Homework 1: Warm-up Exercise Solutions
Eco 603 Fall 2017

Question 1: Normalization

a. The production function is constant returns to scale in K and L. Therefore, we can use the procedure given in the class notes to remove L_t. The next thing to note is that the production function is not constant returns to scale in K and A, therefore dividing through by A_t does not eliminate A_t from the problem. To use the same trick, we need to convert A_t into something that is constant returns to scale with K_t, ie. something to the power $1-\gamma$. This is done by noting that:

$$C_t + K_{t+1} = K_t^{\gamma}\left(A_t^{\frac{1}{1-\gamma}} L_t\right)^{1-\gamma} + (1-\delta) K_t \tag{1}$$

So now A_t augments the productivity of labor. From here:

$$\frac{C_t}{A_t^{\frac{1}{1-\gamma}} L_t} + \frac{K_{t+1}}{A_t^{\frac{1}{1-\gamma}} L_t} = K_t^{\gamma}\left(A_t^{\frac{1}{1-\gamma}} L_t\right)^{-\gamma} + (1-\delta)\frac{K_t}{A_t^{\frac{1}{1-\gamma}} L_t} \tag{2}$$

$$\frac{C_t}{A_t^{\frac{1}{1-\gamma}} L_t} + \frac{K_{t+1}}{A_{t+1}^{\frac{1}{1-\gamma}} L_{t+1}} \frac{A_{t+1}^{\frac{1}{1-\gamma}} L_{t+1}}{A_t^{\frac{1}{1-\gamma}} L_t} = \left(\frac{K_t}{A_t^{\frac{1}{1-\gamma}} L_t}\right)^{\gamma} + (1-\delta)\frac{K_t}{A_t^{\frac{1}{1-\gamma}} L_t} \tag{3}$$

Let $c_t = \frac{C_t}{A_t^{\frac{1}{1-\gamma}} L_t}$, etc. Then:

$$c_t + (1+\eta)(1+\theta)^{\frac{1}{1-\gamma}} k_{t+1} = k_t^{\gamma} + (1-\delta) k_t \tag{4}$$

So we have eliminated A and L from the resource constraint.

Now we have to do the same for the objective function. This is again easy for L, since we maximize per capita utility. But we also have to divide and multiply by A:

$$\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t L_t A_t^{\frac{1-\sigma}{1-\gamma}} \left[\frac{\left(\frac{C_t}{L_t A_t^{\frac{1}{1-\gamma}}}\right)^{1-\sigma} - 1 + 1 - A_t^{\frac{-(1-\sigma)}{1-\gamma}}}{1-\sigma}\right] \tag{5}$$

$$= \max_{c_t,k_{t+1}} \sum_{t=0}^{\infty} \beta^t L_t A_t^{\frac{1-\sigma}{1-\gamma}} \left[\frac{c_t^{1-\sigma}-1}{1-\sigma}\right] + \sum_{t=0}^{\infty} \beta^t L_t \left[\frac{A_t^{-\frac{1-\sigma}{1-\gamma}}-1}{1-\sigma}\right] \tag{6}$$

$$= A_0^{\frac{1-\sigma}{1-\gamma}} L_0 \max_{c_t,k_{t+1}} \sum_{t=0}^{\infty} \left(\beta\left(1+\eta\right)\left(1+\theta\right)^{\frac{1-\sigma}{1-\gamma}}\right)^t \left[\frac{c_t^{1-\sigma}-1}{1-\sigma}\right] + a_0 \tag{7}$$

Here a_0 is a constant (actually an infinite sum of constant terms which converges to a finite sum). Since the optimal decision is not affected by adding a constant or multiplying a constant, we can ignore them. Assuming σ is not one, the discount rate is also changed. Thus we can solve the following problem (which has neither A nor L):

$$= \max_{c_t,k_{t+1}} \sum_{t=0}^{\infty} \hat{\beta}^t \left[\frac{c_t^{1-\sigma}-1}{1-\sigma}\right] \tag{8}$$

Subject to:

$$c_t + \left(1+\eta\right)\left(1+\theta\right)^{\frac{1}{1-\gamma}} k_{t+1} = k_t^{\gamma} + \left(1-\delta\right) k_t \tag{9}$$

Here $\hat{\beta} = \beta\left(1+\eta\right)\left(1+\theta\right)^{\frac{1-\sigma}{1-\gamma}}$.

