

Liquidity and Fire Sales*

David L. Kelly
University of Miami

Stephen F. LeRoy
University of California, Santa Barbara

June 28, 2004

Abstract

A “fire sale” occurs when the owner of a good offers it for sale at a price strictly below the price that some buyers would willingly pay for the good. He does so because the advantage of the quick sale made possible by the lower price outweighs the higher price that other potential buyers would pay, given the likely delay in locating these buyers in the latter case. Fire sales can occur only in illiquid markets. This paper generalizes earlier treatments of illiquid markets by assuming that the asset can be offered for sale at any time, rather than only after its owner loses his capacity to operate it profitably. Also, it specifies that profitability follows a random walk.

In a common usage, a “fire sale” occurs when the owner of a capital (or consumer) good offers it for sale at a price strictly below the price that some buyers would willingly pay for the good. He does so because the advantage of the quick sale made possible by the lower price outweighs the higher price that other potential buyers would pay, given the likely delay in locating these buyers in the latter case. As this description makes clear, fire sales can occur only in illiquid markets. In liquid markets, by definition, a seller can count on locating buyers with the highest valuation virtually immediately, so he or she has no motivation to sell at a price lower than such buyers would pay.

Two earlier papers (John Krainer and Stephen F. LeRoy (2002) and David L. Kelly and LeRoy (2003)) analyzed the valuation of illiquid assets in a search-and-matching setting. In these models potential buyers with different valuations of the asset that is offered for sale arrive one per period. The seller must either sell to the current buyer or incur the costly delay of waiting for a buyer with a higher valuation. These models produce equilibria in which capital assets are sold at fire sale prices as defined above.

However, the authors of the cited papers imposed a highly stylized and unrealistic characterization of the environment. First, it was assumed in both papers that

*We have received helpful comments from our discussant, Anil Kashyap.

owners of capital assets have a “match” that governs the operating profit from the asset (when a match is broken, the owner permanently loses the ability to operate the factory profitably). The realization of a Markov chain determines whether the buyer’s match continues to the next period. The seller was assumed to be able to offer the asset for sale after losing his match, and only then. This assumption is the analogue of the corresponding specification in models of labor markets that workers can search for new jobs only when they are unemployed. The assumption is acceptable in a job search setting since searching for a new job can plausibly be assumed to take more time than is available to an already-employed worker. However, the corresponding assumption in the context of capital markets is unacceptable, except as a crude approximation: it is hardly reasonable to require that a corporation cannot simultaneously operate a capital asset and search for a buyer for that asset.

Second, the cited papers assumed that the operating profit generated by a capital asset is constant as long as the owner’s match continues. As a consequence of this specification the models miss the important point that the extent of fire sale pricing depends on operating profit: the lower the profit, the deeper the price discount.

The model presented below remedies both of these defects: first, owners of the capital asset can offer the asset for sale at all dates, not just after losing their match. In equilibrium they do so whenever the probability of successful sale is nonzero (which is the case only when at least some potential new owners can operate the asset more profitably than the current owner). Second, operating profit is assumed to follow a random walk, rather than being constant as long as the match continues. This specification makes it possible to show how the fire sale discount depends on operating profit and the chance of recovery. The analysis consists of examining an extended example: generalization is obviously possible, but little additional insight would be gained by moving to a more abstract setting.

We begin with introductory discussion of liquidity.

1 Liquidity: An Introduction

Liquidity is one of those terms that is extremely widely used despite (or perhaps because of) the fact that it does not have a single clear meaning. Until recently it would have been fair to say that the term has no clear meaning. However, in the last several years a number of papers have appeared that use the term in a way that is precise, but is still related to earlier and less formal discussions.

It is becoming increasingly clear that no single formal model can capture all of the various meanings of the term “liquidity”. For example, a firm or financial institution may have access to positive net present value investments, but may be unable to obtain financial backing to implement these projects. Such a firm is said to be solvent, because of the availability of profitable investment projects, but illiquid, because it cannot finance them. Most such discussions do not provide a clear explanation of why financing is not forthcoming despite the availability of profitable investment

projects. However, Bengt Holmstrom and Jean Tirole (1998) provided a model that has exactly this property. In their model firms have access to an investment project that is risky, but has positive net present value. Their firms have cash, but they also borrow from outside investors so as to implement their investment projects on a larger scale. After the investment decision has been made the firm will face a random liquidity shock, resulting in abandonment of the project unless new cash is supplied. Further, firm management has the option to manage diligently, resulting in a high probability of success, or shirk, resulting in a low probability of success. Shirking generates a side payment (perks) to management. The assumed parameter values are such that in equilibrium firms manage efficiently, and the equilibrium financial contract involves payments to managers to induce them not to shirk.

Combining liquidity shocks with the possibility of shirking on the part of management results in a surprisingly complex equilibrium. *Ex ante*, the fact that management cannot precommit to managing diligently results in misallocation of capital (positive net present value projects are rejected), reducing the surplus that management would otherwise be able to appropriate. *Ex post*, under intermediate values of the liquidity shock, firms would like to continue the investment project, because it has positive net present value, but they are cash-constrained. They have exhausted their cash, and cannot raise cash from outside investors because the side payment required to induce diligent management renders the project unprofitable to the outside investors. Therefore the firm abandons the project. Holmstrom-Tirole's model is seen to imply a fully explicit account of why firms might be solvent but illiquid.

As Holmstrom-Tirole noted, their model does not give a complete account of the term "liquidity": the term has other connotations that take the analysis in a different direction. In some discussions a market is said to be liquid to the extent that sellers can count on quickly locating buyers who can pay a high price. This was the sense in which the term was used in the introduction. This consideration points toward the literature on search and matching. The paper of Krainer-LeRoy cited above contains a model in which endogenous variables widely associated with liquidity, such as equilibrium expected time to sale and the size of the equilibrium discount for immediate sale, are related to parameters representing search costs. The effects were as expected: the higher the liquidity of the market, the shorter the expected time to sale and the lower the equilibrium discount for immediate sale. In Kelly-LeRoy an equilibrium is derived in which investors purchase illiquid assets using defaultable debt. It turns out that the availability of the default option implies that preexisting debt is not a sunk cost, as it would be if the debt were nondefaultable. The equilibrium strategy for selling the illiquid asset depends on the level of preexisting debt: the higher the debt, the more valuable is the option to default, and therefore the higher is the selling price.

