Stochastic Models in Operations Research
Session 4: Discrete Time Markov Chains

Prof. Christos Zacharias
czacharias@miami.edu

Department of Management Science
School of Business Administration, University of Miami

Coral Gables, Feb 09 2015

Today's agenda

1. Recap
2. Simulating a DTMC
3. Classification of States
4. Limiting Behavior (as $n \to \infty$)

Discrete Time Markov Chains

- A DTMC $\{X_n, n \geq 0\}$ is completely defined by
  1. **State Space S**

     State Space S is a finite or countable set of states that the random variables X_n may take on.

  2. **Transition Probabilities p_{ij}**

     $$p_{ij} = P(X_{n+1} = j | X_n = i), \text{ for all } i, j \in S, \text{ for all } n \geq 0$$

  3. **Initial probability distribution α**

     $$\alpha_i = P(X_0 = i), \text{ for all } i \in S$$

An Inventory Model (cont'd)

- Consider the inventory model from Lecture 3.
- $S = \{0, 1, 2, 3, 4, 5\}$
- Assume that the initial state is randomly chosen, i.e.,

$$\alpha = \left(\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}\right)$$

- Transition Probability Matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{cccccc} 0 & 0 & 0.1 & 0.2 & 0.4 & 0.3 \\ 0 & 0 & 0.1 & 0.2 & 0.4 & 0.3 \\ 0.3 & 0.4 & 0.3 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.4 & 0.3 & 0 & 0 \\ 0 & 0.1 & 0.2 & 0.4 & 0.3 & 0 \\ 0 & 0 & 0.1 & 0.2 & 0.4 & 0.3 \end{array}\right] \end{array}$$

- $p_{ij}^{(n)} = P(X_{k+n} = j | X_k = i)$ is given by the C-K Equations

$$\mathbf{P}^{(n)} = \mathbf{P}^n$$

probability distribution of X_n

- The probability distribution of X_n is

$$\boldsymbol{\pi}_n = \boldsymbol{\alpha}\mathbf{P}^n$$

- We denote $P(X_n = i) = \pi_n(i)$
- The expected value of X_n is given by

$$E(X_n) = \boldsymbol{\pi}_n \cdot S = \sum_{i \in S} i\pi_n(i) \quad \text{(dot product)}$$

$$\begin{aligned}
P(X_2 = 4) &= \pi_2(4) \\
&= \sum_{i=0}^{5} P(X_2 = 4 | X_0 = i) P(X_0 = i) \\
&= \sum_{i=0}^{5} p_{i4}^{(2)} \alpha_i \\
P(X_2 = 4, X_6 \geq 3, X_{11} = 0) &= P(X_{11} = 0 | X_2 = 4, X_6 \geq 3) P(X_2 = 4, X_6 \geq 3) \\
&= P(X_{11} = 0 | X_6 \geq 3) P(X_2 = 4, X_6 \geq 3)
\end{aligned}$$

discrete inverse transform method

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

- Once you get X_{n-1}, simulate X_n, given the value of X_{n-1}
- To sequentially simulate the first n transitions $X_1, X_2, ..., X_n$, we only need to simulate n iid uniforms $U_1, U_2, ..., U_n$

discrete inverse transform method

- Consider again the inventory model

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \color{blue}{\mathbf{3}} \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0.1 & 0.2 & 0.4 & 0.3 \\ 0 & 0 & 0.1 & 0.2 & 0.4 & 0.3 \\ 0.3 & 0.4 & 0.3 & 0 & 0 & 0 \\ \color{blue}{0.1} & \color{blue}{0.2} & \color{blue}{0.4} & \color{blue}{0.3} & \color{blue}{0} & \color{blue}{0} \\ 0 & 0.1 & 0.2 & 0.4 & 0.3 & 0 \\ 0 & 0 & 0.1 & 0.2 & 0.4 & 0.3 \end{bmatrix} \end{array}$$

- Suppose $X_{n-1} = \color{blue}{3}$
- How to simulate X_n?
- Simulate a $U \sim Uniform(0, 1)$.
- Get X_n according to the following table

Simulated U	$0 \leq U \leq 0.1$	$0.1 < U \leq 0.3$	$0.3 < U \leq 0.7$	$0.7 < U \leq 1$	$1 < U$
Simulated X_n	0	1	2	3	4 or 5

General Case

- $S = \{0, 1, ..., m\}$

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & \dots & m \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ m \end{array} & \begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0m} \\ p_{20} & p_{21} & \cdots & p_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m0} & p_{m1} & \cdots & p_{mm} \end{bmatrix} \end{array}$$

- Suppose $X_{n-1} = i$
- How to simulate X_n?
- Simulate a $U \sim Uniform(0, 1)$
- Get X_n according to the following table

Simulated U	$0 \le U \le p_{i0}$	$p_{i0} < U \le p_{i0} + p_{i1}$	$\dots$	$\sum_{j=0}^{m-1} p_{ij} < U \le \sum_{j=0}^{m} p_{ij}$
Simulated X_n	0	1	$\dots$	m

Accessibility

Definition: Accessibility

We say that a state j is **accessible** from state i

$$i \to j$$

if $p_{ij}^{(n)} > 0$ for some $n \geq 0$.

