Learning and Climate Feedbacks:
Optimal Climate Insurance and Fat Tails

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Abstract

We study the effect of potentially severe climate change on optimal climate change policy, accounting for learning and uncertainty in the climate system. In particular, we test how fat upper tailed uncertainty over the temperature change from a doubling of greenhouse gases (the climate sensitivity), affects economic growth and emissions policy. In addition, we examine whether and how fast uncertainties could be diminished through Bayesian learning. Our results indicate that while overall learning is slow, the mass of the fat tail diminishes quickly, since observations near the mean provide evidence against fat tails. We denote as “tail learning” the case where the planner rejects high values of the climate sensitivity with high confidence, even though significant uncertainty remains. Fat tailed uncertainty without learning reduces current emissions by 38% relative to certainty, indicating significant climate insurance, or paying to limit emissions today to reduce the risk of very high temperature changes, is optimal. However, learning reduces climate insurance by about 50%. The optimal abatement policy is strongly influenced by the current state of knowledge, even though greenhouse gas (GHG) emissions are difficult to reverse. Non-fat tailed uncertainty is largely irrelevant for optimal emissions policy.

JEL classification: Q54; Q58; D83; D81; H43

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1 Introduction

Uncertainty is a dominant feature of climate change. Recent research highlights a particular aspect of climate change uncertainty: a relatively small chance of severe climate change exists. In particular, a doubling of greenhouse gases (GHGs) above preindustrial levels may cause a very large steady state increase in temperature.\(^1\) The sensitivity of the temperature to GHG concentrations is known as the climate sensitivity. Uncertainty about the climate sensitivity creates an insurance motive for reducing emissions, in that paying to limit GHG emissions today prevents GHG concentrations from rising, which reduces the probability of very high temperature changes.\(^2\) The prior distribution of the climate sensitivity is known to have a fat upper tail, meaning the upper tail of the distribution of temperature changes declines at a rate slower than exponential. The existence of a fat tail significantly increases the insurance value of current GHG abatement, since households are willing to pay more upfront abatement costs to eliminate fat tailed risk of severe climate change.\(^3\)

However, climate change uncertainty differs from a standard insurance problem in that learning reduces uncertainty over time. If learning resolves climate uncertainty relatively quickly, then the initial need for climate insurance is small, as the planner still has time to increase abatement if learning quickly indicates the climate sensitivity is large. However, if learning resolves climate uncertainty slowly, then the optimal policy calls for aggressive initial abatement for insurance purposes. The central questions for climate policy are then: how fast will learning resolve fat tailed uncertainty about the climate sensitivity, and what is the optimal climate policy with fat tailed climate uncertainty and learning?

The prior literature finds that learning is a slow process. Kelly and Kolstad (1999b) consider uncertainty regarding the heat capacity of the ocean. In their integrated assessment model,\(^4\) stochastic weather shocks obscure the climate change signal in the temperature

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\(^1\)Stocker, Dahe, and Plattner (2013) reviews many studies and finds values higher than 4.5°C cannot be ruled out, and values greater than 6°C can only be ruled out with “medium confidence”, although a value between 1.5 and 4.5°C is “likely”. Weitzman (2009b) averages 22 studies and finds a 5% chance that a doubling of GHGs will cause temperatures to rise more than 7°C. Other papers which estimate the current scientific uncertainty regarding the climate sensitivity include: Lemoine (2010), Newbold and Daigneault (2009), Roe and Baker (2007), Schwartz (2007), and Baker and Roe (2009).

\(^2\)We are referring here to the definition of insurance as protection against a possible adverse outcome, not the purchase of a contract which provides compensation in the event of a loss.

\(^3\)Indeed, fat tailed uncertainty is now arguably the most important current issue in climate change policy, since it is the dominant motivation (either directly or indirectly through the effect of uncertainty on the discount rate) for stringent immediate limits on GHG emissions (see, for example, Weitzman 2007).

\(^4\)An integrated assessment model is broadly defined as a model which combines scientific and socio-economic aspects of climate change to assess policy options for climate control (Kelly and Kolstad 1999a).
data, which slows Bayesian learning. This result has since been confirmed in models with other types of climate uncertainty and different distributional assumptions. In particular, Leach (2007) considers uncertainty over the climate sensitivity and finds Bayesian learning about the climate sensitivity is extremely slow.\(^5\) Roe and Baker (2007) argue that resolving uncertainty regarding the climate sensitivity is difficult, because small uncertainties in climate feedbacks magnify the uncertainty about the climate sensitivity.\(^6\) Keller, Bolker, and Bradford (2004) show that slow learning about the climate sensitivity combined with an uncertain climate threshold, implies significant near term abatement is optimal, to avoid accidentally exceeding the threshold. Lemoine and Traeger (2014) study alternative uncertain thresholds with learning and find somewhat smaller effects on near term abatement. The above literature on learning with thin tails finds that learning is too slow to have much impact initially: near term abatement policy is similar with or without learning.

However, it is possible that the planner learns enough to reject severe climate change with a high degree of confidence quickly, even though the climate sensitivity is difficult to pin down precisely.\(^7\) We define this case as “tail learning.”

To investigate tail learning, we develop a quantitative integrated assessment model in which the planner faces stochastic weather shocks and uncertainty over the first order autoregressive coefficient in the equation governing the evolution of temperature, the climate feedback parameter. Because the climate feedback parameter is uncertain, the climate sensitivity is also uncertain. If the climate feedback parameter is close to one, then GHG “shocks” to temperature are long lived, and an increase in GHG emissions causes high steady state temperature changes. Hence, thin tailed uncertainty in the feedback parameter causes fat tailed uncertainty in the climate sensitivity (Roe and Baker 2007, show that normally distributed uncertainty in the feedback parameter results in climate sensitivity uncertainty which approximates the current uncertainty in the scientific literature). The social planner learns the feedback parameter, and therefore the climate sensitivity, using Bayes rule.

\(^5\)Most of the literature and our paper consider observational learning in the sense that the planner learns from the data on temperature and GHG concentrations. An alternative is to allow learning where the planner pays for R&D. Nonetheless, to fully resolve uncertainty, all R&D must eventually be confirmed in the data.

\(^6\)Climate feedbacks are changes in the climate system brought on by higher temperatures which amplify or diminish the relationship between GHGs and temperature (climate forcing). For example, higher temperatures melt ice, which in turn implies less heat is radiated back into space, which amplifies climate forcing. The magnitude of many climate feedbacks are uncertain (Forest, Stone, and Sokolov 2006).

\(^7\)Strictly speaking, the planner takes the entire current and expected future distribution of uncertainty into consideration when calculating the optimal policy. That is, the planner does not statistically test the hypothesis that climate change is severe. Instead, we show the optimal abatement policy can be intuitively understood as one in which the test statistic plays a dominant role.
We define the lower bound of the fat tail as when a doubling of GHGs implies steady state temperatures increase by 1.5°C more than the mean of the current prior distribution. For example, we calibrate that the mean of the initial prior results in a temperature increase of 2.76°C, so the tail of the distribution equals values greater than or equal to 4.26°C.\(^8\) When the planner rejects the hypothesis that the climate sensitivity implies a steady state temperature increase greater than or equal to the lower bound at the 1% or 0.1% level, we say that tail learning is complete.\(^9\) Such learning is partial in the sense that significant uncertainty typically remains even after a high climate sensitivity is rejected.

Our results show that the social planner rejects that the climate sensitivity is in the upper tail of the prior distribution very quickly. That is, although we confirm results in the previous literature that learning the actual true value \textit{precisely} is a relatively slow process, the planner rejects values of the climate sensitivity in the upper tail of the prior distribution quickly. In fact, tail learning is complete in less than a decade, if the true climate sensitivity is moderate. First, observations near the moderate true value provide evidence against the tail of the distribution. In addition, the density of even a fat tail is not large, so Bayes rule requires relatively few observations to reduce the mass of the fat tail below the critical confidence level. This result is surprising given the common intuition in the literature that reducing uncertainty in the tail of the climate sensitivity prior distribution must be a slow process since climate disasters are rare (see for example, Weitzman 2009b, page 12).

If the true climate sensitivity is relatively high, learning slows. First, Bayes rule requires more observations to move the mean estimate from the prior of a moderate climate sensitivity to the true high value. Second, Bayes rule requires more observations to resolve the difference between a climate sensitivity that is relatively high and a climate sensitivity which is very high. Nonetheless, because a high climate sensitivity is relatively unlikely according to the prior, the possibility that learning is slower due to a high climate sensitivity receives relatively little weight when computing the expected time until tail learning is complete. Integrating over the prior distribution, we find that the expected time, conditional on prior information, until tail learning is complete is only about 17 years at the 0.1% level.

Like Weitzman (2009b), our model considers a high climate sensitivity as a possible scenario with high damages from climate change. Other potentially high damage scenarios exist, including high sea level rise (Nicholls, Tol, and Vafeidis 2008), thermohaline circulation

\(^{8}\)No generally agreed upon value for what constitutes the tail of the prior distribution exists. Nonetheless, much of the literature uses higher values (e.g. Weitzman, 2009 discusses values above 7°C). A larger lower bound would only strengthen our results.

\(^{9}\)See Kelly and Kolstad (1999b) for a justification for using hypothesis tests to measure learning.
collapse (Keller, Bolker, and Bradford 2004), and a reduction in the decay rate of carbon into the ocean (Lemoine and Traeger 2014). Learning in each of these contexts may differ from our results. Nonetheless, fat tailed uncertainty in the climate sensitivity is the most commonly analyzed high damage scenario in the literature, presumably because the fat tail is clearly evident in the scientific priors.

We quantify the effect of uncertainty on near term optimal emissions and abatement policy. With uncertainty, but without learning, in the initial period emissions are about 38% lower, and the carbon tax is $22.94 higher, than under certainty. The planner insures by reducing emissions, paying for more abatement to reduce the probability of high damages that occur if the climate sensitivity is high. However, initial emissions with uncertainty and learning are only about 19% lower than under certainty. The optimal carbon tax with uncertainty and learning is only $8.84 per ton higher than under certainty. Therefore, learning reduces emissions abatement for insurance purposes by about 50%. Further, optimal emissions with uncertainty and learning converge quickly to emissions given perfect information, typically in about 16 years. Uncertainties remain after 16 years, but the remaining uncertainty is not relevant for the optimal emissions policy. The fat tail drives policy, and learning shrinks the mass of the fat tail quickly. Optimal policy under learning converges quickly to the perfect information case even if the true climate sensitivity is very high. This is because with a higher mean estimate, deviations from certainty are driven by the mass of the tail of the new distribution with a higher mean, and the mass of the tail of the new distribution still shrinks quickly.

Fat tailed uncertainty arises naturally when multiple uncertainties exist, or, in the approach taken here, when an uncertain parameter has a multiplicative effect through feedbacks in the climate system. Climate sensitivity uncertainty is then fat tailed, even if the prior distribution for the uncertain feedback parameter is normal. Roe and Baker (2007) show that, given a coefficient of risk aversion greater than or equal to one, the risk premium required to accept an uncertain climate sensitivity with fat tails is infinite (the dismal theorem).

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10 Tol (2009) reviews the findings of the literature on various damages caused by climate change.
11 For example, if the prior distribution for the climate sensitivity conditional on the variance of the weather shocks is the thin tailed normal distribution and the variance of the weather shocks is also unknown, with a gamma prior, then the unconditional prior distribution for the climate sensitivity is the fat tailed t distribution.
12 Roe and Baker (2007) show that a normally distributed feedback parameter results in a distribution for the climate sensitivity which accurately approximates the current state of knowledge.
13 See also Geweke (2001).
However, Costello, Neubert, Polasky, and Solow (2010) show that the dismal theorem is an asymptotic result: truncating the upper tail of the distribution of uncertainty invalidates the dismal theorem. They argue that truncation is reasonable, since infinite temperatures are not physically possible. We therefore set a maximum temperature change of 26.8°C.

