Training Expectations With Least Squares Learning in Non-linear Self-referential Models

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First Version: November 9, 1992
Current Version: December 1, 1995

*The idea for this paper originated from a conversation with Stephen E. Spear. Thanks also to Marvin H. Blachman, Wouter den Haan, Bennett T. McCallum, Albert Marcet, and Shyam Sunder for comments and suggestions. The usual disclaimer applies.

Abstract

In this paper, we examine a general class of non linear self-referential economies in which agents use least squares learning to forecast and update their beliefs, given an arbitrary vector of past state variables and forecasts. We find conditions for which least squares learning converges to stationary rational expectations equilibria. These conditions are similar to conditions found by Grandmont (1985) and Guesnerie and Woodford (1989) for convergence of models with adaptive forecasts to stationary equilibria. We further show these conditions enable expectations to be "trained" in the sense of Marimon, Spear, and Sunder (1991). JEL Classification numbers D83, D84, E32.

1. Introduction

A broad class of models receiving extensive study in the literature are "self-referential" models, in which agents' beliefs about future values of state variables determine the current values of state variables. The Cagen hyperinflation model considered in Marcet and Sargent (1989a), the partial equilibrium model of investment in Lucas and Prescott (1971), and the overlapping generations (OG) model are examples of self-referential economies.[1] Some structure on the beliefs of the agents is necessary to complete self-referential models. The standard assumption for agents beliefs is rational expectations.

Lucas (1986) and others postulate that rational expectations is the final product of some unspecified learning process. In the past, agents used some trial and error method of forecasting until they learned to forecast rationally, without consistent errors. Rational expectations has numerous advantages. If an agent is forecasting rationally, there is no incentive to change how the agent forecasts, therefore rational expectations represents an equilibrium of the learning process. Furthermore, rational expectations is designed so that there is no need to address the learning process, where analysis is more difficult and economic theory offers little intuition on how an agent learns.

There is an extensive body of results on self-referential economies with rational expectations. Spear (1985) and others have shown conditions for the existence of a continuum of rational expectations equilibria in self-referential economies. Therefore, analysis focuses around convergence to stationary rational expectations equilibria (REE). For example Cass, Okuno, and Zilcha (1979) Grandmont (1985), and others give conditions for stability of stationary REE in OG models. Kehoe and Levine (1984) give conditions for stability of the monetary steady state in an OG model. How agents are able to learn to be rational, however, is not clear, especially in non-linear models. Furthermore, authors such as Azariadis (1981) and Woodford (1990) find that self-referential economies often exhibit sunspot equilibria, which are rational, but depend on extrinsic uncertainty. In particular, if the stationary state is stable, then the stationary state is indeterminate, or not locally isolated. Conversely, the stationary state can be determinate, or locally isolated, but the perfect foresight dynamics

[1] See Evans (1987) for other examples of self-referential economies.

may be unstable around the stationary state. Such models have no predictive power unless the model can "select" the sunspot or stationary REE that the economy converges to. In this case, a learning rule may select a REE or sunspot equilibrium.

Authors such as Grandmont (1985) and Guesnerie and Woodford (1989) suppose that agents believe future values of the state variables are determined by some function of past values and past expectations. These adaptive rules are rational if the economy is at one or more stationary points, but otherwise are not rational. Grandmont (1985) and Guesnerie and Woodford (1989) show that stability of stationary REE in such cases often depends upon the specification of the adaptive forecasting rule. Specifically, a stationary REE is stable if the backward perfect foresight dynamics are stable. Hence while a specific adaptive rule may select a REE, the selection depends upon which adaptive forecasting rule is used.

Adaptive expectations imply agents are not fully rational, since the assumption that future values of state variables depend linearly on past values is incorrect. Although such an assumption may seem unsatisfactory, if agents are allowed to update the weights in a linear model, the learning process can converge to stationary rational expectations equilibria, where forecasts are rational. For example Marcet and Sargent (1989b), Bray and Savin (1986), and others demonstrate that least squares learning converges to the rational expectations equilibria. Furthermore, Evans (1987) and Marcet and Sargent (1989b) show that learning can select between REE. In a non-linear model, Woodford (1990) shows that if agents include sunspot variables in the linear forecasting rule, the learning process can converge to a forecasting rule that uses sunspot variables and is rational at stationary sunspot REE using the Robbins-Monroe learning algorithm.

On the other hand, adaptive expectations does not resolve indeterminacy of equilibria. An adaptive forecasting rule implicitly specifies what agents believe determines equilibrium and how equilibrium is determined. To make equilibrium determinate requires knowledge of agents beliefs.

Marimon, Spear, and Sunder (1991) show experimentally that agents attempt to forecast by trying to recognize patterns in the past data in a non-linear OG model. In these experiments, agents forecasted cyclic prices given a cyclic price history and forecasted stationary state prices given a stationary state price history.

In this paper, we show that stability of stationary REE under least squares learning is determined by the stability of the backward perfect foresight dynamics. We find theoretical conditions for which the results of Marimon, Spear, and Sunder (1991) are replicated using least squares learning. To model pattern recognition we assume agents use a linear forecasting rule, and update the weights to forecast either cycles or stationary points. The agents believe that an element of some wide class of forecasting rules determines future state variables, which include linear versions of the adaptive rules of Grandmont (1985) and the linear adaptive rules of Guesnerie and Woodford (1989). We find the stability results of Grandmont (1985) and Guesnerie and Woodford (1989) for adaptive expectations are enough to ensure that agents recognize patterns in the past data.

If agents forecast by detecting patterns in the past data, then the REE that is selected depends on what the agents were "trained" to believe in according to Marimon, Spear, and Sunder (1991). We find that agents can be trained to believe in cyclic or steady state patterns. In our model, training is represented by the initial conditions or priors of the least squares learning algorithm. Least squares learning also has initial conditions corresponding to the confidence the agent places in the prior beliefs. We find that if the agent is not sufficiently confident in the priors, then beliefs are updated such that no pattern is immediately recognized. Hence the training period must be sufficiently long.

Section 2 introduces a general self-referential model and the least squares learning specification. Section 3 gives theoretical conditions on for training expectations with least squares learning. In section 4 we show more general sufficient conditions, which enable expectations to be trained in the model. In section 5, we show that expectations can be trained using two examples from the literature. Section 6 has some concluding remarks.