These papers, although also dealing with liquidity, are unrelated to Holmstrom-Tirole; for example, they do not address the possibility of the joint occurrence of solvency and illiquidity in the sense discussed above. The conclusion is that expecting

any single model of liquidity to address all the meanings that the term has acquired is unrealistic. The model to be presented in this paper views liquidity in the same manner as its predecessors; it does not broaden the field to include considerations such as those analyzed by Holmstrom-Tirole.

2 Perfect Liquidity

The best way to begin developing the connection between illiquidity and fire sale pricing is to analyze a version of our model in which the market for the capital good is perfectly liquid, meaning that the seller of the capital good can immediately locate a buyer with the highest possible valuation.

We assume that there exists a single capital good, a factory, and that the profit accruing to the owner of the factory is a random walk with binomial innovations:

$$x_{t+1} = x_t + \tilde{\varepsilon}_{t+1}, \quad (1)$$

where $\tilde{\varepsilon}_{t+1} = \pm\varepsilon$ with equal probability. At each date t the owner of the capital good will offer it for sale to one potential buyer. The seller sets the price $p(x_t)$, reflecting the presumption that the current owner's profit rate will affect the optimal sale price. Each potential buyer of the factory can generate initial operating profit y that is uniformly distributed between 0 and 1. Potential buyers for whom initial profit equals 1 obviously have the highest valuation of the factory, and in liquid markets sellers assume that they have immediate access to buyers with this valuation. Therefore potential buyers with lower initial profit are irrelevant to the equilibrium (contrary to the case of illiquid markets, as we will see below). If the profit x_t accruing to the current owner equals or exceeds 1, the value of the factory to the current owner equals or exceeds its value to any potential buyer, so the current owner will not offer it for sale (or, equivalently, will offer it for sale at such a high price that the probability of sale is zero). Accordingly, in that case the valuation function $q(x)$ obeys the difference equation

$$q(x) = \beta \left(x + \frac{q(x + \varepsilon) + q(x - \varepsilon)}{2} \right) \quad (2)$$

for $x = 1 + \varepsilon, 1 + 2\varepsilon, \dots$, where all agents are risk neutral and have discount rate β . When $x = 1$ the owner of the factory knows that if the realization of $\tilde{\varepsilon}$ equals $-\varepsilon$ he will attach a lower valuation to the factory than a potential buyer would, since he would generate profit of $1 - \varepsilon$, compared with 1 for potential buyers. In that case he will sell the factory for price $p(1)$. Since immediate sale is assumed, we have $q(1) = p(1)$. Thus the boundary condition for the difference equation (2) is

$$q(1) = \beta \left(1 + \frac{q(1 + \varepsilon) + q(1)}{2} \right). \quad (3)$$

The difference equation (2) with boundary condition (3) has solution

$$q(x) = \frac{\beta x}{1 - \beta} + \left(\frac{\beta}{1 - \beta} \right) \left(\frac{\beta \varepsilon}{1 - \beta + \sqrt{1 - \beta^2}} \right) b_1^{\frac{x-1}{\varepsilon}}, \quad (4)$$

where

$$b_1 = \frac{1 - \sqrt{1 - \beta^2}}{\beta} \quad (5)$$

(see Appendix A). Since $0 < b_1 < 1$, the value $q(x)$ of the factory converges to $\beta x / (1 - \beta)$ as $x \rightarrow \infty$. Figure 1 graphs the value of the factory for $\beta = 0.9$ and $\varepsilon = 0.1$. The vertical distance between $q(x)$ and $\beta x / (1 - \beta)$ measures the value to the owner of the factory of the option to sell it for price $q(1)$. This option, of course, will be exercised when $x = 1$ and $\tilde{\varepsilon} = -\varepsilon$, and only then.

The value of the option is highest for low values of x , as the diagram indicates. The convergence of $q(x)$ to $\beta x / (1 - \beta)$ reflects the fact that the value of the option to sell for price $p(1)$ when $x = 1 - \varepsilon$ has negligible value when x is high. This is so because even though x will eventually drop to $1 - \varepsilon$ with probability 1, this event is likely to occur only in the very distant future. It contributes little to the current value of the factory, due to discounting. Profitability x cannot fall below $1 - \varepsilon$ because the realization $\tilde{\varepsilon} = -\varepsilon$ results in immediate sale when $x = 1$.

There also exists a continuum of solution paths that do not converge to $\beta x / (1 - \beta)$ (these paths occur when the second root b_2 of a characteristic equation has nonzero coefficient; see Appendix A). Along the nonconvergent paths the factory has a positive or negative bubble. Following precedent, we exclude these bubble paths.

This example makes clear that in liquid markets there are no fire sales: for $x > 1$ no price exists that is mutually agreeable both to the current owner and any potential owner, while in the case $x = 1 - \varepsilon$ the owner will sell for price $q(1)$, the maximum price that buyers are willing to pay.

3 Illiquid Markets

In illiquid markets the owner of the factory cannot count on locating a buyer with maximal operating profit immediately. Therefore potential buyers with initial profit less than 1 are relevant to the equilibrium, in contrast to the case of perfect liquidity. Following Krainer and LeRoy (2002) and Kelly and LeRoy (2003), we assume that each period one and only one potential buyer is available. Buyers know the realization of their initial profit rates, whereas the seller knows only the distribution of initial profit rates. If the potential buyer purchases the factory, the profit rate will evolve from the buyer's initial profit level according to (1), with y replacing x_t , while if he does not, the would-be seller will continue to operate the factory, with profit evolving from x_t according to (1).