Our results do not contradict the dismal theorem. The fat tail remains in our model for any finite number of observations, and without truncation, the risk premium is infinite.\textsuperscript{14} However, our results show the importance of results by Costello, Neubert, Polasky, and Solow (2010) for policy. We show that the fat tail is important for near term policy, even with truncation. Nonetheless, we show that learning quickly reduces the upper tail of the distribution to close to an exponential. Therefore, although learning has no effect on the infinite risk premium without truncation, with truncation learning does significantly reduce the risk premium.

Put differently, our results highlight the difference between variance and fat tails. It is possible, through learning, to reduce the variance so that the mass in the upper tail is arbitrarily small. For example, we show that in many cases the planner can reduce the mass in the tail to 1% or 0.1% in just 10-16 years. The planner then easily rejects the hypothesis of a high climate sensitivity. However, tails are still fat in that the rate of decline in the upper tail is eventually slower than exponential. Therefore, with truncation, abatement policy is sensitive to the mass of the fat tail, which learning reduces by reducing the variance.

Considerable debate exists in the literature on the importance of fat tailed uncertainty for near term climate policy when learning is possible. For example, Weitzman (2011) argues that strong inertia in the climate should mean learning is less relevant for near term policy. Given that the stock of GHG emissions is difficult to reduce quickly, it will be difficult to reduce GHG concentrations if we learn climate change is more severe than expected. Conversely, Nordhaus (2011) argues that severe climate change should be evident in the data within the next 50 years, and so time exists to reduce GHG concentrations. Pindyck (2011) points out that the question is inherently quantitative and depends on the cost of insurance, the probability of severe climate change, etc.

This paper is the first to provide a quantitative answer to the above debate.\textsuperscript{15} Reducing emissions initially is important due to the fat tailed uncertainty and the difficulty in reversing GHG stocks. Nonetheless, consistent with Nordhaus’ idea, learning is indeed fast enough so that mid course corrections are possible. For example, we show that if the true climate

\textsuperscript{14}I.e., the key elements of the model satisfy the assumptions of Millner (2013) for an infinite risk premium.

\textsuperscript{15}Subsequent to this paper, Hwang, Reynes, and Tol (2014) consider a version of the Nordhaus DICE model with fat tailed uncertainty and learning.
sensitivity implies a doubling of GHG concentrations causes a 5°C temperature change, then the planner reduces emissions to within 1% of the certainty level in only 14 years.

Note that in all of above results, convergence to the perfect information level of emissions requires only tail learning. Indeed we show that learning as conventionally defined in the literature is not complete for 85 years. Learning is slow, but the remaining uncertainty after the fat tail is statistically rejected is not important for policy. Only the fat tail provides an insurance motivation for near term abatement.

2 Model

The model is similar in spirit to a simplified DICE model (Nordhaus 2007). Economic growth generates GHG emissions, which in turn cause temperatures to rise, reducing productivity. However, we use the abatement cost function from Bartz and Kelly (2008), and use different assumptions about future improvements in the emissions intensity of output. The learning model and stochastic temperature change are related to Kelly and Kolstad (1999b) and Leach (2007), however, here we look at learning the fat tailed climate sensitivity rather than learning the primitives of the temperature model. The damage function is from Weitzman (2009a). Finally, time is annual rather than decadal as in Nordhaus.

2.1 Economic system

The population of $L_t$ identical households have preferences over consumption $C_t/L_t$ given by the period constant relative risk aversion utility function:

$$U\left(\frac{C_t}{L_t}\right) = \left(\frac{C_t}{L_t}\right)^{1-\sigma} - 1 \over 1 - \sigma.$$ (1)

A constant returns to scale technology exists that produces output $Q_t$, from capital $K_t$, and productivity-augmented labor $A_t L_t$. Here $A_t$ is labor productivity, which grows exogenously at rate $\phi$. Population grows at exogenous rate $\eta$. The production technology is such that:

$$Q_t = F(K_t, A_t L_t) = K_t^{\psi} (A_t L_t)^{1-\psi}.$$ (2)

Unabated GHG emissions are an exogenous proportion $1/B_t$ of output. Let $u_t$ denote

\footnote{In this sense, our results are consistent with Kelly and Kolstad (1999b), who estimate learning about the heat capacity was complete after about 90 years.}

the fraction of emissions abated, then \((1 - u_t)/B_t\) is the emissions intensity of output, and emissions, \(E_t\), are:

\[
E_t = (1 - u_t) \frac{Q_t}{B_t}.
\]  

(3)

The cost of abatement is \(\Lambda(u_t)Q_t\). Hence, output net of abatement costs, \(Y_t\), is

\[
Y_t = (1 - \Lambda(u_t))Q_t.
\]  

(4)

We assume a convex cost function:

\[
\Lambda(u_t) = 1 - (1 - u_t)\epsilon.
\]  

(5)

The abatement cost function (5), differs from standard cost functions in the literature. For example, Nordhaus (2008) uses a two parameter function: \(\Lambda(u_t) = \epsilon_1 u_t^2\). The abatement cost function (5) has a particular advantage in that it is consistent with a balanced growth path (Bartz and Kelly 2008), which simplifies the computations considerably. Equation (5) implies the cost of a backstop technology with zero emissions is 100% of GDP. In contrast, in Nordhaus the backstop costs only \(\epsilon_1 < 1\) fraction of output, which declines over time. Therefore, we find lower optimal abatement rates in the far future, although we find similar abatement rates in the near future.\(^1\)

Using equations (2)-(5) to substitute out for \(Q_t\) and \(u_t\) implies output net of abatement costs is Cobb-Douglas:

\[
Y_t = F(K_t, B_t E_t, L_t A_t) = K_t^\theta (B_t E_t)^\epsilon (A_t L_t)^{1-\theta-\epsilon}.
\]  

(6)

Here \(\theta = \psi(1 - \epsilon)\) is the capital share and \(\epsilon\) can thus be interpreted as the emissions share.\(^2\)

A balanced growth path is a steady state where aggregate capital, output, and consumption all grow at the same constant rate \((1 + \eta)(1 + \phi) - 1\). Appendix A shows that a balanced growth path exists with constant emissions, GHG concentrations, and temperature

\(^{17}\)However, in the simulations below the climate sensitivity may turn out to be quite high, in which case high abatement levels are optimal even in the near term. The lack of an inexpensive backstop technology makes fat tailed uncertainty less trivial in the model. In the Nordhaus framework, beyond a certain point in the uncertainty distribution, the low cost backstop technology would be employed, making fat tailed uncertainty and learning less relevant. Finally, the cost function (5) is less convex than Nordhaus for low values of \(u_t\), but more convex for \(u_t\) sufficiently close to one (using \(\Lambda_{uu} \cdot u/\Lambda_u\) to measure convexity). Therefore, for sufficiently low values of \(u_t\), abatement is more sensitive to the resolution of uncertainty in our model (all else equal).

\(^{18}\)Bartz and Kelly (2008) calibrate the emissions share for four air pollutants.
if the exogenous growth rate of $B_t$ equals the growth rate of output:\textsuperscript{19} For this reason, we find lower abatement rates in the far future.

$$B_{t+1} = (1 + \eta) (1 + \phi) B_t.$$ \hfill (7)

Appendix A derives the steady state values of all endogenous state and decision variables. Note that the economy exhibits a relatively fast transition as the economy accumulates capital for a relatively fixed climate, after which the climate and economy will exhibit slow transitional dynamics as the climate changes. Exogenous growth of $B_t$ captures both technological change in abatement and compositional changes in output.

Let capital depreciate at rate $\delta_k$. The resource constraint then sets consumption plus net investment equal to production net of abatement costs after damages, $D(T)$, due to climate change:

$$C_t = (1 - D(T_t)) Y_t + (1 - \delta_k) K_t - K_{t+1},$$ \hfill (8)

where the damage function is:

$$D(T_t) = 1 - e^{-b_1 T_t^{b_2}}.$$ \hfill (9)

Here $b_1$ and $b_2$ are damage parameters.

\subsection*{2.2 Climate system}

Let $M_t$ represent the current accumulation of carbon-equivalent GHG in the atmosphere, and $M_p$ is the preindustrial stock. We assume the ocean and biosphere absorb atmospheric carbon at a constant rate $\delta_m$. The stock of pollution accumulates according to:

$$M_{t+1} - M_p = (1 - \delta_m) (M_t - M_p) + E_t.$$ \hfill (10)

\textsuperscript{19}Note that if $B_t$ grew slower than the rate of output then the returns to emissions savings innovation would approach infinity, while the returns to labor productivity would go to zero, and the reverse if $B_t$ grew faster than the rate of output (see appendix A.3). Therefore, in a more general decentralized economy where innovations require costly R&D, spending on R&D would tend to flow to the sector with the slower growth rate, which tends to equalize the growth rates in the long run (assuming the government gives appropriate incentives for innovation, including protection of intellectual property and an increasingly strict regulation on carbon emissions). Therefore, identical growth rates is a reasonable assumption. In the Nordhaus DICE model, $B_t$ grows slower than $(1 + \eta) (1 + \phi)$, which implies $E_t \rightarrow 0$ (see appendix A.1).
In turn, accumulated GHGs cause radiative forcing, $F_t$ according to:
\[
F_t = \Omega \log_2 \left( \frac{M_t}{M_p} \right). \tag{11}
\]
Here $\Omega$ is the radiative forcing parameter. We use a one equation physical model for temperature:
\[
\hat{T}_t = \hat{T}_{t-1} + \frac{1}{\alpha} \left( F_t - \hat{T}_{t-1} - \Gamma \right) + \nu_t. \tag{12}
\]
Here $\hat{T}_t$ is the annual global temperature (difference in °C between year $t$ and the 1961-1990 average temperature); $\Gamma$ is the preindustrial temperature difference from the 1961-1990 average temperature; $\alpha$ is the heat capacity of the upper ocean; $\lambda$ is the climate sensitivity; and $\nu \sim N(0, \sigma^2)$ is the stochastic weather shock. The climate sensitivity is an uncertain parameter, and the weather shock is not observed.

Equations (11) and (12) vastly simplify large physical models of climate, known as general circulation models (GCMs). We will consider a more complicated climate model with an ocean layer for some results in Section 4.3. Nonetheless, models similar to (11) and (12) are frequently estimated and used for policy analysis.\(^{20}\)

The uncertain parameter is the climate sensitivity $\lambda$, which measures how responsive temperature is to GHG concentrations. The climate sensitivity amplifies the effect of radiative forcing on temperature. To see this, rewrite equations (11) and (12) as:
\[
T_t = \beta_1 T_{t-1} + \beta_2 \log_2 \left( \frac{M_t}{M_p} \right) + \nu_t. \tag{13}
\]
Here $T_t = \hat{T}_t - \Gamma$ is the annual global temperature deviation from preindustrial level, $\beta_1 = (1 - \frac{1}{\lambda \alpha})$ is the climate feedback parameter, and $\beta_2 = \frac{\Omega}{\alpha}$. The climate feedback parameter is positively related to the climate sensitivity. Since the climate sensitivity is uncertain, the climate feedback parameter $\beta_1$ is also uncertain.

Let $\Delta T_{2x}$ be the steady state temperature change that results from a steady state doubling of the GHG concentrations, then from equations (12) and (13):
\[
\Delta T_{2x} = \Omega \lambda = \frac{\beta_2}{1 - \beta_1}. \tag{14}
\]
\(^{20}\)See for example, Andronova and Schlesinger (2001) and Schwartz (2007). Kelly, Kolstad, Schlesinger, and Andronova (1998) discuss some weaknesses of one equation physical models. Traeger (2014) calibrates a one equation model to match near term temperature changes predicted by the Nordhaus DICE model.
Since $\lambda$ is uncertain $\Delta T_{2x}$ is also uncertain. The steady state temperature change from a doubling of CO$_2$ is most straightforward to understand, so we will report most results in terms of the equivalent $\Delta T_{2x}$.