2. Model

Consider an economic model that can be reduced to a first order, one dimensional dynamical system. Let P_t be the state variable of the system, and let the economy evolve according to:

$$P_t = \phi(P^e_{t+1}) \tag{2.1}$$

Equation (2.1) is the law of motion for an economy. The value of P_t today depends on the believed value of the state variable in the future. Economic models, such as the overlapping generations model, can often be reduced to such a system, with P typically representing prices or labor (see section 4). Hereafter, for convenience, we refer to P_t as the "price." There is one regularity assumption on the law of motion:

ASSUMPTION A1 *The function ϕ is C^2 and invertible locally around the stationary state.*

Often ϕ is generated by finding an implicit function, so local invertibility is necessary to reduce the problem to (2.1) regardless. Note that there is no linearity restriction so that in this sense we extend the work of Marcet and Sargent (1989b) to a non-linear model. P_t is a scaler, however, which is more restrictive than Marcet and Sargent (1989b). Furthermore, the law of motion is deterministic, which is an additional simplification of Marcet and Sargent (1989b).[2]

The standard analysis of the law of motion (2.1) begins with the perfect foresight assumption. Since there is no inherent uncertainty in the model, agents can deduce the next period price and forecast without error. Thus:

$$P^e_{t+1} = P_{t+1} \tag{2.2}$$

Perfect foresight gives rise to the following definitions. Let $\phi^k(P)$ denote the kth composition function $\phi(\phi(\ldots\phi(P)))$. Then we have:

Definition 1 A *perfect foresight equilibrium sequence* is a sequence $\{P_i\}$ such that:

$$P_i = \phi(P_{i+1}) \quad \forall\ i \tag{2.3}$$

Definition 2 A *k-state cycle* is a vector $P^k = [P_1 \ldots P_k]'$ such that:

[2]We make the law of motion deterministic in order to better compare the results with Marimon, Spear, and Sunder (1991).

$$P_i = \phi^k(P_i) \quad \forall \quad i = 1 \ldots k \text{ and } P_i \neq P_j \ \forall \ i \neq j \ i, j \in 1 \ldots k \tag{2.4}$$

ASSUMPTION A2 *Consider any k-state cycle P^k. Then $P^k >> 0$.*

We believe assumption (A2) is not restrictive since typically the state variable P is positive. In section 5 we give more general conditions about the existence of strictly positive cycles. In the overlapping generations model with money, we restrict analysis to monetary equilibria. Local neighborhoods around the cyclic points are also restricted to be positive.

Let $P^1 = P^*$ be the stationary state. Equation (1) implies there exists a continuum of perfect foresight equilibria. Therefore analysis focuses on local behavior around the k-state cycle. Grandmont (1985) and others have shown that the condition for stability of a cycle under perfect foresight is that the forward dynamics are stable:

$$\left| \prod_{i=1}^{k} \phi^{-1\prime}\left(P_i^k\right) \right| < 1 \tag{2.5}$$

Grandmont (1985) also demonstrates that a sufficient condition for stability of a k-state cycle under a general adaptive forecasting scheme is that the backward perfect foresight dynamics are stable:

$$\left| \prod_{i=1}^{k} \phi'\left(P_i^k\right) \right| < 1 \tag{2.6}$$

Suppose now agents do not possess perfect foresight, but are in the process of learning how to forecast. Marimon, Spear, and Sunder (1991) show that, in an experimental setting, agents attempt to forecast by trying to recognize patterns in the previous data. One way to model pattern recognition is through a least squares learning algorithm. Let $d_t \equiv \begin{bmatrix} P_t & \ldots & P_{t-r} & P_t^e & \ldots & P_{t-r}^e \end{bmatrix}'$ and suppose agents forecast according to:

$$P_{t+1}^e = \beta_{t-1}' d_t \tag{2.7}$$

Here, β_{t-1} is a vector of weights determined by:

$$\beta_{t-1} = \left(\sum_{s=1}^{t-2} d_s d_s' \right)^{-1} \left(\sum_{s=1}^{t-2} P_{s+1} d_s \right) \tag{2.8}$$

Thus agents forecast adaptively using equation (2.7), but update the parameter β each period with a least squares regression, equation (2.8). Note that since the law of motion (2.1) is usually non-linear, least squares learning cannot converge to (learn) the true law of motion as in Marcet and Sargent (1989b) and others. However, the learning algorithm can learn cyclic patterns in the past data.

Using equation (2.1) to substitute out for the expectations variables gives:

$$d_t = \left[\begin{array}{cccccc} P_t & \dots & P_{t-r} & \phi^{-1\prime}(P_{t-1}) & \dots & \phi^{-1\prime}(P_{t-r-1}) \end{array} \right]' \tag{2.9}$$

That agents forecast by equation (2.7) causes the actual law of motion to be determined by:

$$P_t = \phi\left(\beta_{t-1}' d_t\right) \tag{2.10}$$

The next assumption insures that the agent does not rule out the possibility of a cycle *a priori*.

ASSUMPTION A3 *Suppose equation (2.1) possesses a k-state cycle under perfect foresight. Then $r \geq k$.*

Thus the space of cyclic patterns that the agent believes possible initially is at least as large as the actual number of cycles. Of course r must be finite for a least squares regression to make sense. Assumption (A3) admits a wide class of adaptive rules common in the literature that the least squares algorithm can learn.[3] We also define d^k to be the k-state cycle with $P_t = P_1, \quad P_{t-1} = P_2$ and so on.

Assumption (A3) leads to the following definitions:

Definition 3 An *associated fixed point* of a k-state cycle is a vector β^k satisfying:

$$\sum_{i=1}^{m} \left(\beta_{i\cdot k}^k + \beta_{(r+1)+i\cdot k}^k\right) = 1 \tag{2.11}$$

$$\beta_j^k = 0 \;\forall\; j \neq ik, r+1+ik$$

[3] Similar to Evans (1987), we perturb an adaptive rule by adding an arbitrary number of extra lags. In fact, assumption (A3) is equivalent to assuming that "strong expectational stability" in the sense of Evans (1987) is considered here, except that we consider only price and expectation lags and not sunspot variables.

Here m is the largest multiple of k less than or equal to r. The associated fixed point is simply any vector β^k that "detects" the cycle.

Definition 4 An *associated invariant set* of an k-state cycle is the set, D_c^k, of all associated fixed points of the k-state cycle:

$$D_c^k = \left\{ \beta^k : \sum_{i=1}^{m} \left(\beta_{i\cdot k}^k + \beta_{(r+1)+i\cdot k}^k \right) = 1 \text{ and } \beta_j^k = 0 \; \forall \; j \neq ik, r+1+ik \right\} \tag{2.12}$$

The invariant set forms a subspace of $\Re^{2r+2}$. In the proofs of convergence, we typically speak of convergence to the invariant set, because all elements of D_c^k are associated with the same cycle.

We impose one restriction on the associated invariant set:

ASSUMPTION A4 *Suppose $k = 1$. Then $\phi'(P^*) < \frac{1}{1+\beta_1^*}$.*

Assumption (A4) arises from the presence of the current price in the forecasting function.

The next two assumptions make the forecasting function of the type studied by Grandmont (1985) or Guesnerie and Woodford (1989), respectively, when $\beta = \beta^k$. Let Φ be the local implicit function found by solving (2.7) for P_t.

