Question 8.

a. Using the second definition of velocity (90):

$$\text{VEL} \;=\frac{C_2+C_1}{C_1}=\frac{\$900+\$200+\$100}{\$900+\$200}=\frac{6}{5}, \tag{95}$$

since checks count as cash and the stock purchase is not consumption.

b. Using the precautionary model,

$$V=\frac{pC}{\bar{M}}=\frac{\$1{,}200}{\$1{,}500}=\frac{4}{5}. \tag{96}$$

c. Notice that 1000 of the dollars were spent one time, and the rest were spent zero times. Therefore:

$$V=\frac{\$900+\$1000}{\$1{,}500}=\frac{2}{3}. \tag{97}$$

d. The cash/credit model over-states velocity because it assumes all dollars are spent.

e. The precautionary model over-states velocity because it assumes that all consumption purchases are with cash, thus it incorrectly counts credit transactions as dollars being spent.

Question 9.

a. Bond redemptions and purchases are now added to the CIA constraint, because they are cash and because they are made prior to when the shopper leaves. The CIA constraint is:

$$c_{1t}\leq\frac{M_t}{p_t}+TR_t+\frac{(1+R_{t-1})\,b_t-b_{t+1}}{p_t} \tag{98}$$

Similarly, the government constraint is now:

$$TR_t=\frac{(e^{\mu_t}-1)\,\bar{M}_t}{p_t}-\frac{(1+R_{t-1})\,B_t-B_{t+1}}{p_t} \tag{99}$$

So:

$$c_{1t}\leq\frac{M_t+(e^{\mu_t}-1)\,\bar{M}_t}{p_t}+\frac{(1+R_{t-1})\,(b_t-B_t)-b_{t+1}+B_{t+1}}{p_t} \tag{100}$$

Now let $\hat{b}_t = \frac{b_t}{M_t}$ and the same for $\hat{B}_t$, so that:

$$c_{1t} \leq \frac{\hat{m}_t + e^{\mu_t} - 1}{\hat{p}_t e^{\mu_t}} + \frac{(1 + R_{t-1}) \left(\hat{b}_t - \hat{B}_t \right)}{e^{\mu_t} \hat{p}_t} + \frac{-\hat{b}_{t+1} + \hat{B}_{t+1}}{\hat{p}_t} \tag{101}$$

b. It is easy to see that the workers constraint is unchanged, since all bond transactions are done by the shopper. Thus, assuming $\hat{B} = \hat{B}' = 0$ the value function for this problem is:

$$\begin{aligned} v\left(k, K, \hat{m}, z, \mu, \hat{b}\right) = \max_{k', h, \hat{m}', \hat{b}'} \Bigg\{ &\alpha \log \left[\frac{\hat{m} + e^{\mu} - 1 + (1 + R_{-1}) \left(\hat{b}\right)}{\hat{p} e^{\mu}} + \frac{-\hat{b}'}{\hat{p}} \right] \\ &+ (1 - \alpha) \log \left[e^z F_k (K, H) k + e^z F_h (K, H) h - \frac{\hat{m}'}{\hat{p}} - k' + (1 - \delta) k \right] \\ &- \psi h + \beta \mathrm{E} \left[v\left(k', K', \hat{m}', \rho z + \epsilon', \theta\mu + \zeta', \hat{b}'\right) \right] \Bigg\} \end{aligned} \tag{102}$$

We do not need $\hat{B}$ as a state since it is constant at zero and in equilibrium $\hat{b} = \hat{b}' = 0$. We have only one extra foc and envelope (everything else is unchanged). These will determine the bond rate.

$$\frac{\alpha}{C_1} \frac{1}{\hat{p}} = \beta E \left(v_6 \left(K', K', 1, \rho z + \epsilon', \theta\mu + \zeta', 0 \right) \right) \tag{103}$$

$$v_6 (K, K, 1, z, \mu, 0) = \frac{\alpha}{C_1} (1 + R_{-1}) \frac{1}{\hat{p} e^{\mu}} \tag{104}$$

