Exercise 1: Eco 521 Fall 2009
Math Review
Due: September 6

These problems are simple examples of the types of problems we will see in class. Remember, we need only understand the principles of concavity, optimization, and difference equations to understand the rest of the course. Assume all variables are positive unless otherwise noted.

Question 1.

For each of the following functions r, compute:

a. $\frac{dr}{dx}$.

b. $\frac{d^2r}{dx^2}$.

c. $\frac{d^2r}{dzdx}$

1. $r = u(x, y - z)$ with $y = f(x)$.
2. $r = u(z, 1 - x) + \lambda\left(G\left(\frac{f(x, \alpha x - z)}{x}\right) - y\right)$.
3. $r = \frac{\left(x^\alpha z^{1-\alpha} - y\right)^{1-\sigma}}{1-\sigma}$
4. $r = \sum_{t=T-1}^{T} u(x_t, z_t - x_{t+1})$. Here find the derivatives with respect to x_T rather than x and the same for z.

Question 2.

Consider the utility function $u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}$. Give conditions for concavity of u and prove that $lim_{\sigma \to 0} u(c) = \log c$.

Question 3.

Suppose $U(c)$ is concave and differentiable and $f(k)$ is differentiable and $0 \leq \delta \leq 1$.

a. Give conditions for concavity of $U(f(k) + (1 - \delta)k)$ in k.

b. Assume the above concavity conditions hold and $0 \leq y \leq f(k) + (1 - \delta)k$. Find the first order necessary conditions for a maximum of the problem:

$$\max_y U[f(k) + (1 - \delta)k - y] + U[f(y)] \tag{1}$$

c. Find the comparative static $\frac{dy}{dk}$. Give conditions for y to be increasing in k.

Question 4.

Suppose U is concave. For the problem:

$$\max_{c_t} \sum_{t=0}^{\infty} \beta^t U\left[c_t\right] \tag{2}$$

Subject to:

$$(1+r) x_t \geq c_t + x_{t+1} \qquad c_t \geq 0 \ \forall t \tag{3}$$

Here $x_0 > 0$ is given.

a. Set up the problem in Lagrange form.

b. Find the first order conditions. Interpret the Lagrange multiplier. Solve for and interpret a difference equation in c_t.

c. Solve for c_t for the special case of $U\left(c_t\right) = \log c_t$.

d. Interpret the answer. How do future and current consumption depend on β and r?

Question 5.

For each of the following difference equations, find the steady state(s) and give conditions for stability of the steady state(s).

a. $x_t = x_{t-1}\left(a - bx_{t-1}\right)$.

b. $x_t = \beta\gamma x_{t-1}^{\gamma}$.

c. Suppose linear demand and supply $D_t = \alpha - \beta P_t$, $S_t = -\gamma + \delta P_t$, and prices which respond to changes in inventories $P_{+1} = P_t - \sigma\left(S_t - D_t\right)$. Give the steady state(s) and conditions for stability for P_t, D_t, and S_t.

Question 6.

For each of the functions in Question 4, calculate the growth rate $g\left(x\right) = \frac{x_t - x_{t-1}}{x_{t-1}}$. Is the growth rate necessarily increasing or decreasing?

Question 7.

Consider an alternative growth rate: $\hat{g}\left(x_t\right) = \log\left(\frac{x_{t+1}}{x_t}\right)$.

a. Repeat Question 5 with the alternative growth rate.

b. Show that $\hat{g} \approx g$ (hint: use a Taylor expansion).

c. What value of g and $\hat{g}$, corresponds x doubling each period?