ASSUMPTION A5 *The expectation formula satisfies the following condition:*

$$\left.\frac{\partial \Phi}{\partial P_{t-i}}\right|_{d=d^k, \beta=\beta^k} \equiv \left.\frac{\frac{\partial}{\partial P_{t-i}} \phi\left(\beta_{t-1}' d_t\right)}{\frac{\partial}{\partial P_t}\left(P_t - \phi\left(\beta_{t-1}' d_t\right)\right)}\right|_{d=d^k, \beta=\beta^k} \geq 0 \; \forall i \tag{2.13}$$

Assumption (A5) is assumption (3.h) of Grandmont (1985). Assumption (A5) implies the forecasting function is of the type studied by Grandmont (1985) when $\beta = \beta^k$.

ASSUMPTION A6 *The associated invariant set, D_c^k, satisfies $m = 1$ and $0 \leq\leq \beta^k \leq\leq 1$.*

Assumption (A6) is an alternative to assumption (A5). Assumption (A6) implies when $\beta = \beta^k$ the forecasting function is of the type examined by Guesnerie and Woodford (1989). This assumption is more reasonable in cases where $\phi'(P^*)$ is negative, since assumption (A5) often severely restricts the associated invariant set in this case.

3. Training Expectations

Marimon, Spear, and Sunder (1991) found that, given a law of motion with a two-state cycle and a fixed point, agents examined past data for either of these patterns and then forecasted the next point in either the cycle or the fixed point. Furthermore, agents could be "trained" to believe in either the cycle or the fixed point by giving the agents a data set that had either of these patterns. In this model, the training period corresponds to the initial conditions of least squares learning. Thus for example, if β_0 is in a neighborhood of β^k, the agent priors are that a k-state cycle is the pattern in the past data, or that the agent has been trained to believe in the period k pattern. Hence the following definition:

Definition 5 The following are equivalent statements:

- Agents can be trained to believe in an k-state cycle.
- A k-state cycle is locally stable under least squares learning.
- $\beta_t \to \beta^k$ and $d_t \to d^k$ as $t \to \infty$ for β_0 sufficiently close to β^k and d_0 sufficiently close to d^k.

The interesting case is when, as in Marimon, Spear, and Sunder (1991), there are multiple cycles which the agents can be trained to believe in.

We use the "differential equation approach" developed by Ljung (1975) and used by Marcet and Sargent (1989b) and others to establish conditions on convergence. Specifically, we use the results for non-linear dynamics given in Ljung (1975). Proofs are given in the appendix unless otherwise stated.

First we rewrite equations (2.8) and (2.10) recursively, which is somewhat more intuitive.

LEMMA A *Equation (2.8) can be written in the following form:*

$$x_t = x_{t-1} + \gamma(t)\,\mathcal{Q}(t; x_{t-1}; \varphi_t) \tag{3.1}$$

Here φ_t is the observation vector and x_t is a vector of estimated parameters. The recursive form of the parameter vector is:

$$\beta_t = \beta_{t-1} + \frac{1}{t}\left(R_{t-1}\right)^{-1} d_{t-1}\left(\frac{1}{1+\frac{1}{t}\left(d'_{t-1}R_{t-1}^{-1}d_{t-1}-1\right)}\right)\left[P_t - d'_{t-1}\beta_{t-1}\right] \tag{3.2}$$

$$\text{col}(R_t) = \text{col}(R_{t-1}) + \frac{1}{t}\left[\text{col}(d_{t-1}d'_{t-1}) - \text{col}(R_{t-1})\right] \tag{3.3}$$

The system is in the form of (3.1), with $\mathcal{Q}$ defined from equations (3.2) and (3.3), and:

$$x_t = \begin{bmatrix} \beta_t \\ \text{col}(R_t) \end{bmatrix} \tag{3.4}$$

$$\gamma\left(t\right) = \frac{1}{t} \tag{3.5}$$

$$\varphi_t = \begin{bmatrix} P_t & P_{t-1} & \dots & P_{t-r-1} \end{bmatrix}' \tag{3.6}$$

Here col is the column operator.

Lemma (A) divides the system into input variables, which are parameters to be estimated, and output variables, which are the state variables of the system. R_t is the "precision matrix," which roughly corresponds to how confident the agent is in the weights. Equations (3.2) and (3.3) form a recursive system that is first order in x_t. However, the presence t in the functions causes difficulties in the analysis. Additionally a "forgetting factor" could also be introduced, as in Marcet and Sargent (1989b), which would modify the the γ function. The forgetting factor is omitted here to avoid complications, but the proofs go through as long as the forgetting factor satisfies the conditions given in the appendix. Equations (3.2) and (3.3) also show that there are two initial conditions for the input variables: β_0 and R_0. β_0 corresponds to the agent's priors or what the agent was trained to believe. R_0 is equivalent to the length of the training period in Marimon, Spear, and Sunder (1991). The proofs of convergence implicitly depend on R_0 being sufficiently large.

An important aspect of the problem is the behavior of the precision matrix R_t and the inverse of R_t. First, R_t approaches singularity in the limit by assumption (A2). If the inverse of the precision matrix approaches singularity, the coefficients of the least squares regression become unstable from period to period, since different coefficient vectors predict with approximately the same accuracy. Hence the coefficient vector does not converge in this case. However, Ljung (1975) shows that a small perturbation insures that R_{t-1} does not approach a singular matrix. In particular, let:

$$\mathrm{col}(R_t) = \mathrm{col}(R_{t-1}) + \frac{1}{t}\left[\mathrm{col}(d_{t-1}d'_{t-1}) + \mathrm{col}\,(\epsilon I) - \mathrm{col}(R_{t-1})\right] \tag{3.7}$$

Here ϵ is a small number and I is the identity matrix.

LEMMA B *The observation vector, φ_t, can be written recursively as a function:*

$$\varphi_t = g\,(x_{t-1}; \varphi_{t-1}) \tag{3.8}$$

The output takes the following form:

$$\varphi_t = \begin{bmatrix} P_t \\ P_{t-1} \\ \vdots \\ P_{t-r-1} \end{bmatrix} = \begin{bmatrix} \Phi\,(x_{t-1}, \varphi_{t-1}) \\ P_{t-1} \\ \vdots \\ P_{t-r-1} \end{bmatrix} = g\,(x_{t-1}; \varphi_{t-1}) \tag{3.9}$$

Notice that the vector of state variables includes the current price (forecasts are determined simultaneously with prices). If the forecast function consists of strictly past prices, then:

$$\Phi\,(x_{t-1}, \varphi_{t-1}) = \phi\left(\beta'_{t-1}d_t\right) \tag{3.10}$$

The following is an outline of the idea of the convergence proofs. We fix β_{t-1} at $\bar{\beta}$ in a neighborhood of the associated cycle, β^k. With β fixed, Ljung (1975) shows that $\mathcal{Q}$ converges to a first order differential equation in which the troublesome presence of $\gamma\,(t)$ is eliminated. Ljung (1975) also shows that the difference between the evolution of $\mathcal{Q}$ at $\bar{\beta}$ and the evolution of $\mathcal{Q}$ at β_{t-1} goes to zero in the limit. Thus convergence of the differential equation implies convergence of the input variables. The eigenvalues of the Jacobian of the output of the system must have modulus less than one at $\bar{\beta}$ to establish that the difference goes to zero. But with β fixed, the problem is similar to that of the adaptive system examined by Grandmont (1985) or Guesnerie and Woodford (1989) under assumptions (A5) and (A6), respectively. In particular, the problem is the same except that $\bar{\beta} \neq \beta^k$. The following lemma shows that the conditions given by Grandmont (1985) or Guesnerie and Woodford (1989) are enough to get convergence for $\bar{\beta} \in N\left(\beta^k\right)$ as well.