b. We can interpret the state variable as the per capita capital stock, but where persons are weighted by their labor productivity. Thus, we have capital per productivity adjusted worker.

c. For finite welfare, we need the discount rate to be less than one:

$$\hat{\beta} = \beta\left(1+\eta\right)\left(1+\theta\right)^{\frac{1-\sigma}{1-\gamma}} < 1 \tag{10}$$

We cannot have the technology growth rate be so large that high technology capital in the far future becomes so attractive that households consume zero in order to obtain infinite utility from capital later.

d. In the steady state, y_t approaches a constant $\bar{y}$. Here y_t is output per productivity adjusted person. For output per person, we have:

$$\bar{y} = y_t = \frac{Y_t}{L_t A_t^{\frac{1}{1-\gamma}}} = \frac{\frac{Y_t}{L_t}}{A_t^{\frac{1}{1-\gamma}}}, \tag{11}$$

$$\frac{Y_t}{L_t} = \bar{y} A_t^{\frac{1}{1-\gamma}}, \tag{12}$$

In the steady state, the growth rate of the right and left hand sides must be equal, so:

$$A_t^{\frac{1}{1-\gamma}} = A_0^{\frac{1}{1-\gamma}} (1+\theta)^{\frac{t}{1-\gamma}}, \tag{13}$$

$$g_{\frac{Y}{L}} = (1+\theta)^{\frac{1}{1-\gamma}} - 1. \tag{14}$$

Similarly, for total GDP we have:

$$g_Y = (1+\eta)(1+\theta)^{\frac{1}{1-\gamma}} - 1. \tag{15}$$

Question 2.

For the limiting case, note that:

$$\lim_{\sigma \to 1} u(c) = \frac{0}{0}. \tag{16}$$

Since the limit is indeterminate, we can use L'Hopital's rule. Taking the derivative of the numerator and the denominator, we see that:

$$\lim_{\sigma \to 1} u(c) = \lim_{\sigma \to 1} \frac{-\log(c)\, c^{1-\sigma}}{-1} = \log(c). \tag{17}$$

Note: to take the derivative of $c^{1-\sigma}$ with respect to σ, first rewrite using: $c^{1-\sigma} = \exp((1-\sigma)\log(c))$.

Question 3.

a. With the increase in productivity we have two effects:

- The return to savings rises: next period, savings becomes capital that is more productive and thus generates more output. This effect causes savings to rise. This is a subsititution effect (consumption to savings).
- Consumption becomes less smooth: holding savings constant, consumption becomes less smooth as consumption will rise in the future (higher productivity means more production and therefore more consumption, holding savings constant). Since households prefer smooth consumption, they will reduce savings and increase consumption today to try to smooth consumption. This is an income effect.

b. As the utility function becomes more concave, households have stronger preferences for smooth consumption. The utility from smooth consumption becomes increasingly better than the utility from non-smooth consumption:

$$u(c_\phi) >> \phi u(c_t) + (1-\phi) u(c_{t+1}), \quad c_\phi = \phi c_t + (1-\phi) c_{t+1}, \; 0 < \phi < 1. \tag{18}$$

This increases the income effect.

c. The utility function becomes more concave as σ rises.

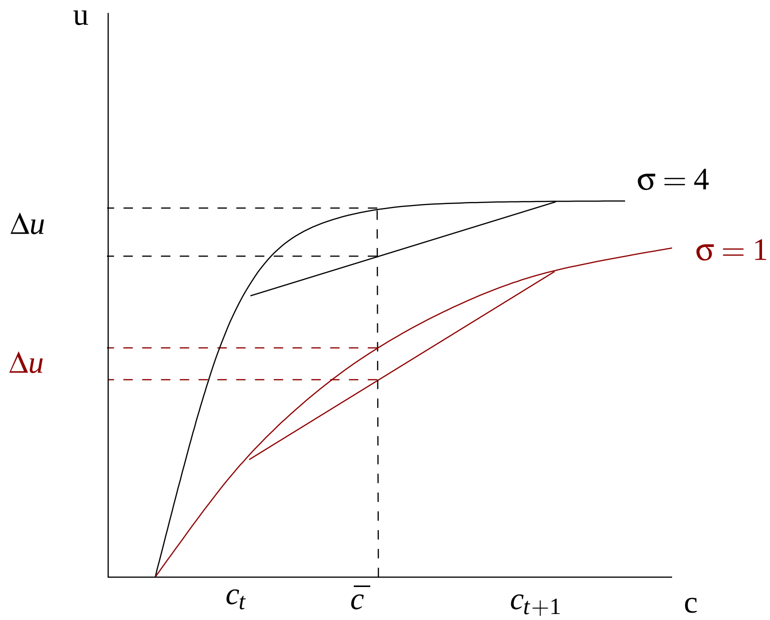

Figure 1: Preference for smooth consumption (ΔC) increases with σ.