The seller posts a take-it-or-leave-it offer, which the buyer will accept if the profit he can generate operating the factory is high enough to justify purchase. The seller will only offer the factory for sale at a price strictly higher than the valuation justified by his own profit rate, since there would be no point in selling otherwise. Consequently, as before, when $x > 1$ no potential buyer can generate as much profit as the current owner of the factory, so the factory will not be sold. When $x < 1$ there exist potential buyers whose initial profit exceeds that of the current owner, implying the possibility of sale. The owner will offer the factory for sale at a price that balances the probability of sale, which depends on the sale price, against the potential capital gain generated as the difference between the price and the seller's valuation. One expects that the lower the value of x_t , the greater the fire sale discount, and this turns out to be the case in equilibrium.

A set of values of x , bounded above by 1, exists such that for those values of x the factory will be offered for sale at a price such that the probability of sale is strictly between 0 and 1. Any of these values of x can occur in equilibrium if earlier attempts to sell fail because of the low initial profits of potential buyers, and if the seller's profit innovations are negative. However, if profitability is negative and sufficiently high in absolute value—denote this level \underline{x} —the owner will set a price such that the factory sells with probability 1 at that level of profitability. The critical level of profitability \underline{x} satisfies

$$q(0) = p(\underline{x}), \quad (6)$$

so that even a buyer with $y = 0$ will break even if he purchases the factory. Since potential buyers' initial profitability is bounded by 0 and 1, it follows that \underline{x} constitutes a lower bound on the profit rates that can occur in equilibrium.

In an illiquid market the seller's problem, to determine the price at which to offer the factory for sale, is nontrivial. To solve it, let $q(x)$ be the value to the seller of the factory when profit equals x . Because the seller chooses sale prices to maximize the value of the factory, $q(x)$ obeys

$$q(x) = \beta x + \max_{p(x+\varepsilon), p(x-\varepsilon)} \frac{\beta}{2} (\lambda(x+\varepsilon) + \lambda(x-\varepsilon)), \quad (7)$$

where $\lambda(x+\varepsilon)$ and $\lambda(x-\varepsilon)$ are the values of the factory conditional on the high and low profit innovations, respectively, given by

$$\lambda(x) = \mu(p(x))p(x) + (1 - \mu(p(x)))q(x). \quad (8)$$

As in the case of liquid markets, the convention on notation presumes that the seller learns the realization of $\tilde{\varepsilon}$ before setting the sale price. The sale price is $p(x+\varepsilon)$ or $p(x-\varepsilon)$ and the probability of sale as a function of the seller's price is $\mu(p(x+\varepsilon))$ or $\mu(p(x-\varepsilon))$. The function $\mu(p)$ is determined by the solving the buyer's problem, and is taken as given by the seller.

The probability of sale is determined by the sale price and the distribution of the random variable \tilde{y} denoting the buyer's initial profit. The probability of sale equals the probability that the buyer's initial profit exceeds a reservation profit rate y^* which is such that the value of the factory to the buyer just equals its sale price. If \tilde{y} is distributed uniformly on $[0, 1]$, we have:

$$\mu(p(x)) = 1 - y^*(p(x)), \quad (9)$$

where y^* satisfies

$$p(x) = q(y^*). \quad (10)$$

We computed the equilibrium, again taking $\beta = 0.9$ and $\varepsilon = 0.1$. An outline of the solution algorithm is provided in Appendix B. Figure 2 shows equilibrium values of $q(x)$ plotted against the upper asymptote $\beta x / (1 - \beta)$ and the lower limit $\beta(x + q(0))$, which is reached when $x = \underline{x}$. Figure 3 plots $q(x)$ against $p(x)$ and $\mu(x)$. As expected, the value of the factory is an increasing function of profitability. For very high values of x we have that $q(x)$ approaches $\beta x / (1 - \beta)$, again reflecting the facts that the option to sell has negligible value for high values of x , and that the solution algorithm excluded equilibria with bubbles. For values of x slightly above 1, $q(x)$ exceeds $\beta x / (1 - \beta)$ by nonnegligible amounts; the reason here is the same as in the preceding section. For $x < 1$, sale of the factory becomes possible. The seller offers it for sale at a price approximately halfway between $q(x)$ and $q(1)$ (when x is near 1), since doing so maximizes the expected profits from sale (a similar phenomenon was noted by Krainer-LeRoy (Krainer and LeRoy, 2002)).

When profitability declines below 1, the sale price is reduced below $q(1) = p(1) = 9.503$. The difference between $p(1)$ and $p(x)$ represents a fire sale discount. For $x = 0$ the seller is aware that all potential buyers can earn higher profit than he can. However, as a monopolistic seller he sets a sale price that will prevent sale to buyers with low (but positive) y so as to exploit potential buyers with high y . Also, the seller makes appropriate allowance for the fact that potential buyers are aware that they too will have the option to offer the factory for sale at a higher price, so even a buyer with zero profitability would assign positive value to the factory based on the fact that he could offer it for sale in the next period.

When profitability declines to $\underline{x} = -1.340$ the seller prices the factory to sell with probability 1. This event, the ultimate fire sale, occurs at a sale price of 5.790. The seller is aware that at sale prices higher than 5.790, potential buyers with zero or very low profitability will be deterred from purchasing the factory, implying that the high rate of losses may continue. Correspondingly, there is no reason to offer the factory for sale for a price lower than 5.790 because even buyers with initial profitability of zero will purchase it at that price.

One advantage of modeling profitability as a random walk is that it allows an easy analysis of the effect of the variability of profits on the value of the factory. Figure

4 displays $q(x)$ for various values of ε . As is clear, for fixed x , $q(x)$ increases with ε . This occurs as a consequence of the convexity of $q(x)$: when ε is high the option to sell is more valuable than when ε is low. The same phenomenon—that the price of an option increases with the variance of the price changes in the underlying security—is familiar from options theory.