LEMMA C *Suppose there exists a k-state cycle under perfect foresight. Suppose assumptions (A1) ... (A4). Suppose further that:*

$$\phi_k \equiv \prod_{i=1}^{k} \phi'(P)\Big|_{P=P_i^k} \in (-1,1) \quad \textit{if } k > 1 \tag{3.11}$$

Finally, suppose that either:

1. *assumption (A5) holds and condition(3.11) holds when $k=1$ or*

2. *assumption (A6) holds and $\phi_1 < 1$ when $k=1$.*

Then there exists a set $D_s^k \supset x^k$ such that for $\bar{x} \in D_s^k$:

$$\frac{\partial g^k\left(\varphi_i^k;\bar{x}\right)}{\partial \varphi_{t-1}} \qquad i = 1 \dots k \tag{3.12}$$

has all eigenvalues of modulus less than one.

We are now in a position to prove the main theorems. Theorem (1) shows that the conditions of Grandmont (1985) or Guesnerie and Woodford (1989) are enough to get convergence of least squares learning. Let $D_r^k \subset D_s^k$ be an open set around the associated invariant set of the k-state cycle, which has all positive elements.

THEOREM 1 *Suppose that the perfect foresight law of motion ϕ possesses a k-state cycle ($k=1$ is the stationary state). Suppose that either set of assumptions for lemma (C) hold. Then the least squares parameters converge to the fixed point or cycle. In particular, there exists a neighborhood D_z such that for $\varphi_0 \in D_z$ and $x_0 \in D_1$, a closed subset of D_r:*

$$\beta_t \to \beta^k \in D_c^k \text{ as } t \to \infty \tag{3.13}$$

Hence the least squares forecasting function converges to the adaptive forecasting function of Grandmont (1985) or Guesnerie and Woodford (1989).

The following corollaries establish conditions for training expectations.

COROLLARY 1 *Suppose the conditions of lemma (C) are satisfied. Then the stationary state or cycle is locally stable. In particular, there exists a neighborhood D_z such that for $\varphi_0 \in D_z$ and $x_0 \in D_1$, a closed subset of D_r:*

$$\varphi_t \to \varphi^k \textit{ as } t \to \infty \tag{3.14}$$

Lemma (C) implies that if the input is converging to the associated fixed point, then the output must converge to the corresponding cycle.

Now suppose ϕ has multiple cycles or stationary states. That is suppose there are k-state cycles indexed $k_1 \dots k_n$. Then, if each cycle satisfies the conditions of lemma (C), convergence depends on the initial conditions. In particular there exists local neighborhoods around each of the cycles such that if the initial condition is inside any of these neighborhoods, convergence to the corresponding cycle occurs. Hence expectations can be trained. We summarize this in the following lemma.

COROLLARY 2 *Suppose there exists k-state cycles $k_1 \dots k_n$ such that $k_i \neq k_j \ \forall \ i \neq j$ and that either set of conditions for lemma (C) are satisfied for each k. Then there exists neighborhoods $D_z^{k_i}$ and $D_r^{k_i}$ such that for $\varphi_0 \in D_z^{k_i}$ and $x_0 \in D_1^{k_i}$, a closed subset of $D_r^{k_i}$:*

$$\varphi_t \to \varphi^{k_i} \ \forall \ i \tag{3.15}$$

In particular, if there exists two or more cycles that are backward stable, then the cycles are all locally stable under least squares learning. Therefore expectations can be trained. Note that as β converges to β^k the problem approaches a model with adaptive forecasting. Thus the condition for convergence is intuitively appealing.

4. Existence of Economic Models With Training

The existence of multiple cycles in the perfect foresight dynamics has been extensively studied. For example Benhabib and Day (1982), Cass, Okuno, and Zilcha (1979), Grandmont (1985), and Marcet and Sargent (1989a) have all given examples of self-referential models that exhibit multiple stationary equilibria. These authors note that the existence of a k-state cycle implies that k'-state cycles also exist, for all $k' < k$. Further, the existence of a 3-state cycle implies cycles of all periods and chaotic trajectories also exist. Finally a sufficient condition for the existence of a 2-state cycle for the law of motion ϕ is that the stationary state is unstable in the backward perfect foresight dynamics, or $\phi'(P^*) < -1$.

The cycles given must also be shown to be consistent with the economic assumptions of the model, such as positive prices. This we assume in assumption (A2). However, in many economic models, this occurs naturally from a strictly increasing utility function.

Authors such as Grandmont (1985) have also examined stability of the perfect foresight dynamics. Grandmont (1985) uses a result by Singer (1978) to show conditions for there to be at most one backward stable k-state cycle. To use the result of Singer (1978), we introduce the following assumption:

ASSUMPTION A7 *Let ϕ be C^3 and let $\phi : X \to X$ where $X = [a, b] \subseteq \Re$.*

Definition 6 *The* Schwarzian derivative*, $S\phi$, of ϕ is:*

$$S\phi = \frac{\phi'''}{\phi'} - \frac{3}{2}\left(\frac{\phi''}{\phi'}\right)^2 \quad \phi' \neq 0 \tag{4.1}$$

As shown in the last section, if there exists at least two backward stable cycles in the perfect foresight dynamics, then expectations can be trained. Sufficient conditions for this are found using a modified version of the Singer (1978) theorem.

THEOREM 2 *Suppose assumption (A7). Suppose ϕ has perfect foresight k-state cycles $k_1 \ldots k_n$ with $n \geq 2$ in the interior of X. Suppose that $S\phi^{-1}(P) < 0$ for all $P \in X$. Finally, suppose ϕ^{-1} has $c < n - 2$ critical points. Then there exists k-state cycles $k_1 \ldots k_{n'}$ with $n' \geq 2$ such that $|\phi_{k_i}| < 1$ $\forall i \in n'$.*

Theorem (2) shows that, given the learning assumptions (A3), (A4), and (A5) or (A6), a negative Schwarzian derivative and a restriction on the number of critical points is sufficient to show that expectations can be trained.

