Production Policies for Multi-Product Systems with Deteriorating Process Condition

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Appendix

The following is a summary of the notation used in the paper:

i, j	$\equiv$	indices that describe machine states; $i, j = 1, ..., N$.
k	$\equiv$	index for products, $k = 1, ..., K$.
a	$\equiv$	index for action; $a = k$ refers to producing product $k = 1, ...K$, and $a = m$ refers to performing maintenance.
r_{ia}	$\equiv$	expected reward earned for taking action a when the machine is in state i. For production actions, r_{ia} represents revenue and is therefore positive; for maintenance, i.e., $a = m$, it represents a cost and is therefore negative. Furthermore, r_{ik} is non-increasing in k for each $i = 1, ..., N$; and, r_{ik} is non-increasing in i for each $k = 1, ..., K$.
p_{ij}^{a}	$\equiv$	probability that the machine is in state j at the next decision epoch given that it is in state i and action a is taken at the current epoch.
τ_a	$\equiv$	mean time to complete action a, where $\tau_1 \geq \tau_2 \geq ... \geq \tau_K$.
$\beta_{k,l}$	$\equiv$	ratio of production times for products k and l, where $k < l$; therefore $\beta_{k,l} = \tau_k/\tau_l > 1$.
c	$\equiv$	a scalar used to designate the rate that the deterioration probabilities change with respect to the processing times.
$RR_{k,l}^{i}$	$\equiv$	ratio of rewards (expected revenues) for products k and l in state i; i.e., $RR_{k,l}^{i} = r_{ik}/r_{il}$.

Proofs of Technical Results:

We present the technical derivations for the core problem that determines the optimal production policy for a two product and three machine state problem. In order to simplify the explanation of the proofs, we first provide the steady-state probabilities for each state, and illustrate how they change from one policy to another. The four policies that can be optimal are: $\mathbf{A}_1 = [1,1,m]$, $\mathbf{A}_2 = [1,2,m]$, $\mathbf{A}_3 = [2,1,m]$, and $\mathbf{A}_4 = [2,2,m]$. We denote each policy by $\mathbf{A}_u$ where $u = 1,2,3,4$ represents the policy number. We also define π_i^u as the steady-state probability of state $i = 1,2,3$ for policy $u = 1,2,3,4$. Let us begin our discussion with the two single-product policies that are used as reference policies. Policy $\mathbf{A}_1 = [1,1,m]$ has the following steady-state probabilities:

$$\begin{aligned}
\pi_1^1 &= \frac{\left(1-p_{22}^1\right)}{\left(1-p_{22}^1\right)+p_{12}^1+\left(1-p_{11}^1\right)\left(1-p_{22}^1\right)}, \\
\pi_2^1 &= \frac{p_{12}^1}{\left(1-p_{22}^1\right)+p_{12}^1+\left(1-p_{11}^1\right)\left(1-p_{22}^1\right)}, \\
\pi_3^1 &= \frac{\left(1-p_{11}^1\right)\left(1-p_{22}^1\right)}{\left(1-p_{22}^1\right)+p_{12}^1+\left(1-p_{11}^1\right)\left(1-p_{22}^1\right)}, \\
EV\left(\mathbf{A}_1\right) &= \frac{r_{11}\pi_1^1+r_{21}\pi_2^1+r_{3m}\pi_3^1}{\tau_1\pi_1^1+\tau_1\pi_2^1+\tau_m\pi_3^1}.
\end{aligned}$$

It should be observed that the denominator for each steady-state probability is the same expression. When these steady-state probabilities are substituted in the expected value expression, the denominator corresponding to the steady-state probabilities appears in all of the terms in the numerator and in the denominator, and therefore, gets cancelled out. Let us then denote the numerator of each steady-state probability by $\widehat{\pi}_i^u$ corresponding to state $i = 1,2,3$ for policy $u = 1,2,3,4$. Therefore, we can rewrite the expected value expression as follows:

$$EV\left(\mathbf{A}_1\right) = \frac{r_{11}\pi_1^1+r_{21}\pi_2^1+r_{3m}\pi_3^1}{\tau_1\pi_1^1+\tau_1\pi_2^1+\tau_m\pi_3^1} = \frac{r_{11}\widehat{\pi}_1^1+r_{21}\widehat{\pi}_2^1+r_{3m}\widehat{\pi}_3^1}{\tau_1\widehat{\pi}_1^1+\tau_1\widehat{\pi}_2^1+\tau_m\widehat{\pi}_3^1}$$