4 Parametrizing Illiquidity

The next exercise is to investigate the effect of specifying more or less illiquidity on the equilibrium pricing strategy. Since we are maintaining the assumption that one and only one prospective buyer arrives per period, the natural way to alter liquidity is to vary the length of the period (here we follow Krainer and LeRoy (2002)). We expect that when the period is short, so that buyers arrive rapidly, the seller will set a high price. In that case sale in any period occurs with low probability. This is acceptable to the seller because if the current buyer declines to buy, a new prospective buyer will be along shortly. Krainer-LeRoy’s finding that the expected time to sale will be low in highly liquid markets also occurs here. In contrast, in illiquid markets sellers who operate the factory unprofitably will set a lower price than they would if liquidity were higher, since failure to sell to the current buyer implies that the current low rate of profit will persist for a relatively long time.

To parametrize illiquidity, we refer to the unit length of time as the year. In the preceding section one buyer arrived per year. Now we assume instead that n buyers arrive per year, so that the next buyer will arrive $1/n$ years after the current buyer. This necessitates several changes in (7). First, the innovations in profitability are $\pm\varepsilon\sqrt{1/n}$ rather than $\pm\varepsilon$, since that assumption preserves the variance of the annual innovation in profitability at ε^2 . Second, we replace the discount factor β by $\beta^{1/n}$ so that the annual discount factor remains equal to β . Third, since we continue to measure profit x at annual rates, it is necessary to divide x by n . Thus if we define $q(x, n)$ as the value of a factory with annual profitability x when the length of the period is n , we have

$$q(x, n) = \beta^{1/n} \frac{x}{n} + \frac{\beta^{1/n}}{2} \max \left\{ \lambda(x + \varepsilon\sqrt{1/n}, n) + \lambda(x - \varepsilon\sqrt{1/n}, n) \right\}, \quad (11)$$

where $\lambda(x, n)$ is given by

$$\lambda(x, n) = \mu(p(x, n))p(x, n) + (1 - \mu(p(x, n)))q(x, n). \quad (12)$$

Here the maximum is taken over all possible values of $p(x + \varepsilon\sqrt{1/n})$ and $p(x - \varepsilon\sqrt{1/n})$.

Figure 5 shows $q(x, n)$ for various values of n . The plots are as expected: for any value of x the factory is worth more in liquid markets than illiquid markets. This makes sense: the factory can be sold more quickly and for a higher price in liquid

markets, so for fixed x it is worth more in liquid markets. For any x the value of the factory less the value of the sale option is $\beta^{1/n}x/[n(1 - \beta^{1/n})]$. This value increases with n , approaching an upper asymptote of $x/\ln(1/\beta)$, as is readily verified using L'Hôpital's rule. This expression is just the value of a perpetuity with flow dividend x , as is evident from the observation that the discount rate ρ is related to the discount factor β by $\beta = e^{-\rho}$.

The level of profitability at which sale occurs with probability 1, $\underline{x}(n)$, approaches 1 as n goes to infinity. The reason is that in highly liquid markets even sellers who can operate the factory with fairly high profit (but below 1) are able to sell it quickly to prospective buyers with still higher profitability. In the limit equilibrium profitability will therefore decline below 1 with probability zero. Finally, we derive in Appendix C the value of the factory in a perfectly liquid market, $q(x, \infty)$, for general x . The result is analogous to Section 2, in that the value of the factory is composed of a term representing profitability and a term representing the value of the option to sell.¹

A fire sale occurs when the factory is sold at a price strictly lower than a prospective buyer who can operate the factory with maximal profitability would be willing to pay. Expressed as a fraction, the discount would be defined by $(p(1, n) - p(x, n))/p(1, n)$. Alternatively, the discount could be defined relative to the price that a buyer with profitability 1 would be willing to pay if markets were perfectly liquid: $(p(1, \infty) - p(x, n))/p(1, \infty)$. There does not seem to be any reason to prefer one of these definitions to the other.

Note that under either definition, the fire sale discount is zero in the case of perfect liquidity.

5 Empirical Evidence

The model just presented generates several empirical implications. First, the prices at which illiquid assets are sold depend on how profitably these assets are operated by their current owners. This is so because the owners of these assets are aware that they may not succeed in selling these assets immediately due to their illiquid nature. Therefore their pricing strategies balance prospective sale proceeds against the operating profit the current owners can expect if they fail to sell. In contrast, if, contrary to the assumption of the present model, the seller knew that there exist

¹The exact representation of perfectly liquid markets in the model of this section differs from that of Section 2. Here perfect liquidity occurs only in the limit, when time is continuous and profitability follows a Wiener process. In Section 2 we maintained the discrete time setting and assumed perfect liquidity directly. The model of Section 2 has the advantage that it is easier to understand, whereas the model of this section has a more explicit justification as a limit of the model for illiquid markets.

Most properties of the equilibrium are the same in the two cases. For example, both specifications imply that in equilibrium profitability falls below 1 with zero probability.

many potential buyers with higher valuations than he has, as in liquid markets, he could sell the assets using an auction with no reservation price. In that case the sale price would not depend on the seller's valuation. Second, an easy extension of the model presented here would show that if the seller has information about different buyers' valuations, the model predicts that that information will be reflected in sale prices. This is so because the seller's guess about the buyer's valuation affects the probability of sale.

Todd C. Pulvino's recent study (Pulvino, 1998) of sales of used aircraft by airline gave strong support to these predictions, although Pulvino interpreted his results differently than we will here. Pulvino defined fire sales as liquidations at prices below fundamental value. The presumption apparently was that fundamental value is the same for all agents.² Such sales occur because of the "financial distress" of the selling firm, a concept that we did not make use of in the present paper. Pulvino adopted the analysis of liquidation developed by Andrei Shleifer and Robert Vishny (1992). Shleifer-Vishny proposed that fire sales will occur when most firms in an industry are cash-constrained, and therefore are willing to sell industry-specific assets to outsiders for less than their fundamental value. Pulvino found that variables purportedly measuring the extent to which airlines were cash-constrained (such as debt ratios) were correlated with the prices of used aircraft: financially constrained airlines received lower prices than those which were not financially constrained. Further, during recessions airlines are more likely to sell aircraft to outsiders, such as financial institutions than during periods of prosperity, and when they do so they receive lower prices.