The proof of the theorem relies on Singer's result that every stable k-state cycle has a critical point attracted to it. Hence, the number of critical points is greater than or equal to the number of stable k-state cycles. Hence restricting the number of critical points in the forward perfect foresight dynamics restricts the number of forward stable k-state cycles. Hence the restriction puts a lower bound on the number of backward stable k-state cycles. The conditions are not necessary since two critical points may be attracted to the same cycle. Generally, the only assumption of the theorem that is difficult to verify is that the Schwarzian derivative is everywhere negative. In the next section we present two examples, the first of which has a negative Schwarzian derivative in the forward perfect foresight dynamics. The second example does not have an everywhere negative Schwarzian derivative, but still has two backward stable cycles.

5. Examples

5.1. A model from Benhabib and Day (1982)

Consider a pure exchange, overlapping generations model. Agents live for two periods and value a non-perishable good, consumption, over their lifetime. Agents are assumed to possess the following utility function:

$$U\left(C_t, C_{t+1}\right) = aC_t - \frac{1}{2}bC_t^2 + C_{t+1} \qquad a, b > 0 \qquad C_t \leq \frac{a}{b} \tag{5.1.1}$$

Additionally, agents receive an endowment in each period of their lifetime, but there is no production. Let w_1 denote the first period endowment, and w_2 the second period endowment. Furthermore, there exists a central authority which extends "credit." The young agents buy goods from the old with checks, and the old deposit checks to settle debts incurred when young. The young, for instance, may wish to borrow to finance a mortgage. Let m_t denote the amount of debt incurred when young. Let $m_t = m_{t+1} = \bar{m}$ be the total checks available in any period, which the old agents present in the first period inherit. Hence the representative consumer faces a budget constraint when young and when old:

$$C_t = w_1 + m_t P_t \tag{5.1.2}$$

$$C_{t+1} = w_2 - m_t P^e_{t+1} \tag{5.1.3}$$

Substituting the budget constraints directly into the utility function, maximizing with respect to m_t, and substituting in the general equilibrium requirement that the total check supply equals the demand for checks, gives:

$$P^e_{t+1} = \phi^{-1}\left(P_t\right) = \left(a - bw_1\right) P_t - b\bar{m} P_t^2 \tag{5.1.4}$$

Equation (5.1.4) is the inverse law of motion in this example.

Benhabib and Day (1982) show that for the case where $a - bw_1 = b\bar{m} \in (3.5, 4)$; the law of motion has cycles of all periods as well as chaotic trajectories. Given the formulation (5.1.4), the law of motion has a monetary stationary state:

$$P^* = \frac{a - bw_1 - 1}{b\bar{m}} \tag{5.1.5}$$

The model also has a two state cycle:

$$P_1^2 = \frac{1 + a - bw_1 + \left(a - bw_1 - 3\right)^{1/2}\left(a - bw_1 + 1\right)^{1/2}}{2b\bar{m}} \tag{5.1.6}$$

$$P_2^2 = \frac{1 + a - bw_1 - \left(a - bw_1 - 3\right)^{1/2}\left(a - bw_1 + 1\right)^{1/2}}{2b\bar{m}}$$

The law of motion specified here may also possess cycles of greater order, depending on the values of a, b, and w_1. However, even the two cycles given by equations (5.1.5) and (5.1.6) are enough to show that expectations can be trained.

THEOREM 3 *The following condition is necessary and sufficient to show that the stationary state (5.1.5) and the two-state cycle (5.1.6) for the law of motion (5.1.4) satisfy the first* *set of conditions of lemma (C).*

$$a - bw_1 > 3.45 \tag{5.1.7}$$

Equation (5.1.7) implies that the stationary state is positive, backward stable, and does not satiate the utility function. Equation (5.1.7) also implies that each element of the two state cycle is positive, real, and does not satiate utility, and that the cycle is backward stable. Thus expectations can be trained if the condition (5.1.7) is satisfied.

5.2. The Marimon, Spear, and Sunder (1991) Model

Consider the pure exchange OG model with fiat money studied by Marimon, Spear, and Sunder (1991):

$$U\left(C_t, C_{t+1}\right) = 2\left(\frac{C_t}{5}\right)^{\frac{1}{2}} - \frac{1}{2}\left(\frac{C_{t+1}}{5}\right)^{-2} + 4 \tag{5.2.1}$$

$$C_t = 10 - \frac{M_t}{P_t} \tag{5.2.2}$$

$$C_{t+1} = \frac{M_t}{P^e_{t+1}} \tag{5.2.3}$$

$$\bar{M} = 25 \text{ per person} \tag{5.2.4}$$

Marimon, Spear, and Sunder (1991) show that the inverse law of motion is the following:

$$P^e_{t+1} = \phi^{-1}\left(P_t\right) = 5^{\frac{3}{2}} P_t^{\frac{-1}{4}} \left(2P_t - 5\right)^{\frac{-1}{4}} \tag{5.2.5}$$

Equation (5.2.5) is sixth order of non-linearity. However, equation (5.2.5) is locally invertible and C^2 and therefore satisfies assumption (A1). Note that the Schwarzian derivative of the forward perfect foresight dynamics is not everywhere negative (for example over the interval $(0, .5]$). However, Marimon, Spear, and Sunder (1991) show that the law of motion has a stationary state and a 2-state cycle.

$$P^* = 5 \qquad P^2 = \begin{bmatrix} 2.56 \\ 14.75 \end{bmatrix}$$

Furthermore, the values that determine the perfect foresight dynamics are:

$$\phi^{-1\prime}\Big|_{P=P^*} = -.75 \qquad \phi^{-1\prime}\Big|_{P=14.75} = -.096\ \phi^{-1\prime}\Big|_{P=2.56} = -64.04$$

Here $\phi_1 = -\frac{4}{3}$ and $\phi_2 = .16$, hence the stationary state and the two-state cycle satisfy the second set of conditions of lemma (C), if we assume that the agent only looks back to the $t-1$ period in his expectations function ($m = 1$). Hence theorem (1) holds in this example for both the stationary state and the two-state cycle. Hence corollaries (1) and (2) hold as well, which replicates the results of Marimon, Spear, and Sunder (1991) on human subjects. Agents can be trained to believe in either a fixed point or 2-state cyclic pattern given the least squares learning specification (2.8).

6. Conclusion

We suggest that a plausible way to model agents beliefs is to suppose that agents start with very general priors, that the economy is linear, but a number of lags may influence the current price. Then agents look at the data and try to extrapolate trends. Such "pattern recognition" learning is advantageous since all information is used, unless there is strong evidence from the data that the information is of no help. Simple adaptive forecasting schemes and many sophisticated learning schemes in effect suppose agents with probability one confidence assume almost all information is of no use, before agents even see a single data point in the information set. Hence a problem with using simple adaptive forecasting or, for example, least squares learning, with a small number of lags is possible outcomes of learning and therefore of the economy as a whole are ruled out *a priori*. Furthermore, adaptive and simple learning schemes cannot fully model other elements of the learning process, such as changes in the structure of agents' beliefs.

The experimental results of Marimon, Spear, and Sunder (1991) suggest that agents use sunspot variables to forecast, but only if there is a pattern in the data that suggests that the sunspot matters. This also suggests that agents start with no strong beliefs about what information to use.