### 2.3 Learning

Each period the social planner observes new statistical records of the climate system and updates beliefs on the uncertain feedback parameter. Bayesian learning characterizes this process. Assume the planner has prior beliefs that the true $\beta_1$ is drawn from a normal distribution, $N(\mu_0, S_0)$. The normal distribution, which assigns positive measure to values of $\beta_1$ outside the unit interval, is theoretically inconsistent with the existence of (for example) a maximum temperature. Nonetheless, Roe and Baker (2007) show that a normal distribution is an accurate approximation of the current state of knowledge.\(^{21}\) We set the initial mean and variance of the prior distribution, $[\mu_0, S_0] = [0.65, 0.13^2]$, to match their estimates. Let:

$$H_t = T_t - \beta_2 \log_2 \left( \frac{M_t}{M_p} \right) = \beta_1 T_{t-1} + \nu_t. \quad (15)$$

Then $H_{t+1} \sim N(\mu_{H,t}, \sigma^2_{H,t})$ combines the stochastic weather shocks and feedback uncertainty into a single random variable, where $\mu_{H,t} = \mu_t T_t$ and $\sigma^2_{H,t} = T^2_t S_t + \sigma^2_{\nu}$. The $t+1$ weather shock occurs at the start of the period, before decisions are made. The social planner observes $H_{t+1}$, and $T_t$ and updates the prior on $\beta_1$. Bayes rule implies that the posterior distribution of $\beta_1$ is also normally distributed with:

$$\mu_{t+1} = \frac{\sigma^2_{\nu} \mu_t + S_t T_t H_{t+1}}{\sigma^2_{\nu} + S_t T^2_t}, \quad (16)$$

$$S_{t+1} = \frac{\sigma^2_{\nu} S_t}{\sigma^2_{\nu} + S_t T^2_t}. \quad (17)$$

Note that from equation (17) that the variance estimate on $\beta_1$ is monotonically non-increasing with time. We use variance instead of the usual precision since the variance is bounded above

\(^{21}\)An even more accurate approximation would assign zero weight to values outside the unit interval by truncating the normal distribution. Our solution algorithm truncates the normal distribution (see Appendix B.2). Therefore, assuming the prior is normal is only a problem in that learning is slightly faster in reality because extra information is available which rules out values of $\beta_1$ outside the unit interval. Allowing the planner in the model access to this knowledge strengthens our results, but also adds considerable complexity because the truncated normal distribution does not have a closed form conjugate prior.
by the initial prior variance, while the precision is unbounded above.\textsuperscript{22} Perfect information implies $S = 0$ and $\mu = \beta_1$.

Roe and Baker (2007) compute the probability density function (PDF) for the climate sensitivity from a Jacobian transformation. Let $\beta_1 = 1 - \frac{1}{\alpha \lambda} \sim N(\mu, S)$, then the density for the climate sensitivity is:

$$h_\lambda(\lambda) = h_{\beta_1}(\beta_1(\lambda)) \left( \frac{\partial \beta_1}{\partial \lambda} \right) = \frac{1}{\sqrt{2\pi S}} \frac{1}{\lambda^2} \exp \left[ -\frac{1}{2S} \left( 1 - \frac{1}{\alpha \lambda} - \mu \right)^2 \right].$$ \hspace{1cm} (18)

A standard definition (Newbold and Daigneault 2009) of a fat tail is that the probability declines to zero at a rate slower than exponential:\textsuperscript{23}

$$\lim_{\lambda \to \infty} \frac{h_\lambda(\lambda)}{\exp(-a \lambda)} > 0, \quad a > 0. \hspace{1cm} (19)$$

It is straightforward to verify that $h_\lambda(\lambda)$ satisfies condition (19). The fat upper tail is clearly visible in Figure 1, which plots $h_\lambda(\lambda; \mu_0, S_0)$.

From equation (18), all moments of climate sensitivity, and therefore all moments of the prior distribution for $\Delta T_{2x}$, are infinite. Nonetheless, the moments of the feedback parameter are always well defined. Some policy experiments in the following sections assume the true feedback parameter equals the mean of the prior, $\mu_0 = 0.65$. The temperature change resulting from a state doubling of GHG concentrations when the true feedback parameter equals the prior, $\Delta T_{2x} = 2.76^\circ$, should not be confused with the mean of the prior distribution for $\Delta T_{2x}$, which is infinite. A contribution of this paper is our methodology for the analysis of learning on fat tailed uncertainty: we perform computations on a non-linear transformation of the fat tailed climate sensitivity, the feedback parameter, which has finite moments and a conjugate family that permits learning.

\textsuperscript{22}The computational solution method requires a bounded state space (see Appendix B.2).

\textsuperscript{23}The PDF also satisfies other definitions. For example, the tail index is one, meaning the PDF is asymptotically equivalent to a Pareto distribution with shape parameter equal to one and all moments including the mean are infinite.
2.4 The Recursive Problem

The social planner maximizes the expected present discounted value of the stream of period utilities weighted by the population size:

\[ W = \max E \sum_{t=0}^{\infty} \beta^t L_t U \left( \frac{C_t}{L_t} \right). \]  

We normalize the problem so that variables are in per labor productivity unit terms. Let \( f(K, E) \equiv F(K, E, 1) \), \( k_t = K_t / (A_t L_t) \) and similarly for \( c_t \) and \( y_t \), and \( m_t = \frac{M_t}{M_p} \). Let \( \gamma = A_0 L_0 E_0 / Q_0 \) be the initial emission intensity coefficient and \( \hat{E} = E / \gamma \). Then, assuming \( \hat{\beta} \equiv \beta (1 + \eta) (1 + \phi)^{1/\sigma} < 1 \), we can write the recursive version of the social planning problem as:

\[
v(k, m, T, \mu, S) = \max_{k', \hat{E}} \left\{ \frac{e^{1-\sigma}}{1-\sigma} \hat{\beta} \int_{H'} v(k', m', T', \mu', S') N(\mu T, T^2 S + \sigma^2) dH' \right\},
\]

subject to:

\[
c = (1 - D(T)) f(k, \hat{E}) + (1 - \delta_k) k - (1 + \eta) (1 + \phi) k',
\]

\[
m' = 1 + (1 - \delta_m) (m - 1) + \left( \frac{\gamma}{M_p} \right) \hat{E},
\]

\[
T' = H' + \beta_2 \log (m'),
\]

\[
\mu' = \frac{\sigma^2 \mu + STH'}{\sigma^2 + ST^2},
\]

\[
S' = \frac{\sigma^2 S}{\sigma^2 + ST^2}.
\]

Equation (21) condenses the double expectation over unknown variables \( \beta_1 \) and \( \nu' \) into an expectation over a single unknown variable \( H' \), which is normally distributed with mean \( \mu T \). Tables 1 and 2 give the parameter and variable definitions for the above problem.

3 Calibration and Solution Method

Table 1 gives the parameter values. For the economic parameters, we chose a risk aversion coefficient of \( \sigma = 1.5 \). The discount factor is consistent with a 5% pure rate of time preference,
which implies the model economy matches the US capital to output ratio. Section 6 performs sensitivity analysis with a lower rate of time preference. The depreciation rate of capital is 4.6%, which implies the model economy matches the US investment to output ratio. The population growth rate is 1% and the growth rate of per capita GDP is 1.8%, which match US data. These values are broadly consistent with the business cycle literature (Bartz and Kelly 2008, Cooley and Prescott 1995).

The emissions share parameter is also the parameter of the abatement cost function. Nordhaus (2008) estimates a cost function using survey data and existing estimates of the cost of a backstop technology. In general, the abatement cost function has only one parameter and therefore cannot match both the low cost of abatement when $u_t$ is low, and the convexity of the Nordhaus (2008) cost function. We chose the calibrated value to match well for low values of $u_t$. Therefore, initial abatement rates, abatement costs, and carbon taxes will tend to be similar to Nordhaus. In later years, the abatement rate grows in both models, but the abatement rate is lower in our model than in the DICE model. Matching near term abatement cost is appropriate for the focus of the paper, which is the effect of uncertainty on optimal near term abatement.

The damage parameters are taken from Weitzman (2009b). The damage parameters and discount factor are set relatively conservatively. A discount factor closer to one or a high damage scenario (for example, Weitzman also considers $b_2 = 4$) would increase both the value of abatement as climate insurance, and the value of learning. We consider the high discount factor case in Section 6.

One issue with the calibration is that the model sets the growth rate in the inverse of the emissions intensity of output ($B_t$) equal to the growth rate of output. This is done so that the balanced growth path exists with constant emissions. In the long run, the growth rate of $B_t$ should equal the growth rate of output otherwise the returns to labor productivity innovations will either be infinite or zero. Nonetheless, in the short run the rate of growth in $B_t$ can differ from that of output. In fact, the growth rate of output has exceeded the growth rate of $B_t$ in recent years. This is not surprising since carbon emissions are still

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$^{24}$Our economy has a balanced growth path. Therefore, by choosing parameters to match a developed economy (e.g. the US), we guarantee that our model in the absence of climate regulation and damages converges to a developed economy with similar properties as developed economies in the data. An alternative (Nordhaus 2008) is to calibrate to match current aggregate world (developed and less developed) data, and assume parameters do not change as countries develop.

$^{25}$In particular, the abatement cost is identical to Nordhaus (2008) for $u_t = 0.31$, and the total abatement cost differs from Nordhaus (2008) by at most 0.0014 for $u_t \leq 0.4$. The cost function becomes more convex than Nordhaus (2008) at $u_t = 0.64$. 

13
largely unregulated in most countries. Allowing for different short run growth rates (as is done in Nordhaus, 2008, for example), would considerably complicate the analysis by adding additional state variables, however.

The variance of the weather shock is taken from Leach (2007). The remaining climate and forcing parameters are set consistent with Nordhaus (2008). Time is in annual increments.

Appendix B gives the details of the computational solution method, which uses a multi-dimensional spline approximation of the value function and value function iteration to solve the problem. We set a maximum temperature of 26.8°C (see Appendix B.2 for details).

4 Results: Learning

4.1 Tail Learning

The solution of the dynamic program is a value function \( v(k, m, T, \mu, S) \) and the corresponding policy functions giving optimal investment and emissions \( k^*(k, m, T, \mu, S) \) and \( E^*(k, m, T, \mu, S) \). The evolution of the state and control variables over time follows:

\[
[k', m', T', \mu', S', E] \equiv G([k, m, T, \mu, S], \lambda, \nu').
\] (27)

Here the function \( G \) is comprised of the policy functions and the laws of motion for the state variables, equations (23)-(26). Table 2 gives the initial conditions, which are set to the year 2005. The initial conditions for the learning parameters are from Roe and Baker (2007), who survey various global circulation models (GCMs). Initial conditions for capital, temperature, and GHG concentrations are from Nordhaus (2008). Given the initial conditions, a simulated time path requires a set of random draws for \( \nu \) over time and a true value of \( \lambda \). Each experiment consists of 150 simulations, for a given true value of \( \lambda \). Each experiment ran until learning was complete in the conventional sense of the literature, that is, until \( \mu \) and \( S_t \) converged to where the hypothesis that values of the climate sensitivity more than 5% away from the true value were rejected at the 95% confidence level (see section 4.2). We report simulations for various hypothetical true values for \( \lambda \), and then take expectations over the results using the prior distribution to get expected results given the current state of information.

We say tail learning is complete if the planner can reject values of the climate sensitivity which are in the tail of the distribution at a given confidence level. That is, tail learning is complete if the planner rejects the hypothesis \( \lambda > T^L/\Omega \) or equivalently \( \Delta T_{2x} > T^L \), where
$T^L$ is the lower border of the tail of the distribution ($^\circ$C).

The lower border of the climate sensitivity distribution which constitutes the “tail” has no precisely agreed upon definition. Further, from equation (21) the optimal decision takes into account the entire distribution in a continuous way. Our hypothesis is that the tail of the distribution drives current optimal abatement policy. If the mass of the tail shrinks quickly, the remaining uncertainty is irrelevant and the optimal policy converges to the certainty case. The first step is therefore to show that learning about the tail of the distribution differs from learning about the mean, which requires a definition of what constitutes the tail of the distribution.