Now imposing that $C_1 = 1/\hat{p}$ implies:

$$\alpha = \beta E \left(v_6 \left(K', K', 1, \rho z + \epsilon', \theta\mu + \zeta', 0 \right) \right) \tag{105}$$

$$v_6 (K, K, 1, z, \mu, 0) = \frac{\alpha}{e^{\mu}} (1 + R_{-1}) \tag{106}$$

Combining these equations gives:

$$\alpha = \beta E \left(\frac{\alpha}{e^{\mu'}} (1 + R) \right) \tag{107}$$

$$1 = \beta E\left(e^{-\mu'}\right)(1+R) \tag{108}$$

Solving for R gives:

$$R = \left(\beta E\left[e^{-\mu_{t+1}}\right]\right)^{-1} - 1 \tag{109}$$

Clearly the nominal rate rises with inflation, since investors must be compensated for the loss of value of a cash bond when inflation is high by a higher nominal return. In the absence of any inflation effects, the bond still earns a rate of return different than capital outside of the steady state since cash bonds may be used to by C_1, whereas the return on capital can be used to be credit goods.

c. In the certainty equivalence steady state, the above equation is:

$$R = \left(\beta e^{-\bar{\mu}}\right)^{-1} - 1 \tag{110}$$

$$1 + R = (1+\rho)\, e^{\bar{\mu}} \tag{111}$$

$$\log(1+R) = \log(1+\rho) + \bar{\mu} \tag{112}$$

$$\approx R \approx \rho + \bar{\mu} \tag{113}$$

$$R = F_k\left(\bar{K}, \bar{H}\right) - \delta + \bar{\mu} \tag{114}$$

This is the Fisher relation.

d. Since $\bar{\mu} = \log\beta$ we have from (111):

$$1 + R = (1+\rho)\,\beta, \quad \rightarrow \quad 1 + R = 1, \tag{115}$$

$$R = 0. \tag{116}$$

We set the nominal rate to zero since we need money and capital to earn the same return for their to be no distortion.

Question 10.

If the constraint does not bind, we can ignore it. We then have only one constraint, which is the budget constraint:

$$\begin{aligned} c_{1t} \quad + \quad & c_{2t} + \frac{\hat{m}_{t+1}}{\hat{p}_t} + k_{t+1} - (1-\delta)\,k_t = r_t k_t + w_t h_t + \frac{\hat{m}_t + e^{\mu_t} - 1}{\hat{p}_t e^{\mu_t}} + \\ & \frac{(1+R_{t-1})\left(\hat{b}_t - \hat{B}_t\right)}{e^{\mu_t}\hat{p}_t} - \frac{\hat{b}_{t+1} - \hat{B}_{t+1}}{\hat{p}_t}. \end{aligned} \tag{117}$$

Using the budget constraint to remove c_2, we are left with a value function whose first order conditions and envelopes include:

$$\frac{1-\alpha}{c_2}\frac{1}{\hat{p}} = \beta \mathrm{E}\left[v_6\left(k', K', \hat{m}', \rho z + \epsilon', \theta\mu + \zeta', 0\right)\right] \tag{118}$$

$$v_6\left(k, K, \hat{m}, z, \mu, 0\right) = \frac{1-\alpha}{c_2}\left(1 + R_{-1}\right)\frac{1}{\hat{p}e^{\mu}} \tag{119}$$

$$\frac{1-\alpha}{c_2}\frac{1}{\hat{p}} = \beta \mathrm{E}\left[v_3\left(k', K', \hat{m}', \rho z + \epsilon', \theta\mu + \zeta', 0\right)\right] \tag{120}$$

$$v_3\left(k, K, \hat{m}, z, \mu, 0\right) = \frac{1-\alpha}{c_2}\frac{1}{\hat{p}e^{\mu}} \tag{121}$$

The latter equations are the first order condition and envelope for money holdings. The above four equations state that money and bonds have exactly the same benefit and the same cost, so holding money must have the same return as holding bonds, which is only true if $R = 0$. To see this formerly, combine the two first order conditions so that:

$$\beta \mathrm{E}\left[v_6\left(k', K', \hat{m}', \rho z + \epsilon', \theta\mu + \zeta', 0\right)\right] = \beta \mathrm{E}\left[v_3\left(k', K', \hat{m}', \rho z + \epsilon', \theta\mu + \zeta', 0\right)\right] \tag{122}$$