For $\sigma = 0$, utility is linear. As σ rises, the difference between the left and right hand sides of equation (18) increases.

d. Savings will fall as σ rises due to the stronger income effect.

e. Less savings is equivalent to discounting the future more (β closer to zero). As σ increases, the discount factor falls, reflecting the stronger income effect. Increasing θ magnifies both the income and substitution effects. Therefore, the effect of θ on the discount rate depends on σ, which determines which effect is stronger.

f. Under log utility, the income and substitution effects exactly cancel, and so the discount rate is unaffected by technological change.

Question 4

a. We have one state variable which is k. So v is a function of the one state variable. After substituting for c using the resource constraint, we have a single control variable which is k'. Thus:

$$v(k) = \max_{k'} \left\{ \frac{1}{1-\sigma} \left[k^{\gamma} + (1-\delta) k - (1+\eta)(1+\theta)^{\frac{1}{1-\gamma}} k' \right]^{1-\sigma} + \hat{\beta} v(k') \right\} \tag{19}$$

b. The first order condition is:

$$c^{-\sigma} = \frac{\hat{\beta}}{\Gamma} v_k(k'), \tag{20}$$

$$\Gamma \equiv (1+\eta)(1+\theta)^{\frac{1}{1-\gamma}}. \tag{21}$$

The left hand side is the marginal utility of consumption and the right hand side is the marginal utility of investment. With one more unit of resources, we should be indifferent between consuming and saving. If consuming gave more utility, then we should go back and reallocate more resources from saving (which gives low utility) towards consumption (which gives higher utility). Therefore, we must be indifferent with the last resource unit. We divide by the growth rates because the savings must be split among a larger productivity adjusted population.

Question 5

a. First, we can build the resource constraint by setting resources equal to allocations. Resources are production and capital after depreciation, as in question 1. For allocations, we have consumption as in question 1, and investment. Since any investment in period t becomes capital in two periods, investment is X_{2t}. So the resource constraint is:

$$C_t + X_{2t} = K_t^{\gamma} \left(A_t^{\frac{1}{1-\gamma}} L_t \right)^{1-\gamma} + (1-\delta) K_t \tag{22}$$

Note that we also have a constraint that investment today is capital under construction in the next period:

$$X_{2t} = X_{1,t+1}. \tag{23}$$

Finally, capital next period is equal to capital under construction today plus capital which has not depreciated:

$$K_{t+1} = X_{1,t} + (1-\delta) K_t. \tag{24}$$

Normalizing as in question 1 results in:

$$c_t + x_{2t} = k_t^\gamma + (1-\delta)\, k_t, \tag{25}$$

$$x_{2t} = \Gamma x_{1,t+1}. \tag{26}$$

$$\Gamma k_{t+1} = x_{1,t} + (1-\delta)\, k_t. \tag{27}$$

b. State variables are given from today's perspective, but change over time. Clearly k_t and capital under construction x_{1t} are given today. The control variables are investment x_{2t} and consumption c_t. So we can write the value function as:

$$v\left(k,x_1\right) = \max_{x_2} \left\{ \frac{1}{1-\sigma} \left[k^\gamma + (1-\delta)\, k - x_2 \right]^{1-\sigma} + \hat{\beta} v\left(k', x_1'\right) \right\}. \tag{28}$$

Notice that we have two variables not defined as either states or controls, x_1' (a control) and k' (a state). We also have two remaining constraints that need to be substituted in. Hence:

$$v\left(k,x_1\right) = \max_{x_2} \left\{ \frac{1}{1-\sigma} \left[k^\gamma + (1-\delta)\, k - x_2 \right]^{1-\sigma} + \hat{\beta} v\left(\frac{x_1 + (1-\delta)\, k}{\Gamma}, \frac{x_2}{\Gamma} \right) \right\} \tag{29}$$