Therefore, we set the lower border of the tail of the distribution conservatively as $T^L_t = \Delta T_{2x, t} + 1.5^\circ$C, where $\Delta T_{2x, t}$ is the climate sensitivity resulting from the current mean estimate of the feedback parameter.\textsuperscript{26} The initial prior mean feedback parameter results in a temperature change of 2.76°C, so the initial lower border of the tail is $T^L = 4.26^\circ$C, which constitutes the upper 16.7% of the mass of the distribution. If the true value is large, say 5°C, then learning will move the mean of the prior higher, and the uncertainty can eventually be partitioned into uncertainty about the exact value of the climate sensitivity (near 5) and uncertainty about the probability of still higher values (even more disastrous values greater than 6.5 are possible).\textsuperscript{27}

The planner’s optimization problem (21) does not specify such a test as part of the optimal policy. Instead, we are developing a hypothesis that the effect of uncertainty on optimal policy can be approximately partitioned into two parts: the effect of uncertainty over the exact value of the climate sensitivity, and the mass of the tail of the distribution of uncertainty.

We consider two confidence levels, 99% and 99.9%. Given the high damages associated with severe climate change, we assume the planner requires a relatively high level of confidence that the climate sensitivity is not large before rejecting the hypothesis that the true climate sensitivity is in the tail of the distribution.

We consider 50 possible true values, $\Delta T_{2x}^i$, indexed by $i$. For each $\Delta T_{2x}^i$, for each simulation $j$, and associated random vector of weather shocks $\nu_j$, we record the first period,

\textsuperscript{26}Values above 1.5° will speed up the learning, values less than 1.5° mean the tail is relatively likely to occur, which is inconsistent with the idea that the true value is in the tail with relatively low probability.

\textsuperscript{27}Alternatives are less attractive. If the lower border of the tail is fixed for all $t$ (say $T^L = 5^\circ$C), then the planner must essentially learn the exact value of the climate sensitivity for true values near the lower border, since in that case the tail of the distribution converges to the upper half of the distribution. As $\Delta T_{2x, t} \rightarrow T^L$, this definition of tail learning requires an arbitrary large number of periods to complete, which is not consistent with the optimal policy in Section 5.
for which the hypothesis that $\Delta T_{2x,jt}^i > T^L$ is rejected and not subsequently not-rejected. We then say that the planner achieves tail learning in period $n^*$, for true value of $\Delta T_{2x}^*$ and simulation $j$.

We then average over all simulations, and weight all $n^*$ using the prior distribution. Mathematically:

$$E[n^*|\mu_0, S_0] = \int_1^\infty \int \frac{1}{\alpha} n^* (\lambda, \nu) N(0, \sigma^2_\nu) h_\lambda(\lambda) d\nu d\lambda.$$  \hspace{1cm} \text{(28)}

Table 3 shows the results. The expected time to complete tail learning varies from 8.99-14.58 periods, depending on the required confidence level. Tail learning is relatively quick for three reasons. First, by definition the initial mass of the upper tail is not large. Therefore it takes relatively few observations away from the tail to reduce the mass to 1% or 0.1%. Second, although the global temperature is subject to random weather shocks which makes the exact value of the climate sensitivity difficult to pin down, the calibrated standard deviation of the shocks is only 0.11°C, which makes it relatively easy to reject high values of the climate sensitivity if the true value is not too large. Third, most of the mass of the prior distribution is relatively close to the mean. Therefore, true values with smaller learning times receive more weight when calculating the expected learning time conditional on the prior distribution. As expected, tail learning at the 99.9% confidence level requires more periods than at the 99% confidence level.

The left panel of Figure 2 shows the learning time as a function of $\Delta T_{2x}^*$ for both confidence levels. For true values of $\beta_1$ near the mean of the prior distribution, learning takes less than 10 periods, but rises to 90 years or more as $\Delta T_{2x}^*$ increases. Tail learning becomes increasingly difficult as $\Delta T_{2x}^*$ increases. Since $\lambda = 1/(\alpha(1 - \beta_1))$, for values of $\beta_1$ near one, small differences in $\beta_1$ generate large differences in $\lambda$. Therefore, the planner must learn $\beta_1$ with increased precision to reject values of $\lambda$ in the tail of the distribution when $\beta_1$ is near one. For example, with $\beta_1$ equal to the prior, the lower bound of the tail ($T^L = 4.26$) corresponds to $\beta_1^L = 0.77$, which is rejected in reasonable time given a true value of $\beta_1^* = 0.65$ and a calibrated standard deviation of the weather shocks equal to 0.11. Conversely, if $\Delta T_{2x}^* = 5$ and $T^L = 6.5$ then $\beta_1^* = 0.81$ and $\beta_1^L = 0.85$, which is much harder to reject.

Put differently, for $\beta_1$ large, precise learning is much more important, because small differences in $\beta_1$ imply large differences in the decay rate of GHG “shocks” to temperature.

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An alternative is to define $n^*$ as the period in which the hypothesis is first rejected. In this case, tail learning is 28-34% faster, but the results are qualitatively similar.
Small differences in $\beta_1$ therefore imply large differences in steady state temperature for a given concentration of GHGs, making a precise estimate of $\beta_1$ more important.

Figures 1 and 2 together show that, for most of the prior distribution, tail learning requires less than 20 periods to complete. The expected number of periods until tail learning is complete conditional on prior information is relatively small. Longer learning times are possible but unlikely.\(^{29}\)

### 4.2 Conventional Learning

The above results constitute learning about the likelihood that the true $\Delta T_{2\times}$ is in the tail of the prior distribution, not learning as is conventionally defined in the literature. To contrast our results with the previous literature (see for example Leach 2007), we define two hypothesis tests, $\Delta T_{2\times} \leq 0.95 \cdot \Delta T_{2\times}^*$ and $\Delta T_{2\times} \geq 1.05 \cdot \Delta T_{2\times}^*$, where $\Delta T_{2\times}^*$ is the true value, and a desired confidence level of 95%. If the planner rejects both hypotheses, we say that conventional learning is complete. The right panel of Figure 2 plots the mean number of periods required until conventional learning is complete. The learning time is increasing in the true value. First, as the true value gets farther from the prior, Bayes rule must reject the prior information in favor of the new observations. Second, as noted above, $\Delta T_{2\times}$ is a nonlinear function of $\beta_1$. When $\Delta T_{2\times}$ is large, the range of values of $\beta_1$ that constitute the plus or minus 5% of the true value becomes narrower, which increases the expected learning time. Integrating over the prior distribution, we find an average learning time equal to 79.67 years, This is roughly consistent with the previous literature (Kelly and Kolstad 1999b, Leach 2007).

The left panel of Figure 3 plots $\Delta T_{2\times} = \Omega / (\alpha (1 - \mu_{it}))$, for each simulation $i = 1 \ldots 150$ and time period $t$. That is, Figure 3 plots an estimate of $\Delta T_{2\times}$ using the mean of the posterior distribution of $\beta_1$. The fat upper tail is clearly visible after the first few periods, but after 10 observations none of the sample paths has a mean estimate above 3.5°. Nonetheless, learning makes little additional progress between 10 and 45 periods. Learning rules out extreme values relatively quickly, but some uncertainty remains after 45 periods. The right panel of Figure 3 repeats the experiment when the true value of $\Delta T_{2\times} = 5$. Both tail and conventional learning are slower. Nonetheless, after 15 periods only one sample path has a mean estimate above 6°. Although the estimate is high, even more extreme values are

\(^{29}\)Note that the priors are based on physical, rather than statistical models of temperature change. Therefore, the results may be interpreted as saying that to confirm or refute physical models which predict a relatively high climate sensitivity will require relatively little additional data.
unlikely.

The intuition for these results follows from the way the estimate of the feedback parameter enters into the denominator of the temperature change from a sustained doubling of GHGs, $\Delta T_{2\times}$. Relatively small weather shocks cause small changes in the posterior mean of the feedback parameter $\mu_t$. Because $\mu_t$ enters into the denominator of $\Delta T_{2\times}$ (that is, because feedback effects magnify over time), small uncertainties in $\mu_t$ create larger uncertainties in $\Delta T_{2\times}$, “fattening” the distribution of $\Delta T_{2\times}$. This effect becomes larger when $\mu_t$ is close to one, which occurs when the true value of $\Delta T_{2\times}$ is large. Hence learning is slower when the true $\Delta T_{2\times}$ is relatively large. However, over time this effect works in reverse. After a few observations, the uncertainty over $\mu_t$ narrows. This has a nonlinear effect: small reductions in uncertainty over $\mu_t$ greatly reduce the probability of very high values of $\Delta T_{2\times}$. Hence, tail learning is faster than conventional learning, because tail learning focuses on high values of $\Delta T_{2\times}$.

The last 3 rows of Table 4 presents the same information for the mean of 150 simulations. The initial mass of the tail is 16.5%, and the 95% confidence interval admits a wide range of possible values for the climate sensitivity, from the benign 0.97 to the disastrous 7.09°C temperature change for a doubling of GHGs. By 2015, however, tail learning is complete at the 99.9% confidence level, whereas conventional learning is not complete: the 95% confidence interval is still fairly wide at almost 1°C. By 2050, the 95% confidence is fairly narrow, although conventional learning is still not quite complete.

4.3 Learning with more complicated climate models

We next check for robustness by reporting how various extensions affect the expected time to complete tail learning. These alternative assumptions complicate our highly simplified climate model, and add additional uncertainties.

First, we simulate the model with an ocean layer. Let $\hat{O}_t$ be the ocean temperature in deviations from the recent average and $\zeta$ and $\tau$ be heat transfer coefficients, then the climate

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30 The intuition implies the only essential assumption for the results is that the tail becomes fat via an increasing and convex transformation of a thin tailed distribution, the exact transformation of Roe and Baker (2007) is not essential. For example, $\Delta T_{2\times} = \exp(-\beta_1)$ produces a log-normal transformation that generates nearly identical results.

31 For computational reasons, we use the emissions and investment policy functions given in equation (27), which assumes the ocean layer is constant. Since the ocean layer changes slowly over time, this assumption is reasonable for the short term.
system (12) now becomes:

\[
\hat{T}_t = \hat{T}_{t-1} + \frac{1}{\alpha} \left( F_t - \frac{\hat{T}_{t-1} - \Gamma}{\lambda} \right) - \zeta \left( \hat{T}_{t-1} - \hat{O}_{t-1} \right) + \nu_t, \tag{29}
\]

\[
\hat{O}_t = \hat{O}_{t-1} + \tau \left( \hat{T}_{t-1} - \hat{O}_{t-1} \right). \tag{30}
\]

Because heat transfer to the ocean is a slow process, the model now takes much longer to reach the steady state. However, Table 3 shows that tail learning is complete in 15.84-24.39 periods, which is only slightly longer than with the one equation model. To learn the true climate sensitivity, the planner must statistically isolate the upward trend in the climate from the stochastic weather shocks. The ability of learning to do so is only marginally affected by the ocean and other climate processes with long lags, since the ocean temperature is essentially constant on a year to year basis. Thus our results are robust to the addition of an ocean layer.

The second extension allows for autocorrelation in the weather shocks.\textsuperscript{32} For example, certain temporary weather phenomena, such as El Niño, exhibit serial correlation. We assume:

\[
\nu_t = \chi \nu_{t-1} + \xi_t, \quad \xi_t \sim N \left[ 0, \sigma^2_\xi \right]. \tag{31}
\]

We obtain estimates of \(\chi\) and \(\sigma^2_\xi\) by estimating equation (13), assuming the correlation structure (31), using the standard HadCRUT4 temperature (Morice, Kennedy, Rayner, and Jones 2012) and GISS forcing data (Hansen, Sato, and et. al. 2007). Table 1 reports the estimates. The estimated parameter \(\chi\) is not significantly different from zero (the \(t\)-statistic is 0.32). The lack of significance favors our original iid specification. Nonetheless, we report the results to show robustness. Table 3 indicates that tail learning is slightly faster than the baseline model with iid weather shocks. As seen in equations (25) and (26) the precision of the weather affects how much information each observation contains, and therefore the speed of learning. For the case of autocorrelated weather shocks, we have:

\[
\bar{\sigma}_\nu = \frac{\sigma_\xi}{\sqrt{1 - \chi^2}} = 0.10. \tag{32}
\]

Here \(\bar{\sigma}_\nu\) is the variance of the invariant distribution for \(\nu\). Since in the original formulation

\textsuperscript{32}Again, for computational reasons we assume the optimal emissions policy ignores the autocorrelation in the weather shocks. Traeger and Jensen (2012) finds weather shocks are largely irrelevant for emissions policy.
\( \sigma_\nu = 0.11 \), the model with autocorrelated weather shocks has less noise. The autocorrelation tends to amplify the weather shocks, but the better fit of the model with autocorrelated shocks means the weather residual is smaller. Thus learning is faster. However, the difference is small and so tail learning is only slightly faster.