Now plug in the envelopes to get:

$$\mathrm{E}\left[\frac{1-\alpha}{c_2'}\left(1+R\right)\frac{1}{\hat{p}'e^{\mu'}}\right] = \mathrm{E}\left[\frac{1-\alpha}{c_2'}\frac{1}{\hat{p}'e^{\mu'}}\right] \tag{123}$$

$$\left(1+R\right)\mathrm{E}\left[\frac{1-\alpha}{c_2'}\frac{1}{\hat{p}'e^{\mu'}}\right] = \mathrm{E}\left[\frac{1-\alpha}{c_2'}\frac{1}{\hat{p}'e^{\mu'}}\right] \tag{124}$$

$$R\mathrm{E}\left[\frac{1-\alpha}{c_2'}\frac{1}{\hat{p}'e^{\mu'}}\right]=0 \tag{125}$$

Thus $R=0$, since the expectation is positive. $R=0$ is therefore necessary for the constraint not to bind, so $R>0$ implies the constraint binds.

Question 11

To make things clear, I rewrite problem (102) with an arbitrary stock of bonds:

$$\begin{aligned} v\left(k,K,\hat{m},z,\mu,\hat{b},\hat{B}\right) = \max_{k',h,\hat{m}',\hat{b}'} &\left\{\alpha\log\left[\frac{\hat{m}+e^{\mu}-1+(1+R_{-1})\left(\hat{b}-\hat{B}\right)}{\hat{p}e^{\mu}}+\frac{\hat{B}'-\hat{b}'}{\hat{p}}\right]\right. \\ &+(1-\alpha)\log\left[e^{z}F_k(K,H)\,k+e^{z}F_h(K,H)\,h-\frac{\hat{m}'}{\hat{p}}-k'+(1-\delta)\,k\right] \\ &\left.-\psi h+\beta\mathrm{E}\left[v\left(k',K',\hat{m}',\rho z+\epsilon',\theta\mu+\zeta',\hat{b}',\hat{B}'\right)\right]\right\} \end{aligned} \tag{126}$$

The CIA constraint, and budget constraint, first order conditions, and envelope equations are:

$$C_1=\frac{1}{\hat{p}} \tag{127}$$

$$C_2=e^{z}F(K,H)+(1-\delta)\,K-K'-C_1 \tag{128}$$

$$\frac{1-\alpha}{C_2}=\beta\mathrm{E}\left[v_1\left(K',K',1,\rho z+\epsilon',\theta\mu+\zeta',\hat{B}',\hat{B}'\right)\right] \tag{129}$$

$$\frac{1-\alpha}{C_2}\frac{1}{\hat{p}}=\beta\mathrm{E}\left[v_3\left(K',K',1,\rho z+\epsilon',\theta\mu+\zeta',\hat{B}',\hat{B}'\right)\right] \tag{130}$$

$$\frac{1-\alpha}{C_2}e^{z}F_h(K,H)=\psi \tag{131}$$

$$\frac{1-\alpha}{c_2}\frac{1}{\hat{p}}=\beta\mathrm{E}\left[v_6\left(k',K',\hat{m}',\rho z+\epsilon',\theta\mu+\zeta',\hat{b}',\hat{B}'\right)\right] \tag{132}$$

$$v_1\left(K, K, 1, z, \mu, \hat{B}, \hat{B}\right) = \frac{1-\alpha}{C_2}\left(e^z F_k\left(K, H\right) + 1 - \delta\right) \tag{133}$$

$$v_3\left(K, K, 1, z, \mu, \hat{B}, \hat{B}\right) = \frac{\alpha}{e^\mu} \tag{134}$$

$$v_6\left(k, K, \hat{m}, z, \mu, \hat{B}, \hat{B}\right) = \frac{\alpha}{c_1}\left(1 + R_{-1}\right)\frac{1}{\hat{p}e^\mu} \tag{135}$$

$$g + TR = \frac{e^\mu - 1}{e^\mu \hat{p}} + \frac{\hat{B}'}{\hat{p}} - \left(1 + R_{-1}\right)\frac{\hat{B}}{e^\mu \hat{p}}. \tag{136}$$

The above 10 equations solve for C_1, K', H, C_2, $\hat{p}$, R_{-1}, $\hat{B}'$, v_1, v_3, and v_6. Here I am assuming the government sets a tax/transfer policy TR and then borrows whatever is needed to satisfy the government constraint.