One important problem with pattern learning is that perfect foresight forecasting rule cannot be attained with least squares learning if the law of motion is non linear. We plan to examine non linear learning rules capable of learning either patterns or the perfect foresight forecasting rule. We hope to see under what conditions does the agent learn to be rational and under what conditions does the agent learn patterns in the past data.

7. Appendix: proofs of main theorems

7.1. Proof of lemma (1.1)

$$\beta_{t-1} = \left(\sum_{s=1}^{t-2} d_s d_s'\right)^{-1} \left(\sum_{s=1}^{t-2} P_{s+1} d_s\right) \tag{7.1.1}$$

$$= \left(\sum_{s=1}^{t-2} d_s d_s'\right)^{-1} \left(\sum_{s=1}^{t-2} d_s d_s' - d_{t-2} d_{t-2}'\right) \beta_{t-2} + \left(\sum_{s=1}^{t-2} d_s d_s'\right)^{-1} P_{t-1} d_{t-2} \tag{7.1.2}$$

$$\Rightarrow \beta_t = \beta_{t-1} + \left(\sum_{s=1}^{t-1} d_s d_s'\right)^{-1} d_{t-1} \left[P_t - d_{t-1}' \beta_{t-1}\right] \tag{7.1.3}$$

Notice that the equation includes the prediction error, but only using one period old regression parameter, and not the most current information. This is due to the time period in which the parameter vector was defined. β_t is parameter vector generated using data in period t, not the parameter vector used in the forecast at period t. Next the precision matrix is also defined recursively.

$$R_t \equiv \frac{1}{t} \left(\sum_{s=1}^{t-1} d_s d_s'\right) \tag{7.1.4}$$

$$= \frac{1}{t}(t-1)\left[\frac{1}{t-1}\left(\sum_{s=1}^{t-2} d_s d_s'\right)\right] + \frac{1}{t} d_{t-1} d_{t-1}' \tag{7.1.5}$$

$$= R_{t-1} + \frac{1}{t}\left[d_{t-1} d_{t-1}' - R_{t-1}\right] \tag{7.1.6}$$

Substituting equation (7.1.4) into (7.1.3) gives:

$$\beta_t = \beta_{t-1} + \frac{1}{t}(R_t)^{-1} d_{t-1}\left[P_t - d_{t-1}'\beta_{t-1}\right] \tag{7.1.7}$$

In order to make the right hand side of (7.1.7) contain only past values of R, a recursive substitution is made. The matrix inversion lemma is then applied. Finally, to conform the precision matrix R to the dimension of the parameter vector, the column operator is used. The resulting system is:

$$\beta_t = \beta_{t-1} + \frac{1}{t}(R_{t-1})^{-1} d_{t-1}\left(\frac{1}{1 + \frac{1}{t}\left(d_{t-1}' R_{t-1}^{-1} d_{t-1} - 1\right)}\right)\left[P_t - d_{t-1}'\beta_{t-1}\right] \tag{7.1.8}$$

$$\mathrm{col}(R_t) = \mathrm{col}(R_{t-1}) + \frac{1}{t}\left[\mathrm{col}(d_{t-1} d_{t-1}') - \mathrm{col}(R_{t-1})\right] \tag{7.1.9}$$

This system is now in the form of (3.1), with $\mathcal{Q}$ defined from equations (3.2) and (3.3), and:

$$x_t = \begin{bmatrix} \beta_t \\ \mathrm{col}(R_t) \end{bmatrix} \tag{7.1.10}$$

$$\gamma(t) = \frac{1}{t} \tag{7.1.11}$$

$$\varphi_t = \begin{bmatrix} P_t & P_{t-1} & \ldots & P_{t-r-1} \end{bmatrix}' \tag{7.1.12}$$

Q.E.D.

7.2. Proof of lemma (1.2)

The first step is to determine the recursive relationship of the output variable P_t. Substituting equation (2.7) into the the law of motion (2.1) to gives:

$$P_t = \phi \left(\beta'_{t-1} d_t \right) \tag{7.2.1}$$

However, since the observation vector at time t is a function of the current periods price, an implicit function must be generated from equation (7.2.1). Let Φ be this function.

$$P_t = \Phi \left(\beta_{t-1}, P_{t-1}, \ldots, P_{t-r-1} \right) = \Phi \left(\beta_{t-1}, \varphi_{t-1} \right) = \Phi \left(x_{t-1}, \varphi_{t-1} \right) \tag{7.2.2}$$

Now we define the recursive relationship of the observation vector using the standard trick.

$$\varphi_t = \begin{bmatrix} P_t \\ P_{t-1} \\ \vdots \\ P_{t-r-1} \end{bmatrix} = \begin{bmatrix} \Phi \left(x_{t-1}, \varphi_{t-1} \right) \\ P_{t-1} \\ \vdots \\ P_{t-r-1} \end{bmatrix} = g \left(x_{t-1}; \varphi_{t-1} \right) \tag{7.2.3}$$

Q.E.D.

7.3. Proof of lemma (1.3)

First suppose that $\beta = \beta^k$. Then we claim that the lemma satisfies the assumptions and requirements in Grandmont (1985), under assumption set (1), or Guesnerie and Woodford (1989), under assumption set (2), for stability with learning. These assumptions are straightforward to verify. Hence, according to the theorems in these papers, the eigenvalues of the Jacobian have modulus less than one.

$$\left| J_k \left(\beta_{t-1} \right) \big|_{\beta_{t-1} = \beta^k} - \lambda I \right| \equiv ch \left(\lambda, \beta^k \right) = 0 \tag{7.3.1}$$

Hence the set of all solutions to equation (7.3.1) satisfy $|\lambda| < 1$. Furthermore, since the Jacobian is continuous in β, the characteristic equation is continuous in β and λ. Hence there exists some neighborhood, N, around β^k such that:

$$\beta_{t-1} \in N\left(\beta^k\right) \Rightarrow |\lambda'| < 1 \tag{7.3.2}$$

for all solutions to:

$$ch\left(\lambda', \beta_{t-1}\right) \tag{7.3.3}$$

Let this neighborhood be D_s^k. Q.E.D.

7.4. Proof of theorem (1.1)

We first prove two lemmas, the first to ensure asymptotic stability of $\mathcal{Q}$ in time and the second to ensure that φ does not increase to much for small changes in x.

LEMMA D *The functions* $\frac{\partial \mathcal{Q}}{\partial \varphi_{t-1}}$ *and* $\frac{\partial \mathcal{Q}}{\partial x_{t-1}}$ *are bounded in* t *for* $x \in D_r$.