The third extension allows for the possibility of longer lags in the temperature equation. For example, some evidence of decadal variability in temperature exists (Solomon 2011). To capture longer lags, we consider the temperature model:

\[
T_t = \beta_1 T_{t-1} + \beta_2 \log_2 \left( \frac{M_t}{M_p} \right) + \beta_3 (O_t - T_t) + \beta_4 T_{t-i} + \nu_t. \tag{33}
\]

Estimating equation (33) individually for various lags between 8 and 12 show that lags 8 and 10 are significant, with \( i = 10 \) providing the best fit (the estimate is \( \beta_4 = 0.19 \) and the standard error is 0.071).

To check robustness of the results to longer lags, we simulate the model assuming (33) with \( i = 10 \), again assuming that the optimal emissions decision is independent of \( T_{t-10} \). Table 3 reports that tail learning is somewhat faster, about 4.72 years versus 15.84 in the baseline model, both with ocean. Our result that tail learning is relatively fast is therefore robust to the addition of longer lags in temperature. The estimate of the variance of \( \nu \) changes little from the original temperature equation. However, the upper bound of admissible ranges of \( \beta_1 \) such that the model is stationary is affected. Stationarity requires \( \beta_1 < 1 - \beta_4 \), so adding the decadal lag implies the maximum value of \( \beta_1 \) is 0.85. Since tail learning time is lowest for values of \( \beta_1 \) close to one, restricting the upper bound of \( \beta_1 \) results in faster learning on average.

Finally, we consider an extension to multiple uncertainties. One uncertain parameter is the rate of ocean heat uptake, \( \zeta \). We take as the prior an estimate of the uncertainty in the ocean heat uptake by Forest, Stone, and Sokolov (2006). Table 2 reports the initial uncertainty. The temperature equation (13) becomes:

\[
H_{t+1} = \beta_1 T_t + \beta_3 (O_t - T_t) + \nu_{t+1} \equiv \beta' X_t + \nu_{t+1}, \quad X_t \equiv \begin{bmatrix} T_t \\ O_t - T_t \end{bmatrix}. \tag{34}
\]

Here since \( \zeta \) is uncertain, so is \( \beta_3 \). The updating rules are now:

\[
\beta = \begin{bmatrix} \beta_1 \\ \beta_3 \end{bmatrix} \sim N(\mu, S), \tag{35}
\]
\[ \mu_{t+1} = S_{t+1} \left( S_t^{-1} \mu_t + \frac{1}{\sigma^2} H_{t+1} X_t \right), \]
\[ S_{t+1} = \left( S_t^{-1} + \frac{1}{\sigma^2} X'_t X_t \right)^{-1}. \]

Here \( \mu \) is now a \( 2 \times 1 \) vector consisting of the mean estimate of the feedback parameter and the mean estimate of the ocean heat uptake parameter, and \( S \) is now a \( 2 \times 2 \) variance-covariance matrix.

Considerably more uncertainty now exists, so conventional learning requires many more periods. Table 3 reports that conventional learning of the true climate sensitivity almost triples to 294.67 periods. Note that even when conventional learning is complete for the climate sensitivity, typically significant uncertainty remains about the ocean uptake parameter. The true parameter is very close to zero and the ocean changes extremely slowly, so it takes many periods until the ocean temperature is high enough to cause a large enough effect on temperature to be seen amidst the noisy weather shocks. However, we are concerned here with tail learning. That is, we are concerned only with ruling out extremely high values of the climate sensitivity, not eliminating all uncertainty. Table 3 shows that tail learning requires only slightly more observations than in the base case. Observations near the mean still provide evidence against the tail, although the planner must also rule out the unlikely possibility that the observation near the mean was the result of a very high climate sensitivity offset by a very strong ocean heat uptake.

In sum, our key result that tail learning is relatively quick is robust to more complicated climate models and added uncertainty. These environments extend the time required for complete learning, but have only small effects on tail learning.

Learning the climate sensitivity precisely is a slow process. According to the prior literature with thin tails, slow learning indicates the optimal policy under learning is unlikely to be much different than the optimal policy without learning. Since learning is slow, the planner acts using current information. However, the above analysis shows that with learning the planner rejects extreme values relatively quickly, unless the true value is large. With fat tails, the extreme values drive current policy. Therefore, the learning is potentially much more policy relevant with fat tails. The next section makes these ideas precise.
5 Results: Optimal Policy

5.1 Optimal Insurance

In this section, we examine how learning and fat tailed uncertainty affect the optimal emissions policy. The upper left panel of Figure 5 plots optimal emissions for the case where the true feedback parameter is equal to the mean of the initial prior \((\Delta T_{2x} = 2.76)\). The circle line corresponds to perfect information, where the initial variance of the prior is set to zero. In this case, optimal emissions increases for a short period of time and is then decreasing. The initial world capital stock is 68% of its steady state level.\(^{33}\) Therefore, the planner postpones most emissions abatement until the capital stock has converged and more resources are available. Both damages and costs are a fraction of world GDP, so an increase in GDP affects both damages and costs equally. However, more wealth implies more consumption, which decreases the marginal utility of consumption. Therefore, abatement becomes more attractive. In addition, each year’s emissions have only a small effect on the GHG concentration, so the planner does not incur much additional damage by waiting.

The line with squares shows the optimal policy under uncertainty with learning. As the planner learns the true value, the emissions under learning and uncertainty converges to emissions given perfect information. Notice that emissions under learning are initially below the perfect information case. The planner insures, emitting less than under perfect information just in case climate change turns out to be severe. Emissions are initially 19.3% lower under learning than under perfect information (Table 4), but are only 1.1% lower by 2021. Uncertainty matters for a relatively short period of time. In Figure 5, the true value of the feedback parameter equals the mean of the prior distribution. Therefore, the planner quickly rejects values of the climate sensitivity in the fat tail, and the policy approximately converges to the perfect information case by 2025.\(^{34}\)

The plus line corresponds to the no learning case. In this case, the learning parameters \(\mu\) and \(S\) are not updated, despite the new information. In addition, the model is solved so that the planner knows \(\mu\) and \(S\) will not be updated. Therefore, the no learning case differs from the learning case, even in the initial period when the state vector is identical for both policies. Emissions are lowest under no learning. The planner must insure more by

\(^{33}\)The capital stock converges quickly as if the temperature were approximately constant. The capital stock then adjusts in response to changes in damages, which change slowly over many years.

\(^{34}\)Note that emissions are slightly above the perfect information case for a short period of time. This is because the planner has under-emitted relative to perfect information during the learning process. The planner can therefore over-emit after rejecting the fat tail to reach the same steady state stock of GHGs.
reducing emissions without learning, because the planner knows that she cannot adjust later as more information is revealed. Therefore, learning reduces the need for climate insurance. Emissions for the no learning case are initially 38% below emissions given perfect information, whereas emissions under learning are 19.3% below perfect information. Therefore, learning reduces climate insurance by about 50% (Table 4).

Emissions under no learning are below the true optimal emissions for the entire time path. The planner under no learning must continue to insure, whereas under learning climate insurance is required for only a short time. Emissions for the no learning case are 48% below perfect information in 2020, whereas emissions under learning are only 1.42% below certainty at this point.

Given a true value of $\beta_1$ equal to the mean of the prior, the planner rejects the tail at the 99.9% confidence level after 7.2 periods (see Figure 2 and Table 4). Emissions under learning is only 4.64% lower than emissions under certainty after 8 periods, and is only 1.07% lower after 16 periods (Figure 5 and Table 4). In contrast, conventional learning is not complete for 63.8 years if $\beta_1$ equals the mean of the prior (see Figure 2 and Table 4). Therefore, optimal emissions policy is more sensitive to tail learning. Although uncertainty is present and the climate sensitivity is difficult to pin down precisely, the planner rejects values in the tail quickly and thereafter proceeds as if the planner was certain that $\beta_1$ equals the mean of the current prior.

The upper right panel of Figure 5 gives the emissions abatement rate, which is increasing over time under certainty. This is similar to the “ramp up” strategies found for example by Nordhaus (2008). The abatement rate under learning is initially more stringent, but converges to the certainty case at approximately the same rate as emissions. Without learning, the abatement rate remains elevated as the planner continues to insure. The initial abatement rate under learning is 19.3%, which is similar to values found in the literature. In contrast, without learning the initial abatement rate is 38.3%, approximately twice as high (Table 4). The lower right panel of Figure 5 shows the carbon tax. The initial carbon tax is $46.1 per ton of carbon, also within the range of typical estimates.

The lower left panel of Figure 5 gives the path of temperature increases. The policies with learning and perfect information differ very little in terms of the GHG stock, since the planner adjusts emissions to keep the economy on the same GHG stock trajectory after learning takes place. Emissions and the GHG stock are lower under no learning due to the insurance, resulting in a smaller temperature increase.

Figure 6 shows the mean abatement for an experiment with the true $\Delta T_{2\times} = 2$ rather
than 2.76. The planner under perfect information reduces emissions by less than one percent initially, which increases to a maximum reduction of only 1.5%. Under learning, the planner begins with the same information set as when the true $\Delta T_{2x} = 2.76$, and therefore chooses the same initial policy as in the upper right panel of Figure 5. As new information arrives which decreases the prior, the planner under learning decreases abatement.

The planner rejects the fat tail sooner here, since a relatively small $\lambda$ implies $\beta^L_1$ is not close to $\beta^*_1$. Emissions under learning converge to within 0.4% of emissions under perfect information after 4 periods. Similarly, tail learning is complete after 3.65-5.07 periods at the 99% and 99.9% confidence levels, respectively (Figure 2, left panel). In contrast, conventional learning is not complete until after 57.0 periods (Figure 2, right panel). After ruling out the fat tail, the planner proceeds along a path very close to perfect information, even though uncertainty remains.

Without learning, the initial conditions are identical as when the true value is 2.76. Therefore, the initial abatement without learning is identical in the upper right panel of Figure 5 and Figure 6. Since the planner does not update the prior, abatement is above perfect information indefinitely. In fact, abatement without learning is quite similar in Figures 5 and 6. The insurance motivation is the main determinant of emissions policy without learning, and differences in emissions policy caused by the different temperature trajectories are minor.

Figure 7 gives the results of the experiment with the true $\Delta T_{2x} = 5$. Given that GHG concentrations are projected to more than double, this represents a high damage case. Under certainty, the planner responds to the high $\Delta T_{2x}$ by severely limiting emissions. The upper right panel of Figure 7 indicates the planner reduces emissions initially by 52.4% and by 72.2% in the steady state under certainty.

Initial optimal emissions policy with learning is unchanged from the previous cases, since the initial beliefs are unchanged. Emissions fall over time as the planner increases the mean belief of the climate sensitivity over time in response to higher than expected temperatures. The emissions policy under learning converges to within 1.1% of emissions policy under certainty in 17 periods. From Figure 2, tail learning is complete after 16.09 periods at the 99.9% confidence level. In contrast, learning is not complete for about 105.6 years (see Figure ??) for a true value of $\Delta T_{2x} = 5$. The planner learns that the climate sensitivity is much higher than the mean of the initial prior, but also learns that still higher values are unlikely.

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35However, the optimal emissions path under certainty limits GHG concentrations to only 55% above preindustrial levels when $\Delta T_{2x} = 5$. 

24
even if the exact value is difficult to pin down. Optimal emissions converges to the certainty case, and the remaining uncertainty is not relevant for policy.

Abatement under learning is about 0.6%-0.7% above abatement under certainty from 2022 to 2270. Under learning the planner has over-emitted relative to perfect information in the first 16 years, and must therefore under emit relative to certainty to bring GHG concentrations to the same steady state trajectory. The planner smooths out the error correcting over a considerable period because abatement costs are convex and because of consumption smoothing.

