now real balances are affected by the money growth rate. Equations (4.1.2) and (4.1.7) solve for V_m and $\hat{p}$, depending on μ. The Euler equation is:

$$\frac{e^{\mu}}{\hat{p}\left(\mu\right)}u_c\left(C,1-H\right)=\beta E\left[\frac{1}{\hat{p}\left(\mu'\left(\mu\right)\right)}u_c\left(C',1-H'\right)\right] \tag{4.1.8}$$

We need to show $\hat{p}$ is increasing in μ. In the non-binding region, we have shown that C and H are independent of μ. We can use an induction argument here. Write $1/\hat{p}=G\left(\mu\right)$ recursively as:

$$G_i\left(\mu\right)u_c\left(C,1-H\right)=e^{-\mu}\beta E\left[G_{i-1}\left(\mu'\left(\mu\right)\right)u_c\left(C',1-H'\right)\right] \tag{4.1.9}$$

Now, suppose G_{i-1} is decreasing. Then an increase in μ increases μ', which decreases G_{i-1}, which decreases the right hand side. Now since $e^{-\mu}$ is decreasing, the right hand side decreases again. Therefore, the left hand side must also decrease, and so G_i decreasing. By induction, G will be non-increasing in μ.

Given that $\hat{p}$ is increasing in μ, we can see from (4.1.5) that there exists a range of μ for which a solution with a non-binding constraint exists. That is: $\Lambda=0$ if $\mu\leq\mu^*$, where:

$$\frac{1}{\hat{p}\left(\mu^*\right)}=C. \tag{4.1.10}$$

B Binding constraint

For $\mu>\mu^*$, the constraint binds. In this case, $\Lambda>0$ and money shocks affect the economy. In fact, we will have $C=\frac{1}{\hat{p}(\mu)}$, so C is decreasing in μ.

Assume that $u_{cl}=0$. I claim that $u_c\left(C\right)-\Lambda$ is decreasing in μ. Suppose not, then from the first order condition for H (3.6.15), we have:

$$\left(u_c\left(C\right) - \Lambda\right) e^z F_h\left(K, H\right) = u_l\left(1 - H\right) \tag{4.2.11}$$

Now an increase in μ increases the left hand side by increasing $u_c - \Lambda$. Since K is fixed, to equate the left and right hand sides we need H to increase, which increases the right hand side and decreases the left hand side. Therefore H is increasing in μ. Next, the first order condition for investment (3.6.13) simplifies to:

$$u_c\left(C\right) - \lambda = \beta E\left[V_k\left(K', K', 1, \rho z + \epsilon', \theta\mu + \zeta'\right)\right] \tag{4.2.12}$$

An increase in μ increases the left hand side of (4.2.12). Let us assume θ is small, so that the effect of μ on V_k through $\theta\mu$ is also small. The only variable to adjust on the right hand side is K'. Since the value function must be concave in K', we must have K' decreasing in μ to equate the left and right hand sides of (4.2.12). Finally, from the resource constraint, we have:

$$C + X = e^z F\left(K{,}H\right). \tag{4.2.13}$$

An increase in μ therefore decreases C and X, decreasing the left hand side, but also increases H, increasing the right hand side. We thus contradict that the resource constraint must hold. Therefore $u_c\left(C\right) - \Lambda$ is decreasing in μ.

Given that $u_c\left(C\right) - \Lambda$ is decreasing in μ, it is easy to reverse the arguments and see that H is decreasing in μ and K' is increasing in μ.

So in summary, in the range where the constraint is binding, an increse in μ causes a decrease in C, an increase in K', a decrease in H, an increase in $\hat{p}$, and an increase in Λ.

Intuitively, suppose now the money shock is greater, that $\mu > \mu^*$. So the government announces it will transfer a huge load of printed dollars to workers to carry forward. Workers desire less cash, and so will demand a large amount of cash in exchange for consumption goods, which they could instead keep for investment use. Hence inflation rises (the value of money falls) and real money balances fall. Remember nominal money balances are fixed at $M_t = \bar{M}_t$, until the end of the period.

But falling real balances creates a problem with the constraint. Since $\frac{1}{\hat{p}_t} \geq c_t$, if $1/\hat{p}_t$ falls by enough, the constraint will bind, and $\lambda > 0$. Consumers cannot consume as much as they want and are faced with a cash crunch. The left over goods the worker produces are used for investment. Hours fall since the household knows it must allocate too much income to investment relative to consumption, making working less attractive.

Definition 27 *The* Tobin Effect *occurs when high inflation causes households to substitute into investment from real balances.*

In Tobin's model, the effect is planned in the sense that households desire to hold less cash and more capital in the face of inflation, since the liquidity advantage of cash is outweighed by the inflation cost of holding money. Here it is in part accidental. But the prediction is for higher growth in high inflation countries, ignoring reverse causality in the sense that often high inflation results from a tax shortfall which results from lower growth.

V Other effects

A Lead-Lags

A positive shock to money supply persists. Hence the effect on investment and consumption lasts for many periods. The price level adjusts very quickly, however.