We first show:

$$\lim_{t\to\infty} \frac{\partial \mathcal{Q}}{\partial x_{t-1}} = \lim_{t\to\infty} \begin{bmatrix} \frac{\partial \mathcal{Q}_1}{\partial \beta_{t-1}} & \frac{\partial \mathcal{Q}_1}{\partial \operatorname{col}(R_{t-1})} \\ 0 & I \end{bmatrix} < \infty \tag{7.4.1}$$

Equation (7.4.1) holds if and only if both elements of the top row have finite limits since the other elements are independent of t. Therefore the problem reduces to first showing:

$$\lim_{t\to\infty} \frac{\partial \mathcal{Q}_1}{\partial \beta_{t-1}} = -\left(R_{t-1}\right)^{-1} d_{t-1} d'_{t-1} < \infty \tag{7.4.2}$$

Thus equation (7.4.2) is finite for $x \in D_r$. For the second element:

$$\lim_{t\to\infty} \frac{\partial \mathcal{Q}_1}{\partial \operatorname{col}\left(R_{t-1}\right)} \tag{7.4.3}$$

$$= \lim_{t\to\infty} \frac{-1}{t} \left(1 + \frac{1}{t}\left(d'_{t-1} R_{t-1}^{-1} d_{t-1} - 1\right)\right)^{-2} \frac{\partial}{\partial \operatorname{col}\left(R_{t-1}\right)} \left(d'_{t-1} R_{t-1}^{-1} d_{t-1}\right) = 0 < \infty \tag{7.4.4}$$

Thus $\frac{\partial \mathcal{Q}}{\partial x_{t-1}} < \infty$.

For the derivative of $\mathcal{Q}$ with respect to φ_{t-1}, we have that:

$$\lim_{t\to\infty}\frac{\partial\mathcal{Q}}{\partial\varphi_{t-1}}=\lim_{t\to\infty}\begin{bmatrix}\frac{\partial\mathcal{Q}_1}{\partial P_t} & \frac{\partial\mathcal{Q}_1}{\partial d_{t-1}}\\ 0 & \frac{\partial\mathcal{Q}_2}{\partial d_{t-1}}\end{bmatrix} \tag{7.4.5}$$

Again only the top two elements of equation (7.4.5) are a function of time. Consider first:

$$\lim_{t\to\infty}\frac{\partial\mathcal{Q}_1}{\partial P_t}=\lim_{t\to\infty}R_{t-1}^{-1}d_{t-1}\left(1+\frac{1}{t}\left(d_{t-1}'R_{t-1}^{-1}d_{t-1}-1\right)\right)^{-1}=R_{t-1}^{-1}d_{t-1} \tag{7.4.6}$$

Equation (7.4.6) is finite for $x\in D_r$. Second, consider $\lim_{t\to\infty}\frac{\partial\mathcal{Q}_1}{\partial d_{t-1}}$, which is finite if and only if:

$$\lim_{t\to\infty}\frac{-1}{t}\left(P_t-d_{t-1}'\beta_{t-1}\right)R_{t-1}^{-1}d_{t-1}\left(1+\frac{1}{t}\left(d_{t-1}'R_{t-1}^{-1}d_{t-1}-1\right)\right)^{-2}\frac{\partial}{\partial d_{t-1}}\left(d_{t-1}'R_{t-1}^{-1}d_{t-1}\right)$$

$$=0<\infty \tag{7.4.7}$$

Thus the derivative of $\mathcal{Q}$ with respect to φ_{t-1} is bounded in t. Q.E.D.

LEMMA E *Suppose assumption (A1). Suppose further that* $\varphi_n=\bar{\varphi}_n(\bar{x})$. *Then there exists a constant* C *such that for* $t>n$:

$$||\varphi_t-\bar{\varphi}_t(\bar{x})||<C\max_{n\le k\le t}||\bar{x}-x_k|| \tag{7.4.8}$$

Consider first the $n+1$ term.

$$||\bar{\varphi}_{n+1}(\bar{x})-\varphi_{n+1}||=||g(\bar{\varphi}_n(\bar{x});\bar{x})-g(\varphi_n;x_n)||$$

$$=|\Phi(\varphi_n;\bar{x})-\Phi(\varphi_n;x_n)|$$

From the mean value theorem we know that:

$$|\Phi(\varphi_n;\bar{x})-\Phi(\varphi_n;x_n)|\le\left|\left|\left.\frac{\partial\Phi}{dx}\right|_{\hat{x}}\right|\right|\;||\bar{x}-x_n|| \tag{7.4.9}$$

$$\leq \left|\left| \left.\frac{\partial \Phi}{dx}\right|_{\hat{x}} \right|\right| \max_{n\leq k\leq n+1} ||\bar{x} - x_k|| \tag{7.4.10}$$

for some $\hat{x} \in [\bar{x}, x_n]$. Therefore choose:

$$C = \left|\left| \left.\frac{\partial \Phi}{dx}\right|_{\hat{x}} \right|\right| + \epsilon \tag{7.4.11}$$

Thus the lemma holds in the $n+1$ case since Φ is C^2 on the interval $[\bar{x}, x_n]$. For the $t > n$ case we can repeat the above procedure and choose the constant:

$$C = (t - 1 - n) \max_{n\leq k\leq t-1} \left|\left| \left.\frac{\partial H}{dx_k}\right|_{\hat{x}_k} \right|\right| + \epsilon \tag{7.4.12}$$

Where $\hat{x}_i \in [\bar{x}, x_i] \forall i$ and H is the composite function $\Phi\left(g^i\left(.\right)\right)$. The constant C is finite valued since $\hat{x} \in D_r$ is finite valued and φ_n is finite valued and continuity is preserved under composite continuous functions. Q.E.D.

Proof of theorem 1.1 This is a direct application of theorem 8 of Ljung (1975). The theorem is restated for convenience. For the proof, the above lemmas are used to show that the conditions of Ljung's theorem are satisfied.

THEOREM 4 *Suppose a dynamical system of the form:*

$$x_t = x_{t-1} + \gamma\left(t\right) \mathcal{Q}\left(t; x_{t-1}; \varphi_t\right) \tag{7.4.13}$$

$$\varphi_t = g\left(x_{t-1}; \varphi_{t-1}; e_t\right) \tag{7.4.14}$$

Suppose further that the following assumptions hold:

L1. $||g\left(\varphi, x, e\right)|| < C \qquad \forall \;\; x \in D_r, \varphi, e$

L2. The functions $\mathcal{Q}$ *and* g *are continuously differentiable with respect to* x *and* φ*.*

L3. The scaler gain sequence, $\gamma\left(t\right)$ *satisfies the following properties.*

$$\lim_{t\to\infty} \gamma\left(t\right) = 0 \tag{7.4.15}$$

$$\lim_{t\to\infty}\sum_{s=1}^{t}\gamma(s)=\infty \tag{7.4.16}$$

$$\lim_{t\to\infty}\sum_{s=1}^{t}\gamma(s)^{P}<\infty \textit{ for some } P>0 \tag{7.4.17}$$

$$\lim_{t\to\infty}\sup\left(\gamma^{-1}(t)-\gamma^{-1}(t-1)\right)<\infty \tag{7.4.18}$$

$$\gamma(t) \textit{ is a decreasing sequence} \tag{7.4.19}$$

L4. Let: $\bar{\varphi}_t(\bar{x})=g(\bar{\varphi}_{t-1}(\bar{x}),\bar{x},e_t)$ $\quad\bar{\varphi}_0=\varphi_0$ *Then lemma (D) holds.*