For all three true values of $\Delta T_{2x}$, the upper tail of the distribution is initially policy relevant. Abatement under learning is significantly above abatement when the climate sensitivity is known with certainty (upper right panel of Figure 5). However, learning quickly reduces the mass of the upper tail of the distribution, and emissions quickly converge to the certainty case.

### 5.2 Fat Tails and Variance

Figures 1 and 4 show the evolution of the posterior PDF for $\Delta T_{2x}$ when the true value is 2.76°, after 0, 10, and 50 periods, respectively. We also plot the normal distribution with identical parameters on each graph, to emphasize the fat tail of the distribution of the climate sensitivity. Contrasting Figures 1 and 4, we see that after only 10 periods the mass of the fat tail shrinks considerably. Values of $\Delta T_{2x}$ above 3.5 are very unlikely.

However, as Weitzman (2009b) points out for the case of the $t$ distribution, the tail remains fatter than normal for any finite number of observations. Here our distribution is not $t$, but the same result applies. Condition (19) holds for any $S > 0$, so the tails remain fat for any finite number of observations. After 50 years, the mass of the tail barely visible on Figure 4, but nonetheless, the tail is still fatter than normal.

Section 5.1 shows fat tails are relevant initially for optimal emissions policy. Emissions with learning are 19.3% below the certainty case. Further, when a climate sensitivity above 4.26 is rejected at the 99.9% level, emissions are within 1% of certainty, even though fat tails remain for all finite observations. Therefore, what is important for emissions policy is not the existence of fat tails, but the mass of the fat tail, which is a function of the variance. Learning is relevant since learning affects the variance.

---

36 Note that the variance of $\lambda$ is infinite. Figures 1 and 4 plot $h(\lambda; \mu, S)$ along with a normal distribution with mean $\mu$ and variance $S$. 
6 Sensitivity Analysis

The climate sensitivity is considered a major source of uncertainty in integrated assessment models (Kelly and Kolstad 1999b). Nonetheless, other sources of uncertainty also exist, especially the level and convexity of damages (Weitzman 2009b) and the discount factor (Nordhaus 2008). Higher or more convex damages or a discount factor closer to one all increase the benefits of abatement. Therefore, we examine a detailed sensitivity analysis which varies the discount factor, and briefly report a sensitivity analysis with respect to a damage parameter, which has similar effects.

6.1 Discount Factor

The first sensitivity analysis decreases the pure rate of time preference from 0.05 to 0.03, or alternatively increasing $\beta$ to 0.97. Figure 8 graphs the mean optimal abatement rate for the high discount factor case. Contrasting the upper right panel of Figure 5 with the upper right panel of Figure 8, we see that optimal abatement is higher with the higher discount factor, as expected. Optimal current emissions under certainty in the initial period fall by 24.33% when the discount factor increases from 0.95 to 0.97. Emissions under learning and no learning fall by 39.63% and 35.37%, respectively, when the discount factor increases.

Since damages are potentially greater, optimal insurance also increases. Table 5 indicates current emissions under no learning are 47.31% lower than emissions under certainty for the high discount factor case, versus 38.31% for the low discount factor case. Similarly, emissions under learning are 35.62% lower under learning than emissions under certainty for the high $\beta$ case, versus 19.3% for the low $\beta$ case. Although damages are potentially greater, learning becomes somewhat less important, since the high discount factor means large emissions reductions are optimal for a wide range of outcomes of the learning process. Learning reduces the need for insurance by only 24.71% in the high $\beta$ case, versus about 50% in the low $\beta$ case.

Both tail and conventional learning are slightly slower for the high $\beta$ case. Kelly and Kolstad (1999b) prove that more restrictive climate policies slow learning because the climate change signal is less pronounced amidst the noisy weather. Table 5 shows that this effect is small, however. The mass of the fat tail is 1.4% in 2010 when $\beta = 0.97$, versus 1.3% when $\beta = 0.95$. The 95% confidence interval is [2.37, 3.31] in 2015 when $\beta = 0.97$ versus [2.39, 3.30] when $\beta = 0.95$. Although learning is slightly slower, emissions under learning converges faster to perfect information when $\beta = 0.97$. Table 5 and the upper right panel
of Figure 8 indicate emissions under learning converges to within 1% of emissions under certainty in 7 years, versus about 17 period when $\beta = 0.95$.

## 6.2 Damage Parameter

The level and convexity of damages from climate change is also uncertain (Weitzman 2009a). Both parameters in the damage function affect both the convexity and the level of damages. Here we alter $b_1$ so that the damage from a 2°C increase in the temperature by 1% (1.19% to 2.19%). Increasing $b_1$ also makes the damage function more convex.

Table 5 shows the results are similar to the high discount factor case (we omit the graphs of the high damage case, which are similar to the high discount factor case). The possibility of higher damages raises the optimal insurance: current emissions under no learning are 48.45% of emissions under certainty versus 38.31% using the baseline parameters. Learning becomes less important as higher damages imply large emissions reductions regardless of the outcome of learning. Learning reduces the current optimal insurance by only about 15%. The speed of learning is very similar to the high discount factor case, slightly slower than the baseline case, because higher abatement reduces the climate change signal amidst the noisy weather.

## 6.3 Cost Parameter

The last sensitivity analysis decreases the cost parameter, $\epsilon$, by 7%. By decreasing $\epsilon$, the cost function becomes less convex, but abatement becomes less expensive. When the true value is the mean of the prior, abatement rises under learning from 19.3% to 25.4%, while abatement under no learning rises from 38.3% to 42.3%. Therefore, learning reduces climate insurance by 40% as opposed to a 50% reduction given the base parameters. Nonetheless, even if costs are lower, fat tailed uncertainty remains important for near term climate policy and learning significantly reduces the need for climate insurance.

Overall, sensitivity analysis indicates that if the discount factor is closer to one, damages are greater or more convex, or costs are less convex, the main results remain. Fat tails are initially policy relevant, with high abatement for insurance purposes, learning significantly reduces the need for insurance, and learning is fast in that the fat tail is rejected quickly, although significant uncertainty remains for decades, that uncertainty is not relevant for optimal policy. However, changing the parameters in a way which increases the optimal abatement reduces the importance of learning as high abatement becomes optimal regardless
of what is learned.

7 Concluding Remarks

In this paper, we study the effect of a possible high climate sensitivity on near term optimal climate change policy, accounting for learning and uncertainty in the climate system. We find three major results. First, fat tails are initially policy relevant in that near term GHG emissions policy is much more restrictive when the planner accounts for fat tailed uncertainty in the climate sensitivity (a 38% reduction in emissions). Second, when the planner accounts for learning, the near term emissions reduction falls by half to only 19.3%. Third, although learning as conventionally defined in the literature is slow, learning quickly reduces the mass of the fat tail. Optimal emissions policy is much more sensitive to the mass of the fat tail than the uncertainty around the mean of the prior. Therefore, optimal emissions policy converges quickly to perfect information, even though some uncertainty remains for decades.

The planner knows values of the climate sensitivity in the tail of the prior distribution will be rejected quickly at a high level of confidence if the true climate sensitivity is moderate. If the climate sensitivity is high the planner can quickly reject still higher values, and quickly adjusts emissions to get back on the optimal temperature trajectory. The planner has an option to essentially purchase climate insurance: by paying to limit GHG emissions today, the planner prevents GHG concentrations from rising, which in turn prevents the possibility of very high temperature changes. Without learning, the planner takes out a significant amount of insurance. However, with learning the planner insures about 50% less. First, learning quickly rejects values of the climate sensitivity in the fat tailed part of the prior distribution, if the true climate change is moderate. Second, the planner has time to adjust emissions to keep the economy on the same GHG stock trajectory. Climate insurance under learning in most cases falls to less than 1% after about 17 years as the planner reduces the mass of the tail end of the distribution and the remaining uncertainty is not important for emissions policy.

Our results on the speed of tail learning stand in contrast to the large uncertainties found by aggregating large physical global circulation models (GCMs). Learning in our framework is observational, whereas GCMs have only recently been tested against the data (Nordhaus 2009). Our results indicate that further empirical testing of GCMs has the potential to quickly rule out relatively extreme models, and that doing so has important
policy implications.

Several caveats are in order. First, for computational reasons, our model of the climate system is highly simplified. For example, we do not include a separate ocean layer. Nonetheless, we computed an optimal policy assuming the ocean temperature is constant, but simulated the model and learning with an ocean layer. The results are not much different since learning here is about isolating the upward trend in the atmospheric temperature from the stochastic weather shocks. The ability of learning to do so is only marginally affected by the ocean and other climate processes with long lags, since the ocean temperature is essentially constant on a year to year basis. Thus our results are robust to the addition of longer lags to the climate model.

Second, we consider only a single uncertainty, the climate sensitivity. Climate change has many uncertainties, including the parameters of the damage function, the heat capacity of the ocean, etc. In general, multiple simultaneous uncertainties slows learning. Section 4.3 indicates that conventional learning slows with multiple uncertainties much more than tail learning.

Third, our model has no irreversibilities, tipping points, etc. The existence of irreversibilities makes the planner much more cautious, which increases insurance with or without learning. Learning would certainly still reduce climate insurance in this case, but by less as the planner may not be able to correct a mistake of initially over-emitting. We leave this interesting extension to future research.

Fourth, in our model the planner estimates climate feedbacks using current data. If the climate is subject to regime shifts which occur in the far future, then it might be difficult to learn about the existence of regime shifts today. However, if the process which causes the regime shift is observable today, then our model still applies. Suppose for example, the climate sensitivity is different if the polar ice caps melt as sunlight no longer reflects back into space as efficiently (the albedo effect). If one can estimate the albedo effect by observing changes in ice cover and changes in temperature, then the planner can learn the albedo effect before a regime shift to a world without polar ice caps occurs.

Regardless, our main results are likely robust to any of these extensions. Fat tails matter for climate policy, even if the distribution has a truncation point. Nonetheless, we show that learning significantly reduces the influence of fat tails, especially over the near term. Given these results it is important for policy makers to maintain policy flexibility, and to stand ready to quickly adjust the emissions policy as new information arrives.

Finally, our results have interesting potential implications for recent research which finds
fat tailed uncertainty in other contexts (equity market returns, banking crises, etc.). Fat tails in other contexts is typically modeled as an exogenous property of the return distribution rather than an endogenous implication of parameter uncertainty. Our results show that if the exogenously imposed fat tails are in fact the result of parameter uncertainty, then learning has the potential to reduce fat tailed uncertainty over time, which limits the risk premium of fat tailed uncertainty. We leave this interesting possibility to future research.
Appendix A  Balanced Growth

A.1 Existence

We first show that a necessary condition for the existence of a balanced growth path is that the inverse of the emissions intensity grows at the same rate as the productivity adjusted population. From equation (6),

\[ y_t = \frac{Y_t}{A_t L_t} = \left( \frac{B_t}{A_t L_t} \right)^\varepsilon \left( \frac{K_t}{A_t L_t} \right)^\theta E_t^\varepsilon, \]

\[ = \left( \frac{B_t}{A_0 L_0 (1 + \phi)(1 + \eta)} \right)^\varepsilon k_t^\theta E_t^\varepsilon. \]  

From equation (39), if \( B_t \) grows at a rate greater than \( (1 + \eta)(1 + \phi) - 1 \), then output net of abatement spending per productivity adjusted person goes to zero. However, \( y_t = 0 \) implies no resources are available for consumption, which is inconsistent with the existence of a balanced growth path. Emissions is proportional to output which is growing at the same rate as the productivity adjusted population. So from equation (3), either \( u_t \to 1 \) or \( E_t \to \infty \). Equations (10), (12), and (8) imply the later results in infinite temperature and damages, which implies output net of damages is zero. Equation (5) implies the former requires all income to be spent on emissions abatement, and so output net of abatement spending, \( y_t \), is zero.

Similarly, if \( B_t \) grows at a rate less than the productivity adjusted population, then \( y_t \to \infty \), which is inconsistent with the existence of a balanced growth path for \( y_t \). Equation (3) implies that emissions approach zero as \( t \to \infty \) for any \( u_t \). Hence \( u_t \to 0 \) is optimal, \( Y_t \to Q_t \), and the economy approaches the standard balanced growth path in the Ramsey growth model without climate change. This case is uninteresting because exogenous emissions saving technology improvements solve the climate change problem and also inconsistent with the idea that R&D efforts flow to the sector where innovations are the most productive (see section A.3).