B Velocity

We have from the quantity theory that consumption velocity is found via:

$$\bar{M}V = Pc \tag{5.2.1}$$

$$qV = c \tag{5.2.2}$$

Our model has

$$q \geq c \tag{5.2.3}$$

If the shock is high and real balances fall, then consumption velocity rises up towards one. Hence the model predicts velocity is less than one and rises with μ. Further, for $\mu > \mu^*$, velocity equals one.

This contrasts with the cash good credit good model, which has velocity greater than one (see homework). Velocity less than one matches the data.

VI Certainty Equivalence Steady State

In certainty equivalence, all variables are evaluated at their mean value. Since the CIA constraint binds some of the time, Λ is positive some times and zero other times. Thus the mean of Λ is positive. Similarly the long run average consumption will be below it's value when the CIA constraint does not bind, etc. We have:

$$u_c\left(C,1-H\right)-\Lambda=\beta\left[u_c\left(C,1-H\right)-\Lambda\right]\left[F_k\left(K,H\right)+1-\delta\right] \tag{6.0.1}$$

$$\rho=F_k\left(K,H\right)-\delta \tag{6.0.2}$$

The inflation tax is a tax on purchases made in cash, which is consumption. Now with a constant money growth rate, the inflation tax will be paid eventually. One can try to evade the inflation tax by increasing investment, but eventually that investment will be turned into consumption, at which point the inflation tax will be paid anyway. Therefore, the inflation tax does not distort the steady state investment/consumption decision, and therefore the steady state K (except through changes in H).

$$e^{\mu}\left[u_c\left(C,1-H\right)-\Lambda\right]=\beta u_c\left(C,1-H\right) \tag{6.0.3}$$

$$\left[u_c\left(C,1-H\right)-\Lambda\right]F_h\left(K,H\right)=u_l\left(C,1-H\right) \tag{6.0.4}$$

Steady state hours also falls due to the binding constraint.

$$C=F\left(K,H\right)-\delta K \tag{6.0.5}$$

$$z=0 \tag{6.0.6}$$

$$\mu=\theta\mu+\mathrm{E}\left[\zeta\right] \tag{6.0.7}$$

Recall $\mathrm{E}\left[\zeta\right]=\left(1-\theta\right)\bar{\mu}$ so:

$$\mu-\theta\mu=\left(1-\theta\right)\bar{\mu} \tag{6.0.8}$$

$$\mu = \bar{\mu} \tag{6.0.9}$$

$$\frac{1}{\hat{p}} = C \tag{6.0.10}$$

Here the constraint binds on average (or, alternatively, with certainty equivalence there is no need for precautionary money holdings), from equation (6.0.3), so $\frac{1}{\hat{p}} = C$. We have 7 equations for unknowns K, H, Λ, C, $\hat{p}$, z, and μ.

A Superneutrality

Obviously not superneutral, since the argument given after equation (4.1.9) shows that real balances depend on μ. The economy is neutral only if the CIA constraint does not bind at the certainty equivalence steady state ($\Lambda = 0$). In general however, equation (6.0.3) imlies:

$$\Lambda = \left(1 - \beta e^{-\mu}\right) u_c \left(C, 1 - H\right). \tag{6.1.11}$$

Hence in general $\Lambda \neq 0$ and we have real effects.

B Optimal Monetary Policy

There is no money in the utility function. Thus money is irrelevant to the social planner. The planner does not require money to facilitate exchange, and money is costless to produce. So no optimal quantity of real balances exists. However, an optimal growth rate of money may exist. The planning problem produces a steady state of:

$$\rho = F_k \left(K, H\right) - \delta \tag{6.2.12}$$

$$u_c \left(C, 1 - H\right) F_h \left(K, H\right) = u_l \left(C, 1 - H\right) \tag{6.2.13}$$

$$C = F \left(K, H\right) + \left(1 - \delta_k\right) K - K' \tag{6.2.14}$$

$$z = 0 \tag{6.2.15}$$

This is just the certainty equivalence steady state of the stochastic growth model. Clearly, monetary policy is optimal if $\Lambda = 0$, or if the constraint never binds. How may this be acheived? From equation (6.0.3),

$$\Lambda = u_c\left(C{,}1-H\right)\left(1-\frac{\beta}{e^{\mu}}\right) \tag{6.2.16}$$

We would like $\Lambda = 0$, which implies:

$$\pi = e^{\mu} - 1 = \beta - 1 = -r \tag{6.2.17}$$

This is just the Friedman rule, which is says that the optimal nominal rate is zero.

VII Calibration

Most of the calibration will not change from the RBC model.

A Money growth rate parameter

New parameters include the money growth rate parameters. One could use a procedure similar to the other shock:

$$\bar{M}_{t+1} = e^{\mu_t}\bar{M}_t \tag{7.1.1}$$

$$\mu_t = \log\left(\bar{M}_{t+1}\right) - \log\left(\bar{M}_t\right) \tag{7.1.2}$$

So given data on M_t, it is easy to calculate μ_t. Then we can estimate the autoregressive equation:

$$\mu_{t+1} = \theta\mu_t + \zeta_{t+1} \tag{7.1.3}$$

Cooley (chapter 7) gets: $\theta = .49$ and $\sigma_\zeta = .009$. Hence money supply is not very stochastic.