Pulvino interpreted these findings as supporting the Shleifer-Vishny account of fire sales. However, his results can also be interpreted from the vantage of the model developed here, in which firms are never financially distressed in the sense of Shleifer and Vishny. Variables that Pulvino interpreted as measuring financial distress are surely highly correlated with profitability: why would an airline's balance sheet deteriorate if it is able to operate its aircraft profitably? If balance sheet variables are interpreted as measuring profitability, the model of this paper predicts exactly the negative correlation between indebtedness and sale prices that Pulvino found.

Pulvino's finding that sales of aircraft to financial institutions occurred at low prices is also as expected. When the buyer of used aircraft is an airline, the seller reasonably presumes that the buyer can make profitable use of the aircraft being sold, and sets their prices accordingly. On the other hand, when the buyer is a financial institution which is planning on mothballing the aircraft until the market turns around, the seller knows that the buyer's decision is based entirely on projected future resale value, resulting in lower prices. The model of this paper could readily be extended to cover this case by specifying that some prospective buyers can credibly precommit to some fixed profitability rate, such as zero, that is below the average of

²In contrast, in the model of this paper successful sale occurs at a price that is below the fundamental value of the buyer but above that of the seller, so that both parties gain from the sale.

other buyers' profitability. Depending on the sellers' profitability, sellers might be willing to sell to such buyers at lower prices.

6 Conclusion

The results of this paper show that fire sale discounts occur in illiquid markets because the seller, who is unable to operate the factory as profitably as other owners could, wishes to locate a buyer more quickly than would be likely to occur in the absence of a discount. In illiquid markets fire sales may involve a sizeable discount, whereas in liquid markets they involve only a small discount, since a small discount is sufficient to ensure quick sale.

Appendix A: Solution to Model of Section 2

The evolution of the value of the factory under perfect liquidity is described by

$$q(x) = \beta x + \frac{\beta}{2}q(x + \varepsilon) + \frac{\beta}{2}q(x - \varepsilon). \quad (\text{A.1})$$

Let $t = x/\varepsilon$. Then (A.1) becomes

$$q(\varepsilon t) = \beta \varepsilon t + \frac{\beta}{2}q(\varepsilon(t + 1)) + \frac{\beta}{2}q(\varepsilon(t - 1)). \quad (\text{A.2})$$

Defining $q(\varepsilon t)$ as q_t , it is seen that (A.2) is equivalent to the difference equation

$$0 = 2\varepsilon t + q_{t+1} - \frac{2}{\beta}q_t + q_{t-1}. \quad (\text{A.3})$$

We can write the boundary condition (3) for $x = 1$ (or $t = 1/\varepsilon$):

$$q_{1/\varepsilon} = \beta \left(1 + \frac{q_{1/\varepsilon+1} + q_{1/\varepsilon}}{2} \right), \quad (\text{A.4})$$

or

$$q_{1/\varepsilon+1} = \left(\frac{2}{\beta} - 1 \right) q_{1/\varepsilon} - 2. \quad (\text{A.5})$$

Equation (A.3) has complementary function

$$q_t^c = A_1 b_1^t + A_2 b_2^t, \quad (\text{A.6})$$

where $b_1 = (1 - \sqrt{1 - \beta^2})/\beta$ and $b_2 = (1 + \sqrt{1 - \beta^2})/\beta$, are the roots of the polynomial

$$0 = b^2 - \frac{2}{\beta}b + 1. \quad (\text{A.7})$$

Equation (A.3) has particular solution

$$q_t^p = \frac{\beta}{1-\beta}\varepsilon t. \quad (\text{A.8})$$

The general solution is

$$q_t = q_t^p + q_t^c = \frac{\beta}{1-\beta}\varepsilon t + A_1 b_1^t + A_2 b_2^t. \quad (\text{A.9})$$

Assuming that bubbles can be excluded, we are interested in the solution for which q_t converges to $\beta\varepsilon t/(1-\beta)$. Since $b_2 > 1$, this can only occur if $A_2 = 0$.

By setting $t = 1/\varepsilon$ and $t + 1 = 1/\varepsilon + 1$ in (A.9), we have

$$q_{1/\varepsilon} = \frac{\beta}{1-\beta} + A_1 b_1^{1/\varepsilon} \quad (\text{A.10})$$

and

$$q_{1/\varepsilon+1} = \frac{\beta(1+\varepsilon)}{1-\beta} + A_1 b_1^{1/\varepsilon+1}, \quad (\text{A.11})$$

respectively. Solving (A.10) for A_1 results in

$$A_1 = \left(q_{1/\varepsilon} - \frac{\beta}{1-\beta} \right) b_1^{-1/\varepsilon}. \quad (\text{A.12})$$

Equating the right-hand sides of (A.5) and (A.11), using (A.12) and solving for $q_{1/\varepsilon}$ gives

$$q_{1/\varepsilon} = \left(\frac{\beta}{1-\beta} \right) \left(\frac{1 - \beta(1-\varepsilon) + \sqrt{1-\beta^2}}{1-\beta + \sqrt{1-\beta^2}} \right). \quad (\text{A.13})$$

We then have:

$$A_1 = \left(\frac{\beta}{1-\beta} \right) \left(\frac{\beta\varepsilon}{1-\beta + \sqrt{1-\beta^2}} \right) b_1^{-1/\varepsilon}. \quad (\text{A.14})$$

Thus the solution evolves according to

$$q_t = \frac{\beta}{1-\beta}\varepsilon t + A_1 b_1^t, \quad (\text{A.15})$$

$$= \frac{\beta}{1-\beta}\varepsilon t + \left(\frac{\beta}{1-\beta} \right) \left(\frac{\beta\varepsilon}{1-\beta + \sqrt{1-\beta^2}} \right) b_1^{t-1/\varepsilon}. \quad (\text{A.16})$$

Setting $t = x/\varepsilon$ and $q_t = q(x)$ results in

$$q(x) = \frac{\beta x}{1-\beta} + \left(\frac{\beta}{1-\beta} \right) \left(\frac{\beta\varepsilon}{1-\beta + \sqrt{1-\beta^2}} \right) b_1^{\frac{x-1}{\varepsilon}} \quad (\text{A.17})$$

which converges to

$$q(x) = \frac{\beta x}{1 - \beta} \quad (\text{A.18})$$

as x increases since $b_1 < 1$.