A.2 Derivation

First, equation (17) implies \( S_t \to 0 \). Next, equations (15) and (16) imply:

\[ \mu_{t+1} = \frac{\sigma^2 \mu_t + S_t \left( \beta_1 T_t^2 + T_t \nu_{t+1} \right)}{\sigma^2 + S_t T_t^2}, \]
\[
E[\mu_{t+1}] = \frac{\sigma^2_t E[\mu_t] + S_t \beta_1 T_t^2}{\sigma^2_t + S_t T_t^2}.
\] (41)

Therefore, the mean of the invariant distribution \(E[\mu_{t+1}] = E[\mu_t] = \bar{\mu}\) satisfies \(\bar{\mu} = \beta_1\). Hence the prior distribution converges to certainty. Therefore we can compute the balanced growth path assuming certainty.

All environmental variables are constant in the balanced growth path. Therefore equations (13) and (23) imply:

\[
\bar{m} = \frac{\gamma}{\delta_m M_p} \hat{\bar{E}},
\] (42)

\[
\bar{T} = \frac{\beta_2}{1 - \beta_1} \log (\bar{m}).
\] (43)

Next, the first order conditions for (21) are:

\[
U_c (c) (1 + \eta) (1 + \phi) = \hat{\bar{E}} [v_k (s')],
\] (44)

\[
U_c (c) D (T) f_E (k, \hat{\bar{E}}) = -\hat{\bar{E}} \left[ v_m (s') + \frac{v_T (s') \beta_2}{m'} \right] \frac{\gamma}{M_p}.
\] (45)

The envelope equations are:

\[
v_k (s) = U_c (c) \left( D (T) f_k (k, \hat{\bar{E}}) + 1 - \delta_k \right),
\] (46)

\[
v_m (s) = \hat{\bar{E}} \left[ v_m (s') + \frac{v_T (s') \beta_2}{m'} \right] (1 - \delta_m).
\] (47)

\[
v_T (s) = U_c (c) D_T (T) f (k, \hat{\bar{E}}) + \hat{\bar{E}} [v_T (s') \beta_1].
\] (48)

Assuming certainty equivalence, in the steady state \(s = s' = \bar{s}\) and the same for \(k, m, T,\) and \(\hat{\bar{E}}\). Using (46)-(48) to eliminate the derivatives of the value function, equations (44) and (45) reduce to:

\[
D (\bar{T}) f_k (\bar{k}, \hat{\bar{E}}) + 1 - \delta_k = \frac{(1 + \phi) \sigma}{\beta},
\] (49)

\[
\bar{E} = \Gamma \bar{m} \hat{\bar{E}}^{\delta_2 - 1},
\] (50)

\[
\Gamma \equiv \frac{M_p \left( 1 - \hat{\beta} (1 - \delta_m) \right) (1 - \hat{\beta} \beta_1) \epsilon}{\beta_2 \gamma b_1 b_2}.
\] (51)

Equations (42), (43), (49), and (50) constitute the steady state for \(k, m, T,\) and \(\hat{\bar{E}}\), with \(\bar{m} = \beta_1, \bar{S} = 0, \bar{y} = f (\bar{k}, \hat{\bar{E}}),\) and \(\bar{c}\) given by the resource constraint (8). Since \(\bar{y} = \frac{Y}{A_L}\) is
constant, $Y$ must grow at the same rate as $AL$ in the balanced growth path, and the same for $C$ and $K$.

### A.3 R&D Flows and Productivity

Suppose an innovation in labor-savings productivity that leads to a marginal increase in $A_tL_t$. From equation (6), the value of such an innovation is:

$$\frac{\partial Y_t}{\partial A_tL_t} = (1 - \theta - \epsilon) \frac{Y_t}{A_tL_t} = (1 - \theta - \epsilon) y_t. \quad (52)$$

Similarly, for an emissions-saving innovation:

$$\frac{\partial Y_t}{\partial B_t} = \epsilon \frac{Y_t}{B_t} = \epsilon k_t^\theta E_t^{\epsilon} \left( A_tL_t \right)^{1-\epsilon}. \quad (53)$$

Let $g_B$ denote the growth rate of $B_t$. From section A.1, if $B_t$ grows slower than $A_tL_t$, then $y_t \rightarrow 0$ and the return to labor-savings innovations approaches zero. Therefore, $g_B \geq (1 + \eta)(1 + \phi)$ is necessary for positive spending on labor-saving R&D. Conversely, equation (53) implies that if $B_t$ grows faster than $A_tL_t$, the return to emissions-savings productivity approaches zero. Hence $g_B \leq (1 + \eta)(1 + \phi)$ is necessary for positive spending on emissions-savings R&D. Therefore, if $g_B < (1 + \eta)(1 + \phi)$, all R&D spending is eventually emissions savings, and if $g_B > (1 + \eta)(1 + \phi)$, all R&D spending is eventually labor saving. Combining these two results, a long run equilibrium requires $g_B = (1 + \eta)(1 + \phi)$.

### Appendix B Computational Solution Method

We use a multidimensional spline approximation of the value function and value function iteration to solve the dynamic program. That is, we replace the value function on the right hand side of the Bellman equation with a spline approximation $\hat{v}(x; p_m)$, where $x \equiv [k, m, T, \mu, S]$ is the vector of state variables and $p_m$ denotes the vector of parameters of the spline at iteration $m$. We use Gaussian quadrature to approximate the integral in (21). Let $\{b_{ij}\}_{j=1}^J$ denote the base points and $\{w_{ij}\}_{j=1}^J$ the weights. The approximation to the Bellman’s equation is then:

$$v_{m+1}(x_i) = \max_{k', E} \left\{ \frac{c^{1-\sigma}}{1-\sigma} + \beta \sum_{j=1}^J \hat{v}(x'_i(b_{ij}); p_m) N \left(b_{ij}, \mu_iT_i, T_i^2S_i + \sigma_v^2\right) w_{ij} \right\}, \quad (54)$$
subject to (22)-(26).

### B.1 Summary of the Algorithm

Below is a summary of the algorithm:

1. Initialization. We form a grid $\tilde{x} = \{x_i\}_{i=1}^I$ of feasible state variables. Table 6 gives the collocation nodes which form grid. The curse of dimensionality limits the number of grid points such that the model solves within a reasonable amount of time. Therefore, the selection of collocation nodes cannot be arbitrary. We use a relatively large number of collocation nodes for $T$ and $\mu$ since uncertainty over the climate sensitivity with fat tails implies that a very wide range of values for $T$ and $\mu$ are possible outcomes. In contrast, the range of $S$ is small and decreases monotonically after each time period and therefore requires the least number of grid points. In addition, the value function has significant curvature in the $k$, $T$, and $\mu$ dimensions, which requires additional grid points to estimate accurately. In general, solving the model is a time consuming process where we first solve the model on an evenly spaced grid. Then, slice plots of the model were examined which hold some state variables fixed and vary others. Grid points were added/subtracted where the estimate value function exhibited/did not exhibit significant curvature. Then the model was re-solved and the grid adjusted again. It is also important to allocate grid points near the initial condition, $x_0$.

2. Spline Initialization. We use a cubic spline approximation of the value function. The cubic spline has 3 parameters for each collocation node ($I \cdot 3^{\text{dim}(x)}$ total parameters), which ensure that the spline fits the value function exactly at each grid point, and that the spline is twice continuously differentiable. We choose the initial approximation parameters $p_0$ to fit an initial value function $u(d_0, x)$, where $d_0 = [k'_0, E_0] = [0, m'^{-1}(m, m)]$ are the initial guesses for the decision rule (no savings and enough emissions to keep the stock of GHGs constant).

3. Maximization. For each grid point $x_i$, we use $\hat{v}(x_i; p_m)$ to find $v_{m+1}(x_i)$ using the approximate Bellmen equation (54).

4. Numerical Integration. Each numerical integration uses $J = 8$ base points and the upper and lower bounds of integration are set to the mean of $H' \pm 3$ standard deviations. Since the mean and standard deviation are state dependent, so are the upper and lower bounds.
5. Approximation. We update the approximation parameters $p_{m+1}$ using the data $x_i$ and $v_{m+1}(x_i)$.

6. Termination. The algorithm stops if $\|v_{m+1}(x_i) - v_m(x_i)\| \leq 0.0001$, otherwise we increment $m$ by one and return to step 3.

B.2 Upper Limits of the Integration and the State Variables

The accuracy of the solution method relies in part on a bounded state space. If a particular value for the choice variables being evaluated by the optimization algorithm implies a future state variable $x'$ exceeds the maximum collocation node, then the spline forecast $\hat{v}(x')$ is “out of sample.” The forecast is out of sample in the sense that the spline forecasts using grid points smaller than $x'$, which is generally less accurate than interpolating between grid points.

Most state variables have natural bounds. The capital stock is bounded by the maximum sustainable capital stock. The variance is monotonically decreasing. Hence the initial value is an upper bound. The upper bound for the stock of GHGs is the steady state level of GHGs given no abatement and the maximum sustainable capital stock. An upper bound for $T$ exists so long as $\mu$ is bounded below 1.

Establishing a bound for $\mu$ is problematic. Since $H'$ is a normally distributed random variable with mean $\mu T$, equation (25) implies that $\mu'$ is normally distributed with mean $\mu$ (a requirement for rationality is that $E[\mu'] = \mu$). Let $\bar{\mu}$ denote the maximum collocation point for $\mu$ (here 0.9). Then, for $\mu = \bar{\mu}$, half the distribution of $\mu'$ is outside the grid, regardless of the maximum collocation point. One solution is to assume that any observed $H'$ which implies $\mu > \bar{\mu}$ in fact implies $\mu = \bar{\mu}$. The interpretation would be that the planner knows that $\mu$ cannot exceed $\bar{\mu}$, so the planner knows that large values of $H'$ must be caused by large weather shocks. However, if $\mu = \bar{\mu}$ and the planner believes that $\mu' = \bar{\mu}$ with probability 0.5, then much of the uncertainty is eliminated. In reality, if $\mu = \bar{\mu}$, then a reasonable model of beliefs would admit the possibility that $\beta_1$ is larger than $\bar{\mu}$, even though such a large value of $\mu$ is unlikely given the current prior in which $\mu = 0.65$.

Another problem is computational: if $\mu$ is bounded, then $H'$ is no longer the sum of normal distributions, creating a computational burden. The solution algorithm must perform two integrals for each evaluation of the Bellman equation, instead of one, greatly increasing the computation time.

For these reasons, we instead bound the temperature at 26.8°C. The mass of the distri-
bution above 26.8°C is collected into a mass point at 26.8°C. Establishing an upper bound
to temperature is equivalent to establishing an upper bound for $\beta_1$ in the steady state (the
chosen upper bound for temperature of 26.8°C corresponds to a maximum value of $\beta_1$ equal
to 0.964). However, outside the steady state, $\mu' > \bar{\mu}$ with positive probability, in which case
the temperature exhibits faster acceleration to the steady state, for example. Therefore, the
model maintains non-trivial uncertainty with respect to the expected future temperature
outside the steady state even if $\mu = \bar{\mu}$.

The results are not sensitive to the exact truncation point. See Costello, Neubert, Polasky,
and Solow (2010) for a justification of truncating the prior climate sensitivity distribution.