For Ricardian equivalence, consider a change in lump sum taxes (TR) financed by bonds. That is, $\hat{B}'$ adjusts to the change in TR so that the government budget constraint (136) remains satisfied. The right hand side of the envelope equations (133)-(135) are independent of $\hat{B}$. Therefore the derivatives of the value function v_1, v_3, and v_6 are independent of the stock of bonds. Therefore, the remaining equations (the first order conditions, CIA constraint, and resource constraint) solve for C_1, K', H, C_2, $\hat{p}$, and R_{-1} independent of $\hat{B}'$. Thus a bond financed change in lump sum taxes does not affect any equilibrium variables.

Question 12

a. For the conventional deficit, we can use equation (136), but consider interest payments as an expense (like g):

$$\left(g + \frac{R_{-1}\hat{B}}{pe^\mu}\right) + TR = D_c = \frac{e^\mu - 1}{e^\mu \hat{p}} + \frac{\hat{B}'}{\hat{p}} - \frac{\hat{B}}{e^\mu \hat{p}}. \tag{137}$$

Notice the bond is defined slightly differently in this model versus the notes. In the notes, a each bond entitled the owner to \$1 in period t (and cost $Q_t < 1$ in the previous period). Here each bond entitles the owner to $1 + R_{-1}$ in period t and costs \$1 in the previous period.

b. From equation (136) the primary deficit is:

$$g + TR = D_p = \frac{e^\mu - 1}{e^\mu \hat{p}} + \frac{\hat{B}'}{\hat{p}} - \left(1 + R_{-1}\right)\frac{\hat{B}}{e^\mu \hat{p}}. \tag{138}$$

The primary deficit excludes interest payments in the expense column (interest payments are part of the deficit).

c. We are considering a permanent (steady state) deficit without seniorage ($\mu = 0$), so equation (138) becomes:

$$D_p = \frac{\hat{B}}{\hat{p}} - (1 + R_{-1}) \frac{\hat{B}}{\hat{p}} = -R_{-1} \frac{\hat{B}}{\hat{p}}. \tag{139}$$

So a permanent primary deficit is not possible without seniorage. In fact, we must have a primary surplus.

d. Repeating but allowing $\mu > 0$ gives:

$$D_p = \frac{e^{\mu} - 1}{e^{\mu} \hat{p}} + \frac{\hat{B}}{\hat{p}} - (1 + R_{-1}) \frac{\hat{B}}{e^{\mu} \hat{p}}. \tag{140}$$

From equation (110):

$$D_p = \frac{e^{\mu} - 1}{e^{\mu} \hat{p}} + \frac{\hat{B}}{\hat{p}} - \frac{e^{\mu}}{\beta} \frac{\hat{B}}{e^{\mu} \hat{p}}, \tag{141}$$

$$D_p = \frac{e^{\mu} - 1}{e^{\mu} \hat{p}} - \frac{\hat{B}}{\hat{p}} \left(\frac{1 - \beta}{\beta} \right). \tag{142}$$

Apparently, a permanent steady state primary deficit is possible if the seniorage is large enough to cover the principle and interest payments.

e. A permanent (steady state) conventional deficit without seniorage ($\mu = 0$), from equation (137) satisfies:

$$D_c = \frac{\hat{B}}{\hat{p}} - \frac{\hat{B}}{\hat{p}} = 0. \tag{143}$$

So we cannot have a permanent conventional deficit without seniorage, instead the budget must be balanced.

f. Repeating but with $\mu > 0$:

$$D_c = \frac{e^{\mu} - 1}{e^{\mu} \hat{p}} + \frac{\hat{B}}{\hat{p}} - \frac{\hat{B}}{e^{\mu} \hat{p}}. \tag{144}$$

$$D_c = \frac{e^{\mu} - 1}{e^{\mu}\hat{p}} \left(1 + \hat{B}\right). \tag{145}$$

If $\mu > 0$, a permanent conventional deficit is possible.