Appendix B: Solution Algorithm

We first transformed the seller's decision from setting a sales price to an equivalent choice of the minimum valuation buyer who will buy the factory. Given a sales price of p^h , the buyer with the lowest valuation who will buy the factory has valuation x^h , where $q(x^h) = p^h$, and the probability of sale is $1 - x^h$. Rewriting the seller's problem (7) gives:

$$q(x) = \beta x + \frac{\beta}{2} \max_{x^h, x^l} \{ (1 - x^h)q(x^h) + x^h q(x + \varepsilon) + (1 - x^l)q(x^l) + x^l q(x - \varepsilon) \}. \quad (\text{B.1})$$

This change makes possible an analytic solution to the buyer's problem, in contrast to the formulation of Equation (7), which requires a numerical solution.

Our solution algorithm begins with the initial valuation function $q_0(x)$, given by

$$q_0(x) = \max \left\{ \frac{\beta x}{1 - \beta}, \beta x \right\}. \quad (\text{B.2})$$

Here $q_0(x)$ is a lower bound for $q(x)$ (see Fig. 2). Given $q_0(x)$, we computed the optimal offers x^h and x^l for a grid of possible values of x and substituted the solutions into the objective function to obtain $q_1(x)$:

$$q_1(x) = \beta x + \frac{\beta}{2} \max_{x^h, x^l} \{ (1 - x^h)q_0(x^h) + x^h q_0(x + \varepsilon) + (1 - x^l)q_0(x^l) + x^l q_0(x - \varepsilon) \}. \quad (\text{B.3})$$

We then approximated $q_1(x)$ with a function $\widehat{q}_1(x)$ that is differentiable and that converges to βx plus a constant for low values of x and to $\beta x / (1 - \beta)$ for high values of x . The following function works:

$$\widehat{q}_i(x) = \frac{\beta x}{1 - \beta} g(x) + (\beta x + A_i) (1 - g(x)) + g(x) (1 - g(x)) h(x; B_i) \quad (\text{B.4})$$

$$g(x) = \frac{1}{1 + \exp(-x)} \quad (\text{B.5})$$

$$h(x; B_i) = \sum_{j=0}^n B_{ij} x^j \quad (\text{B.6})$$

Here A_i is a scalar and B_i is a vector of parameters which are estimated from $q_i(x)$. We then computed

$$q_i(x) = \beta x + \max_{p(x+\varepsilon), p(x-\varepsilon)} \beta (\lambda(x; \widehat{q}_{i-1}) + \nu(x; \widehat{q}_{i-1})), \quad (\text{B.7})$$

and repeated until $\|q_i(x) - q_{i-1}(x)\|_x < 0.0001$ under the sup norm.

Appendix C: Solution to Model of Section 4

We consider two cases, $x < 1$ and $x \geq 1$. For the former case, note that as n increases, the number of buyers seen by the seller in any positive period of time approaches infinity. Thus the probability of finding in any positive period of time a buyer with profitability greater than any fixed $y < 1$ approaches 1. Therefore the seller will optimally choose strategies $x^h(n)$ and $x^l(n)$ that are arbitrarily close to 1 as $n \rightarrow \infty$, and the probability of finding a buyer in any finite period of time approaches 1. In the limit for $x < 1$ the factory can be sold in negligible time to a buyer with valuation equal to one. Hence we have

$$q(x, \infty) = q(1, \infty), \quad x < 1. \quad (\text{C.1})$$

For $x \geq 1$, the factory will not be sold. Therefore $q(x, n)$ obeys

$$q(x, n) = \beta^{1/n} \frac{x}{n} + \frac{\beta^{1/n}}{2} \left(q \left(x + \varepsilon \sqrt{1/n}, n \right) + q \left(x - \varepsilon \sqrt{1/n}, n \right) \right), \quad x > 1 \quad (\text{C.2})$$

and

$$q(1, n) \approx \beta^{1/n} \frac{1}{n} + \frac{\beta^{1/n}}{2} \left(q \left(x + \varepsilon \sqrt{1/n}, n \right) + q(1, n) \right) \quad \text{for } n \text{ large.} \quad (\text{C.3})$$

Notice that these equations take the same form as in the perfect liquidity case of Section 2, and are identical in the special case $n = 1$. Therefore we can follow the same steps as in Appendix A to get

$$q(x, n) = \frac{\beta^{1/n} x}{n(1 - \beta^{1/n})} + \frac{\beta^{1/n} x}{n(1 - \beta^{1/n})} \left(\frac{\beta^{1/n} \varepsilon \sqrt{1/n}}{1 - \beta^{1/n} + \sqrt{1 - \beta^{2/n}}} \right) b_1^{\frac{x-1}{\varepsilon \sqrt{1/n}}}, \quad (\text{C.4})$$

where b_1 is defined by

$$b_1 = \frac{1 - \sqrt{1 - \beta^{2/n}}}{\beta^{1/n}}. \quad (\text{C.5})$$

Both of these expressions agree with their counterparts in Appendix A for $n = 1$. The limit is

$$\begin{aligned} \lim_{n \rightarrow \infty} q(x, n) &= x (\log(1/\beta))^{-1} + 2^{-1/2} (\log(1/\beta))^{-3/2} \varepsilon \cdot \\ &\exp \left\{ -2^{1/2} (\log(1/\beta))^{-1} \left(\frac{x-1}{\varepsilon} \right) \right\}, \quad x \geq 1. \end{aligned} \quad (\text{C.6})$$