- We say that j is accessible from i if there is a possibility of ever reaching j from i.
- If j is not accessible from i, then

$$p_{ij}^{(n)} = 0 \ \text{ for all } n \geq 0$$

  and thus the chain started from i never reaches j.
- By definition, i is accessible from i, i.e., $i \to i$

Communication

Definition: Communication

We say that states i and j **communicate**

$$i \leftrightarrow j$$

if i is accessible from j, and j is accessible from i, i.e.,

$$j \rightarrow i \text{ and } i \rightarrow j$$

- We say that two states communicate if they are accessible from one another.
- By definition, i communicates with itself, i.e., $i \leftrightarrow i$.
- Communication is **transitive**:

$$\text{If } i \leftrightarrow j \text{ and } j \leftrightarrow k, \quad \text{then } i \leftrightarrow k$$

Communication Classes: Example 1

- Consider again the inventory model

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0.1 & 0.2 & 0.4 & 0.3 \\ 0 & 0 & 0.1 & 0.2 & 0.4 & 0.3 \\ 0.3 & 0.4 & 0.3 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.4 & 0.3 & 0 & 0 \\ 0 & 0.1 & 0.2 & 0.4 & 0.3 & 0 \\ 0 & 0 & 0.1 & 0.2 & 0.4 & 0.3 \end{bmatrix} \end{array}$$

- All states communicate with each other. The Markov Chain is irreducible.

s-S policy

- Consider again the inventory model

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0.1 & 0.2 & 0.4 & 0.3 \\ 0 & 0 & 0.1 & 0.2 & 0.4 & 0.3 \\ 0.3 & 0.4 & 0.3 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.4 & 0.3 & 0 & 0 \\ 0 & 0.1 & 0.2 & 0.4 & 0.3 & 0 \\ 0 & 0 & 0.1 & 0.2 & 0.4 & 0.3 \end{bmatrix} \end{array}$$

s-S policy: Limiting Behavior

$$\mathbf{P}^{10} \approx \begin{array}{c c} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0.0909 & 0.1556 & 0.2310 & 0.2156 & 0.2012 & 0.1056 \\ 0.0909 & 0.1556 & 0.2310 & 0.2156 & 0.2012 & 0.1056 \\ 0.0909 & 0.1556 & 0.2310 & 0.2156 & 0.2012 & 0.1056 \\ 0.0909 & 0.1556 & 0.2310 & 0.2156 & 0.2012 & 0.1056 \\ 0.0909 & 0.1556 & 0.2310 & 0.2156 & 0.2012 & 0.1056 \\ 0.0909 & 0.1556 & 0.2310 & 0.2156 & 0.2012 & 0.1056 \end{bmatrix} \end{array}$$

- The matrix elements appear to converge to some constant probabilities
- The rows become identical
- Why? What determines that limit?

Stationary Distribution

Definition: Stationary Distribution

Let $\{X_n, n \geq 0\}$ be a Markov Chain with initial probability distribution $\boldsymbol{\alpha}$, state Space S, and transition probability matrix $\mathbf{P}$. A probability distribution $\boldsymbol{\pi}$ is called **stationary distribution** for the markov chain if

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P} \tag{1}$$

and

$$\sum_{i \in S} \pi_i = 1.$$

- The stationary distribution $\boldsymbol{\pi}$ does not depend on $\boldsymbol{\alpha}$.
- Equation (1) can be written as

$$\pi_j = \sum_{i \in S} \pi_i p_{ij} \quad \text{for all } j \in S$$

Stationary Distribution: in MATLAB

- Suppose $S = \{1, 2, ..., m\}$, a finite state space

$\boldsymbol{\pi}$ is a probability distribution, therefore

$$\sum_{i=1}^{m} \pi_i = 1 \Rightarrow (\pi_1, \pi_2, ..., \pi_m) \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \boldsymbol{\pi} \cdot \mathbf{1} = 1$$

$\boldsymbol{\pi}$ is stationary, therefore

$$\begin{aligned} \boldsymbol{\pi} &= \boldsymbol{\pi}\mathbf{P} \\ \Rightarrow \boldsymbol{\pi}\mathbf{I} &= \boldsymbol{\pi}\mathbf{P} \\ \Rightarrow \boldsymbol{\pi}(\mathbf{I} - \mathbf{P}) &= \mathbf{0} \in \mathbb{R}^{m \times 1} \end{aligned}$$

- We end up with the system

$$\boldsymbol{\pi} A = \mathbf{b}$$

where

$$\boldsymbol{\pi} \in \mathbb{R}^{1 \times m}, \quad \mathbf{A} = (\mathbf{I} - \mathbf{P}, \mathbf{1}) \in \mathbb{R}^{m \times (m+1)}, \quad \mathbf{b} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} \in \mathbb{R}^{1 \times (m+1)}$$

Stationary Distribution: in MATLAB

```
%%STATIONARY DISTRIBUTION
 P=[0 0 0.1 0.2 0.4 0.3
    0 0 0.1 0.2 0.4 0.3
    0.3 0.4 0.3 0 0 0
    0.1 0.2 0.4 0.3 0 0
    0 0.1 0.2 0.4 0.3 0
    0 0 0.1 0.2 0.4 0.3];

n=size(P,1);

A=[eye(n)-P,ones(n,1)];

b=[zeros(1,n),1];

%We want to solve the system pi*A=b
pi=b/A
```

Stationary Distribution

Theorem

(under assumptions) Every irreducible DTMC has a unique stationary distribution. The stationary distribution corresponds to the long run proportion of time that the DTMC spends on each state.

- Consider for example, the inventory model
- We can verify this theorem using simulation
- The long run proportion of days that the closing inventory is zero is

$$\pi_0 = 0.0909$$