Finally, the numerical quadrature integrates over a finite interval $[b_{i1}, b_{iJ}]$. We set the
integration range to $\pm 3$ standard deviations from the mean, which varies with the state
vector.
### Appendix C  Figures and Tables

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Calibrated Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>Discount factor</td>
<td>$1.05^{-1}$</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Coefficient of risk aversion</td>
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<td>$\psi$</td>
<td>Capital share</td>
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<tr>
<td>$\epsilon$</td>
<td>Emissions share</td>
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<td>$\delta_k$</td>
<td>Capital depreciation rate</td>
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<td>$\eta$</td>
<td>Population growth rate</td>
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<td>$\phi$</td>
<td>Productivity growth rate</td>
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<td>$\delta_m$</td>
<td>GHG stock decay</td>
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<td>$b_1$</td>
<td>Damage function parameter</td>
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<td>$b_2$</td>
<td>Damage function parameter</td>
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<td>$\gamma$</td>
<td>Initial emissions intensity (GtC/trillion 2005 $)</td>
<td>4.66</td>
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<tr>
<td>$M_p$</td>
<td>Preindustrial GHG concentrations (GtC)</td>
<td>596.4</td>
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<td>$\alpha$</td>
<td>Heat capacity of the ocean</td>
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<td>$\Gamma$</td>
<td>Preindustrial temperature</td>
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<td>$\Omega$</td>
<td>Forcing parameter ($W/M^2$)</td>
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<td>$\sigma_\nu$</td>
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<td>$\max \Delta T_2$</td>
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<th>Additional parameters from robustness checks (Section 4.3)</th>
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<td>$\zeta$</td>
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<td>$\tau$</td>
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<td>$\chi$</td>
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<tr>
<td>$\sigma_\xi$</td>
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<td>$\beta_4$</td>
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Table 1: Calibrated parameter values.
State Initial Value Definition (Units)

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<th>State</th>
<th>Initial Value</th>
<th>Definition (Units)</th>
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<tr>
<td>$K$</td>
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<td>Capital stock (trillions of 2005 dollars)</td>
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<tr>
<td>$m$</td>
<td>1.36</td>
<td>GHG concentration (fraction relative to preindustrial)</td>
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<tr>
<td>$T$</td>
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<td>Atmospheric temperature ($^\circ$C above preindustrial)</td>
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<td>$\mu$</td>
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<td>Prior mean, feedback parameter (Watts per square meter, $W/M^2$)</td>
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<td>$\sqrt{S}$</td>
<td>0.13</td>
<td>Prior standard deviation, feedback parameter ($W/M^2$)</td>
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Additional initial conditions from robustness checks (Section 4.3)

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<th>Initial Value</th>
<th>Definition (Units)</th>
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<td>$O$</td>
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<td>Ocean temperature ($^\circ$C above preindustrial)</td>
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<td>$\nu_0$</td>
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<td>Weather shock ($^\circ$C above preindustrial)</td>
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<td>$T_{t-10}$</td>
<td>0.62</td>
<td>Lagged temperature ($^\circ$C above preindustrial)</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>0.066</td>
<td>Prior mean, ocean uptake ($W/M^2$)</td>
</tr>
<tr>
<td>$S_{22}$</td>
<td>0.0088</td>
<td>Prior variance, ocean uptake ($W/M^2$)</td>
</tr>
</tbody>
</table>

Table 2: Initial conditions corresponding to 2005. Dollar units are adjusted for purchasing power parity. Initial values for $K$, $m$, and $T$ are from Nordhaus (2008). Initial values for $\mu$ and $S$ are from Roe and Baker (2007) The initial mean and variance of the prior distribution for $\Delta T_{2x}$ are infinite. For the robustness checks, the initial ocean and atmospheric temperatures are from Nordhaus (2008), and the prior variance of the ocean uptake parameter is from Forest, Stone, and Sokolov (2006).

<table>
<thead>
<tr>
<th>Model \ Confidence level</th>
<th>Tail</th>
<th>Conventional</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base model</td>
<td>8.99</td>
<td>14.58</td>
</tr>
<tr>
<td>Base model with ocean layer</td>
<td>15.84</td>
<td>24.39</td>
</tr>
<tr>
<td>Base model with ocean layer and autocorrelated shocks</td>
<td>12.94</td>
<td>20.76</td>
</tr>
<tr>
<td>Base model with ocean layer and decadal temperature lag</td>
<td>4.72</td>
<td>6.56</td>
</tr>
<tr>
<td>Base model with ocean layer and multiple uncertainties</td>
<td>15.66</td>
<td>25.42</td>
</tr>
</tbody>
</table>

Table 3: Expected years until tail learning time is complete conditional on current information. Expected number of years until the hypothesis $\Delta T_{2x} \geq \Delta T_{2x}^* + 1.5$ is rejected (and not subsequently not accepted) at the given confidence level, where $\Delta T_{2x}^*$ is the true value. Column 4 is the expected number of years until conventional learning is complete.
<table>
<thead>
<tr>
<th>Year</th>
<th>2005</th>
<th>2010</th>
<th>2015</th>
<th>2020</th>
<th>2050</th>
</tr>
</thead>
<tbody>
<tr>
<td>Emissions: Learning (% below certainty)</td>
<td>19.30</td>
<td>7.68</td>
<td>3.69</td>
<td>1.42</td>
<td>-0.49</td>
</tr>
<tr>
<td>Emissions: No learning (% below certainty)</td>
<td>38.31</td>
<td>46.3</td>
<td>48.17</td>
<td>48.02</td>
<td>45.72</td>
</tr>
<tr>
<td>Abatement: Learning (% below no learning)</td>
<td>49.55</td>
<td>81.36</td>
<td>81.36</td>
<td>75.54</td>
<td>54.76</td>
</tr>
<tr>
<td>Carbon tax: Learning (% below no learning)</td>
<td>23.44</td>
<td>40.71</td>
<td>45.50</td>
<td>46.72</td>
<td>45.96</td>
</tr>
</tbody>
</table>

| Probability $T \geq 4.26$ | 0.17 | 0.013 | 0.0002 | $< 10^{-6}$ | $< 10^{-6}$ |
| 95% confidence interval lower bound, $\Delta T_{2x}$ | 0.97 | 2.07 | 2.39 | 2.51 | 2.71 |
| 95% confidence interval upper bound, $\Delta T_{2x}$ | 7.09 | 3.57 | 3.30 | 3.19 | 3.00 |

Table 4: Difference in optimal emissions policy, learning, no learning, and perfect information. The true value is the mean of the initial prior. The first two rows give the percent difference between emissions under certainty and emissions under learning and no learning. That is the first cell of the table indicates emissions under learning are 19.3% lower than emissions under certainty. Rows 3-4 give the percent difference in policies under no learning and the policies under learning. The last two rows give the progress of tail learning versus conventional learning. All results are the mean of 150 simulations.

<table>
<thead>
<tr>
<th>Year</th>
<th>2005</th>
<th>2020</th>
<th>2050</th>
<th>2005</th>
<th>2020</th>
<th>2050</th>
</tr>
</thead>
<tbody>
<tr>
<td>Emissions: Learning (% below certainty)</td>
<td>35.62</td>
<td>-0.11</td>
<td>-0.24</td>
<td>34.65</td>
<td>-1.41</td>
<td>-0.45</td>
</tr>
<tr>
<td>Emissions: No learning (% below certainty)</td>
<td>47.31</td>
<td>44.63</td>
<td>42.96</td>
<td>48.45</td>
<td>45.02</td>
<td>42.35</td>
</tr>
<tr>
<td>Abatement: Learning (% below no learning)</td>
<td>11.99</td>
<td>31.42</td>
<td>25.07</td>
<td>14.97</td>
<td>32.67</td>
<td>25.02</td>
</tr>
<tr>
<td>Carbon tax: Learning (% below no learning)</td>
<td>18.07</td>
<td>44.01</td>
<td>42.95</td>
<td>21.01</td>
<td>45.13</td>
<td>42.67</td>
</tr>
<tr>
<td>Probability $T \geq 4.26$</td>
<td>0.17</td>
<td>$&lt; 10^{-5}$</td>
<td>$&lt; 10^{-6}$</td>
<td>0.17</td>
<td>$&lt; 10^{-5}$</td>
<td>$&lt; 10^{-6}$</td>
</tr>
<tr>
<td>95% confidence interval lower bound, $\Delta T_{2x}$</td>
<td>0.97</td>
<td>2.50</td>
<td>2.69</td>
<td>0.97</td>
<td>2.49</td>
<td>2.68</td>
</tr>
<tr>
<td>95% confidence interval upper bound, $\Delta T_{2x}$</td>
<td>7.09</td>
<td>3.21</td>
<td>3.02</td>
<td>7.09</td>
<td>3.21</td>
<td>3.03</td>
</tr>
</tbody>
</table>

Table 5: Difference in optimal emissions policy, learning, no learning, and perfect information, when $\beta = 0.97$ and $b_1 = 0.006$. The true value is the mean of the initial prior. The first two rows give the percent difference between emissions under certainty and emissions under learning and no learning. Rows 3-4 give the percent difference in policies under no learning and the policies under learning. The last two rows give the progress of tail learning versus conventional learning. All results are the mean of 150 simulations.
<table>
<thead>
<tr>
<th>State variable</th>
<th>Number of Collocation nodes</th>
<th>Collocation nodes</th>
<th>Normalized 2005 value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>35</td>
<td>[0.001; 0.0025; 0.005; 0.0075; 0.01; 0.02; 0.025; 0.03; 0.04; 0.05; 0.065; 0.08; 0.1; 0.13; 0.16; 0.25; 0.34; 0.44; 0.49; 0.54; 0.59; 0.64; 0.74; 0.84; 1; 1.1; 1.2; 1.4; 1.6; 1.8; 2.5; 5; 7; 10; 11.18]</td>
<td>0.69</td>
</tr>
<tr>
<td>$m$</td>
<td>6</td>
<td>[1.01; 1.36; 1.99; 2.97; 3.95; 4.93]</td>
<td>1.3563</td>
</tr>
<tr>
<td>$T$</td>
<td>12</td>
<td>[0.01; 0.73; 2.45; 4.89; 7.33; 9.76; 12.2; 14.64; 17.08; 19.52; 21.96; 26.83]</td>
<td>0.73</td>
</tr>
<tr>
<td>$\mu$</td>
<td>12</td>
<td>[0.01; 0.11; 0.21; 0.31; 0.41; 0.5; 0.6; 0.65; 0.7; 0.8; 0.85; 0.9]</td>
<td>0.65</td>
</tr>
<tr>
<td>$\sqrt{S}$</td>
<td>4</td>
<td>[0; 0.01; 0.16; 0.2]</td>
<td>0.13</td>
</tr>
</tbody>
</table>

Table 6: Collocation nodes. The grid consists of all combinations of the collocation nodes ($I = 120,960$ total grid points). Capital is further normalized so that the steady state capital is one.
Figure 1: Prior PDF for the climate sensitivity. For comparison, the figure also plots the normal distribution $\lambda \sim N(\mu_0, S_0)$, with $\Delta T_{2x} = \Omega \lambda$.

Figure 2: Learning time as a function of $\Delta T_{2x}^*$, tail and conventional learning (mean of 150 and 1200 simulations, respectively, for each value of $\Delta T_{2x}^*$). Tail learning is the mean time to reject $\Delta T_{2x} \geq \Delta T_{2x}^* + 1.5^\circ C$ at the critical value. Conventional learning is the mean time required to reject $\Delta T_{2x} \leq 0.95 \cdot \Delta T_{2x}^*$ and $\Delta T_{2x} \geq 1.05 \cdot T_{2x}^*$ with 95% confidence.
Figure 3: Sample posteriors. Each point is the value of $\Delta T_{2X}$ implied by the mean of the posterior distribution given a sample path of weather shocks (true $\Delta T_{2X} = 2.76^\circ$). Each figure plots 150 sample paths.

Figure 4: Posterior PDF of $\Delta T_{2X}$ after 10 and 50 periods, mean of 150 simulations, true $\Delta T_{2X} = 2.76^\circ$. Each figure also plots a normal PDF, $\Delta T_{2X} = \Omega \lambda$, $\lambda \sim N(\mu_t, S_t)$, $t = 10, 50$. 

42
Figure 5: Emissions (GtC), abatement rate, temperature (°C above preindustrial), and carbon tax ($/ton), true $\Delta T_{2\times} = 2.76$. Mean of 1000 runs.
Figure 6: Optimal emissions abatement rate, true $\Delta T_{2x} = 2$. Mean of 1000 runs.
Figure 7: Emissions (GtC), abatement rate, and temperature (°C above preindustrial), true
$\Delta T_{2x} = 5$. Mean of 1000 runs.
Figure 8: Optimal fraction of emissions abated, $\beta = 0.971$. Each graph is the mean of 1000 simulations.
References