L5. There exists a set $D_s^k\supset x^k$ *such that:*

$$||\bar{\varphi}_{1t}(\bar{x})-\bar{\varphi}_{2t}(\bar{x})||<C(\bar{\varphi}_{1s}(\bar{x}),\bar{\varphi}_{2s}(\bar{x}))\lambda^{t-s}(\bar{x}) \qquad \textit{where } t>s \textit{ and } \lambda(\bar{x})<1$$

L6. The following limit exists for $\bar{x}\in D_r$ *(the expectation is over* e_t*):*

$$f(\bar{x})\equiv\lim_{t\to\infty}EQ(t;\bar{x};\bar{\varphi}_t(\bar{x}))$$

L7. e_t *is a sequence of independent random variables.*

L8. The differential equation:

$$\frac{dx}{dt}=f(x)$$

has an invariant set D_c^k *with domain of attraction* $D_A^k\subset D_c^k$*.*

Then $x_t\to D_c^k$ *as* $t\to\infty$*.*

First note that the dynamical system is equivalent to the one used in theorem (4) when perturbed by an independent, mean zero, variance zero random variable e_t, according to lemmas (A) and (B). Thus we need only verify the assumptions of Ljung's theorem.

The assumption (L1) holds trivially for e_t. Since the closure of D_r is compact and g is continuous, the first assumption holds for $\bar{x}$ as well. Lemma (C) implies that, for φ_0 sufficiently close to φ^k, there exists a finite set D_z such that $\varphi_t \in D_z\ \forall t$.[4] Thus the first assumption holds for φ as well.

Given assumption (A1) ϕ is locally continuously differentiable and invertible, the implicit function theorem implies that Φ and therefore g is locally continuously differentiable with respect to φ_{t-1}. The implicit function theorem also implies that g is continuously differentiable with respect to β and therefore x. By inspection, Q is continuously differentiable for $P \geq 0$ and $R_0 > 0$.

Given the assumptions of the theorem (A1)...(A6), lemma (D) holds.

Clearly $\gamma(t) = 1/t$ is a geometric sequence which satisfies the properties required for the theorem. Note that one might assume that the agent discounts very old observations in which case these assumptions might not be trivial.

The set D_s^k is the region of asymptotic stability of g and exists by lemma (C).

The limit $f(\bar{x})$ can be calculated:

$$f(\bar{x}) \equiv \lim_{t\to\infty} EQ\left(t;\bar{x};\bar{\varphi}_t(\bar{x})\right) = \lim_{t\to\infty} Q\left(t;\bar{x};\bar{\varphi}_t(\bar{x})\right)$$

$$= \begin{bmatrix} \bar{R}^{-1}\bar{d}\left(\bar{P} - \bar{d}'\bar{\beta}\right) \\ \operatorname{col}\left(\bar{d}\bar{d}'\right) - \operatorname{col}\left(\bar{R}\right) \end{bmatrix} \tag{7.4.20}$$

Here $\bar{P}$ and $\bar{d}$ are the corresponding elements of:

$$\lim_{t\to\infty} \bar{\varphi}_t(\bar{x}) \tag{7.4.21}$$

[4]The analysis is the same as in Woodford (1990).

Note that as in Grandmont (1985) and Guesnerie and Woodford (1989), for φ_0 sufficiently close to φ_k, we have that:

$$\lim_{t\to\infty} \bar{\varphi}_t \left(x^k\right) = \varphi^k \tag{7.4.22}$$

Equation (7.4.20) is finite by lemma (C), assuming R is non-singular. Ljung (1975) discusses problems that arise when R is converging to a singular matrix, as must be the case here. He shows that with a small perturbation, R^{-1} is bounded and the theory applies. The last assumption is trivially satisfied in the e_t perturbation. Thus Ljung's theorem holds and convergence occurs if the differential equation (L8) is stable. Ljung (1975) shows that the stability of the differential equation is determined by the stability of the first element of equation (7.4.20):

$$\frac{d\beta}{dt} \equiv f_1(\beta) = \bar{R}^{-1}\bar{d}\left(\bar{P}(\beta) - \bar{d}'\beta\right) \tag{7.4.23}$$

To evaluate the stability of equation (7.4.23), the standard technique is used. Linearizing f_1 around the cyclic point gives:

$$\left.\frac{df_1}{d\beta}\right|_{\beta=\beta^k} = \bar{R}^{-1}d^k\left(\frac{dP}{d\beta} - d^{k'}\right) \tag{7.4.24}$$

$$\left.\frac{dP}{d\beta}\right|_{\beta=\beta^k} = \phi_k d^{k'} \quad k > 1 \tag{7.4.25}$$

$$\left.\frac{dP}{d\beta}\right|_{\beta=\beta^*} = \frac{\phi_1}{1-\phi_1\beta_1^*}d^{*'} \quad k = 1 \tag{7.4.26}$$

Here we have used equation (7.4.22), the implicit function theorem, and that $\beta^{1k} = 0$ for $k > 1$. Simplifying equation (7.4.24) gives:

$$\left.\frac{df_1}{d\beta}\right|_{\beta=\beta^k} = (\phi_k - 1) I \tag{7.4.27}$$

And similarly for the stationary state. Thus the differential equation is stable if (7.4.27) has eigenvalues of modulus less than zero. Since (7.4.27) is a multiple of the identity matrix, all eigenvalues are equal to $\phi_k - 1$. Thus we have stability if $\phi_k < 1$. but this is assumed to be the case. For the stationary state we have stability for:

$$\phi_1 < \frac{1}{1 + \beta_1^*} \tag{7.4.28}$$

Equation (7.4.28) holds by assumption. Thus the differential equation is stable. Thus $\beta_t \to \beta_k$ by Ljung's theorem. Q.E.D.

7.5. Proof of theorem 1.2

Given the assumptions of the theorem, following the steps of Singer's theorem, we find that every stable cycle of ϕ^{-1} has a critical point attracted to it. Given that there are at most $n-2$ critical points of ϕ^{-1}, there are at most $n-2$ forward stable k-state cycles. Let $k_1 \ldots k_{n'}$ index the unstable cycles. Then $n' \geq 2$ and:

$$\left|\phi_{k_i}^{-1}\right| > 1 \ \forall i \in n' \tag{7.5.1}$$

Since:

$$\phi^{-1\prime}(x) = \frac{1}{\phi'(\phi(x))} \tag{7.5.2}$$

Combining these two and using definition (2) gives:

$$|\phi_{k_i}| < 1 \ \forall i \in n' \tag{7.5.3}$$

Q.E.D.

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