Homework 1: Warm-up Exercises
Eco 603 Fall 2017
Due: August 31

Consider the following social planning problem in which total factor productivity (A_t) grows along with population.

$$\max_{C_t, K_{t+1}} \sum_{t=0}^{\infty} \beta^t L_t \left[\frac{\left(\frac{C_t}{L_t}\right)^{1-\sigma} - 1}{1-\sigma} \right] \tag{1}$$

Subject to a resource constraint:

$$C_t + K_{t+1} = A_t K_t^{\gamma} L_t^{1-\gamma} + (1-\delta) K_t \tag{2}$$

and exogenous growth rates:

$$L_{t+1} = (1+\eta) L_t \qquad A_{t+1} = (1+\theta) A_t \tag{3}$$

Here $\sigma > 0$ with $\sigma = 1$ being log utility.

Question 1: Normalization

a. Normalize the model: rewrite the model defining new variables so that L_t and A_t are removed from the problem, and no remaining variables are growing without bound exogenously.

b. Interpret the normalized variables.

c. What assumption on β is required so that social welfare is finite?

d. Assume all normalized variables are constant. What is the growth rate of output and output per capita?

Question 2.

Prove that the limiting case is log utility:

$$\lim_{\sigma \to 1} \frac{(C_t)^{1-\sigma} - 1}{1-\sigma} = \log C_t. \tag{4}$$

Question 3 (Productivity Growth).

Growth in productivity implies (all other things equal) an income effect in that we are wealthier in the future and a substitution effect in the form of greater returns to saving, since capital is more productive in the future.

a. Explain how the above income and substitution effects affect today's savings decision.

b. Explain how the income and substitution effects are affected by the concavity of the utility function (a graph of the utility function will help).

c. Explain the relationship between σ and the concavity of the utility function.

d. Use the above results to explain how σ and θ affect the normalized discount rate.

e. Finally, explain what happens to the normalized discount rate under log utility.

Question 4 (Recursive Form)

a. Write the problem recursively using the value function (Bellman's equation).

b. Write the first order condition(s). Interpret.

Question 5 (Recursive Form)

Suppose a variant of the problem where capital takes two periods to build. Let X_{2t} denote net investment today which becomes capital in period $t+2$. Let $X_{1,t+1} = X_{2t}$ denote capital under construction in $t+1$. Capital in $t+1$ is then undepreciated capital plus capital under construction today: $K_{t+1} = X_{1,t} + (1-\delta) K_t$. All other assumptions are the same as question one.

a. Normalize the problem.

b. Write the value function.