Appendix D: Figures

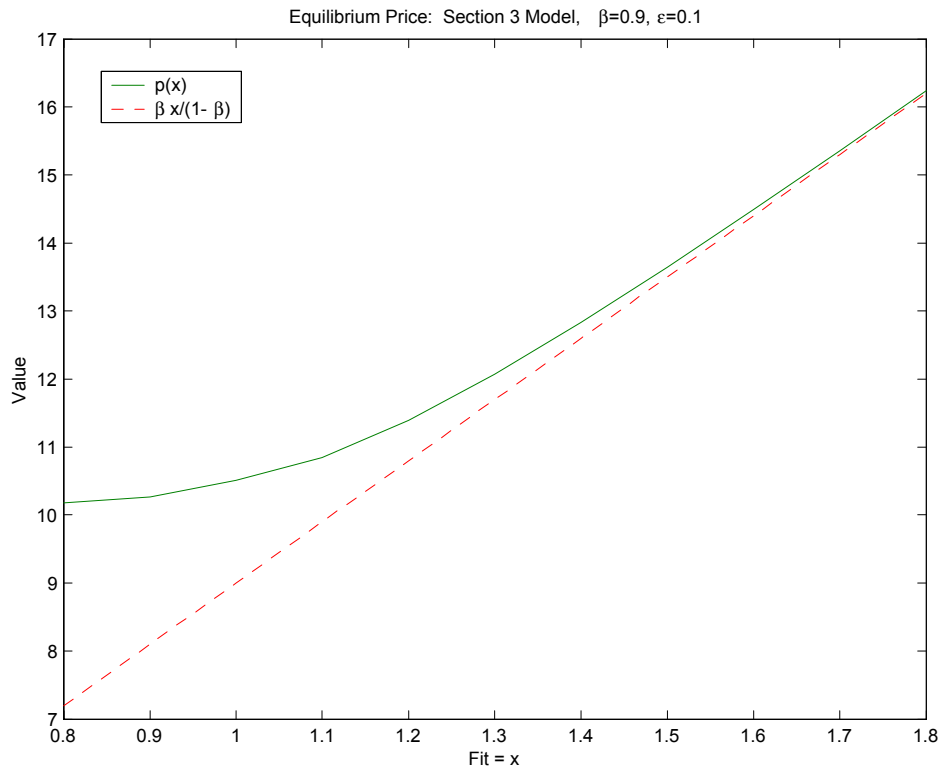


Figure 1: Equilibrium Price; Perfect Liquidity

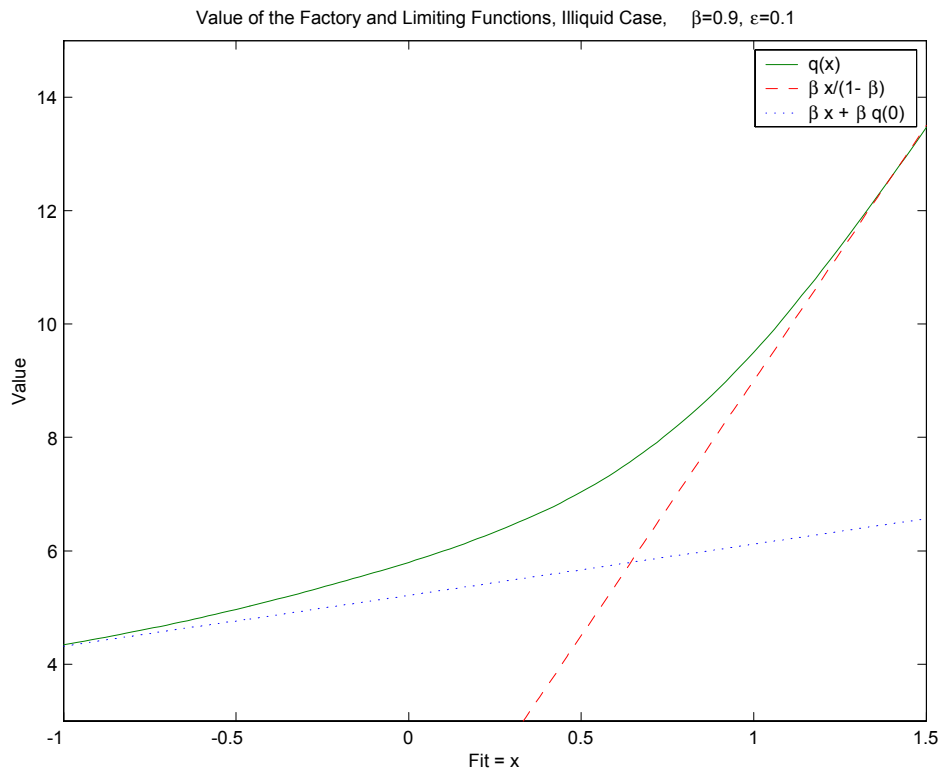


Figure 2: Equilibrium Value; Illiquidity

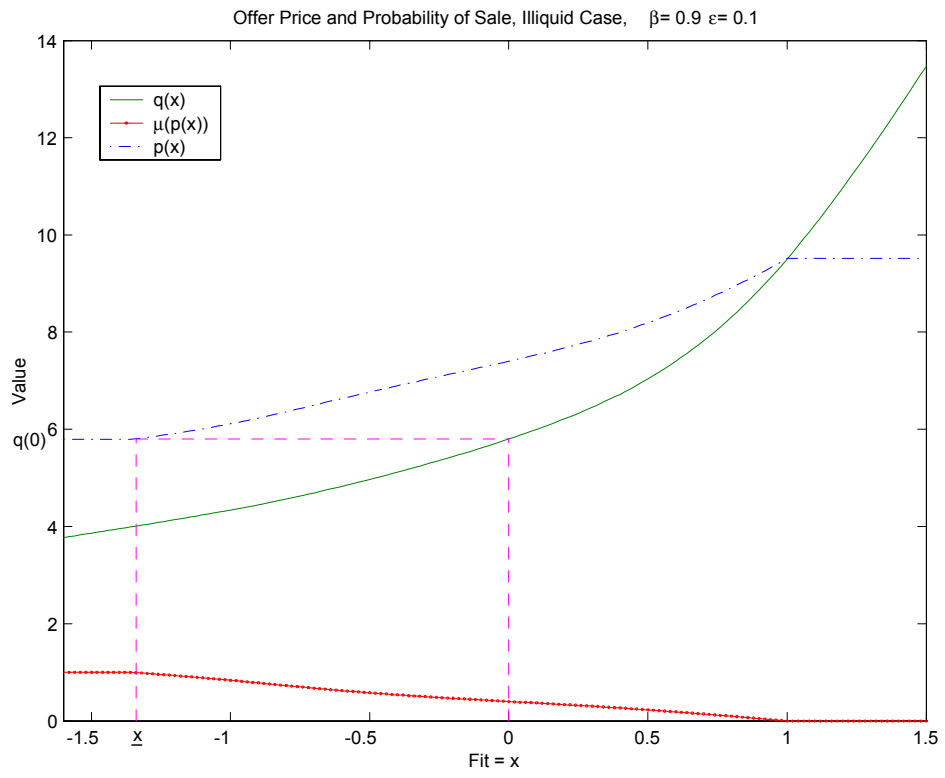


Figure 3: Value, Sale Price and Probability of Sale