Note: the construction of the money supply data prevents this. Money innovations are not generally treated as shocks. Instead the gov't creates a new monetary statistic and reverse engineers the data backwards. Hence the government could create a new statistic M^6, with the new innovation, and then estimate backwards what it would have been had the innovation been around previously. Hence if one uses just M^6, there will be few shocks.

B β, γ, ρ, δ, σ_ϵ^2

Clearly, δ may be calibrated from the law of motion for investment, in an identical way to the RBC model. Further, the capital share equation derived from (6.0.2) is unchanged. Therefore the calibration of γ, ρ, and β are unchanged from the RBC model. Finally, the parameters governing the law of motion for the productivity shocks, ρ and σ_ϵ^2 are also calibrated in identical fashion as the RBC model.

C Elasticity of Substitution Parameter

For the function form of the utility function, we can use:

$$u = \alpha \log c + (1-\alpha)\log(1-h)\,. \tag{7.3.4}$$

Combining (6.0.3) and (6.0.4) gives:

$$\beta e^{-\mu} u_c\,(C, 1-H)\,F_h\,(K,H) = u_l\,(C, 1-H)\,. \tag{7.3.5}$$

Using the functional forms, and solving for the consumption share α gives:

$$\alpha = \left[\beta e^{-\mu}(1-\gamma)\frac{Y}{C}\frac{1-H}{H} + 1\right]^{-1}. \tag{7.3.6}$$

Notice that we have to adjust the consumption share up as μ rises. If μ is high, the consumption/output ratio will fall not because households do not value consumption, but because working is taxed through inflation whereas leisure is not.

VIII Observed Facts versus Model

Overall, we have:

Parameters		Data	Precautionary	Two good
μ	c	positive	negative	negative
μ	x	negative	positive	positive
μ	P	negative	positive	positive
μ	π	negative	positive	positive
μ	h	negative	negative	neg, weak
μ	y	negative	negative	neg, weak
μ	V	pos, weak	positive	positive

A Consumption and Investment

The precautionary model has that unexpected high money growth causes a cash crunch which decreases consumption and increases investment spending. This is reinforced also because money growth acts like a tax on consumption since only consumption goods are bought with cash.

The cash/credit model is like a tax on consumption of cash consumption goods. So a smaller decrease in overall consumption and increase in investment spending.

The data goes the opposite way.

B Prices and Inflation

An increase in the supply of money decreases the price of money $\frac{1}{P}$. So the correlation is negative in both models. Because shocks are correlated over time, future prices are higher as well, increasing inflation. The data has a negative correlation with prices, but a mysterious negative correlation with inflation.

C Hours and output

In both models, wages must be converted to cash first and the inflation tax paid before getting consumption goods. Thus working is not as attractive and leisure rises as money growth rises.

Capital is fixed, but since hours falls, output falls, so we have a negative correlation between money growth and output.

The data has a negative correlation as well.

D Velocity

In the precautionary model, velocity rises with inflation as the cash crunch reduces liquidity holdings. In the cash/credit model an increase in money growth increases the tax on the cash good, increasing credit good purchases relative to cash. Thus we have:

$$\frac{c_1 + c_2}{c_1} = 1 + \frac{c_2}{c_1} \tag{8.4.7}$$

So velocity rises also in the cash good/credit good model. The models differ in that predicted velocity is greater than one in the cash/credit and less than one in the precautionary.

In the data the correlation is near zero, but in higher inflation countries the correlation is positive.

E How Much of the Business Cycle is Explained by Monetary Shocks?

By comparing the model with one of constant money growth, we can see how monetary shocks affect the business cycle. The standard deviation of GNP without monetary shocks is 1.69%. The actual economy is 1.72%. These are close due to the indivisible labor assumption (which probably overstates the correlation, since it allows for no part time work). With monetary shocks, the standard deviation of GNP is still 1.69%. Monetary shocks have virtually no effect on output.

F Implied Phillips Curve

Note that money growth is positively correlated with inflation and negatively correlated with hours and therefore output. Thus both models predict an implicit negative correlation between inflation and output. This is a Phillips curve, but in the wrong direction.

In the data, a Phillips curve seems to exist since the inflation output correlation is 0.34.

Relatively poor job of explaining facts. In the two good model, the real sector is virtually unchanged in terms of standard deviations. Possible problems:

- FED policy on money growth is more complicated, depending on Y and π, not just μ_{t-1}.
- Price rigidities are not modeled effectively?
- Missing channels? Since cash is required for some consumption goods, inflation acts kind of like a sales tax, which has minimal real effects. If for example, firms used cash to pay factors, then inflation would have stronger effects on labor and capital demand, and therefore on hours and investment.