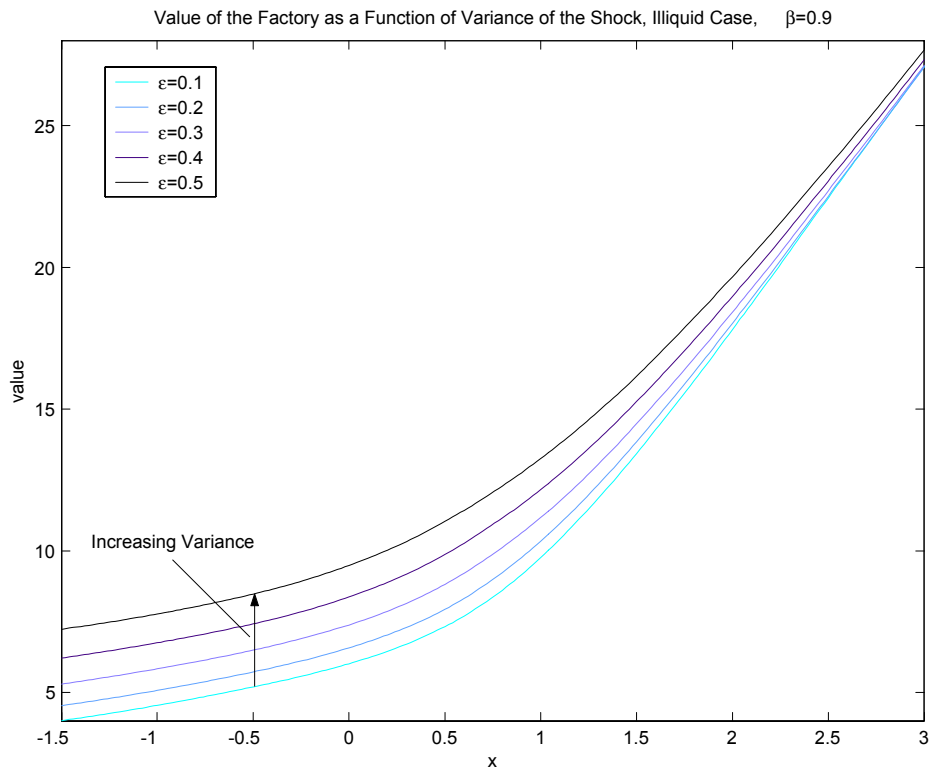


Figure 4: Equilibrium Value; Variance of Profit Innovations

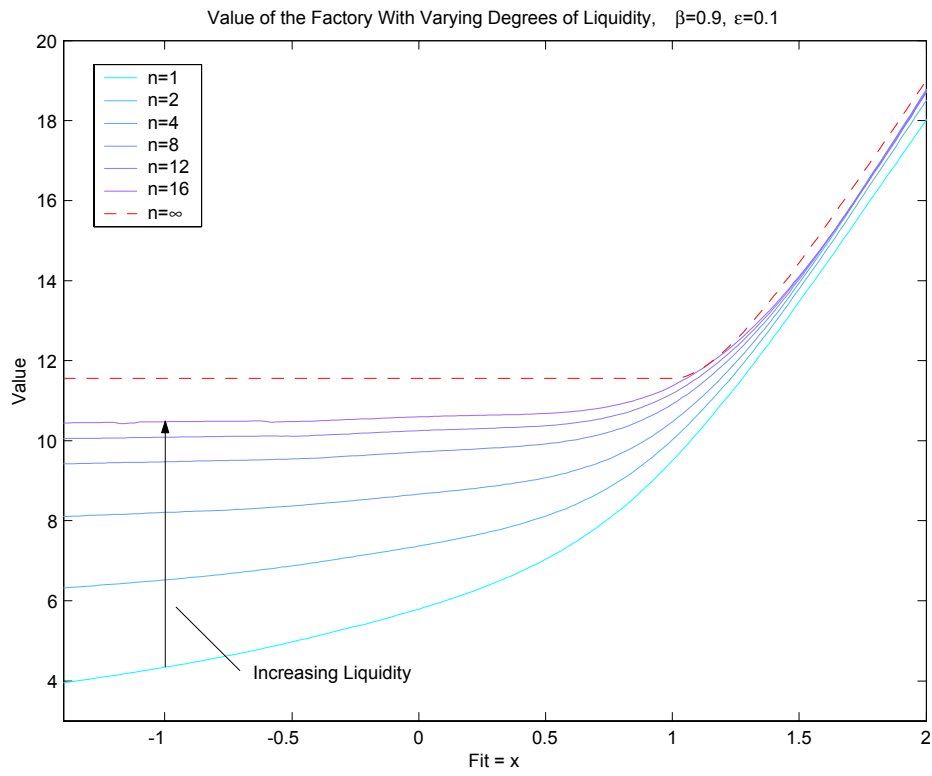


Figure 5: Equilibrium Value, varying liquidity

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