The other single-product policy $\mathbf{A}_4 = [2, 2, m]$ has the following steady-state probabilities:

$$\begin{aligned}
\pi_1^4 &= \frac{\left(1-p_{22}^2\right)}{\left(1-p_{22}^2\right)+p_{12}^2+\left(1-p_{11}^2\right)\left(1-p_{22}^2\right)}, \\
\pi_2^4 &= \frac{p_{12}^2}{\left(1-p_{22}^2\right)+p_{12}^2+\left(1-p_{11}^2\right)\left(1-p_{22}^2\right)}, \\
\pi_3^4 &= \frac{\left(1-p_{11}^2\right)\left(1-p_{22}^2\right)}{\left(1-p_{22}^2\right)+p_{12}^2+\left(1-p_{11}^2\right)\left(1-p_{22}^2\right)}, \\
EV\left(\mathbf{A}_4\right) &= \frac{r_{12}\pi_1^4+r_{22}\pi_2^4+r_{3m}\pi_3^4}{\tau_2\pi_1^4+\tau_2\pi_2^4+\tau_m\pi_3^4} = \frac{r_{12}\widehat{\pi}_1^4+r_{22}\widehat{\pi}_2^4+r_{3m}\widehat{\pi}_3^4}{\tau_2\widehat{\pi}_1^4+\tau_2\widehat{\pi}_2^4+\tau_m\widehat{\pi}_3^4}.
\end{aligned}$$

We next consider policies $\mathbf{A}_2 = [1, 2, m]$ and $\mathbf{A}_3 = [2, 1, m]$ and express how their steady-state probabilities are related to those developed for the single-product policies. Recall that the relationship between the deterioration probabilities of two products is characterized by equation (1); under equal variances in processing times, this reduces to

$$p_{ij}^1 = c\beta_{1,2}p_{ij}^2 \text{ for all } 1 \leq i < j \leq 3.$$

Utilizing the above equation, we explain how the steady-state probabilities change when the production is altered in a state. This enables us to illustrate how the expected value of the new policy changes with respect to the single-product policies. Let us begin with policy $\mathbf{A}_2 = [1, 2, m]$:

$$\begin{aligned}
\pi_1^2 &= \frac{\left(1-p_{22}^2\right)}{\left(1-p_{22}^2\right)+p_{12}^1+\left(1-p_{11}^1\right)\left(1-p_{22}^2\right)}; \widehat{\pi}_1^2 = \widehat{\pi}_1^4 = \frac{1}{c\beta_{1,2}}\widehat{\pi}_1^1, \\
\pi_2^2 &= \frac{p_{12}^1}{\left(1-p_{22}^2\right)+p_{12}^1+\left(1-p_{11}^1\right)\left(1-p_{22}^2\right)}; \widehat{\pi}_2^2 = c\beta_{1,2}\widehat{\pi}_2^4 = \widehat{\pi}_2^1, \\
\pi_3^2 &= \frac{\left(1-p_{11}^1\right)\left(1-p_{22}^2\right)}{\left(1-p_{22}^2\right)+p_{12}^1+\left(1-p_{11}^1\right)\left(1-p_{22}^2\right)}; \widehat{\pi}_3^2 = c\beta_{1,2}\widehat{\pi}_3^4 = \frac{1}{c\beta_{1,2}}\widehat{\pi}_3^1, \\
EV\left(\mathbf{A}_2\right) &= \frac{r_{11}\pi_1^2+r_{22}\pi_2^2+r_{3m}\pi_3^2}{\tau_1\pi_1^2+\tau_2\pi_2^2+\tau_m\pi_3^2} = \frac{r_{11}\widehat{\pi}_1^2+r_{22}\widehat{\pi}_2^2+r_{3m}\widehat{\pi}_3^2}{\tau_1\widehat{\pi}_1^2+\tau_2\widehat{\pi}_2^2+\tau_m\widehat{\pi}_3^2}.
\end{aligned}$$

We next consider policy $\mathbf{A}_3 = [2, 1, m]$ and its steady-state probabilities:

$$\begin{aligned}
\pi_1^3 &= \frac{\left(1-p_{22}^1\right)}{\left(1-p_{22}^1\right)+p_{12}^2+\left(1-p_{11}^2\right)\left(1-p_{22}^1\right)}; \widehat{\pi}_1^3 = c\beta_{1,2}\widehat{\pi}_1^4 = \widehat{\pi}_1^1, \\
\pi_2^3 &= \frac{p_{12}^2}{\left(1-p_{22}^1\right)+p_{12}^2+\left(1-p_{11}^2\right)\left(1-p_{22}^1\right)}; \widehat{\pi}_2^3 = \widehat{\pi}_2^4 = \frac{1}{c\beta_{1,2}}\widehat{\pi}_2^1, \\
\pi_3^3 &= \frac{\left(1-p_{11}^2\right)\left(1-p_{22}^1\right)}{\left(1-p_{22}^1\right)+p_{12}^2+\left(1-p_{11}^2\right)\left(1-p_{22}^1\right)}; \widehat{\pi}_3^2 = c\beta_{1,2}\widehat{\pi}_3^4 = \frac{1}{c\beta_{1,2}}\widehat{\pi}_3^1, \\
EV\left(\mathbf{A}_3\right) &= \frac{r_{12}\pi_1^3+r_{21}\pi_2^3+r_{3m}\pi_3^3}{\tau_2\pi_1^3+\tau_1\pi_2^3+\tau_m\pi_3^3} = \frac{r_{12}\widehat{\pi}_1^3+r_{21}\widehat{\pi}_2^3+r_{3m}\widehat{\pi}_3^3}{\tau_2\widehat{\pi}_1^3+\tau_1\widehat{\pi}_2^3+\tau_m\widehat{\pi}_3^3}.
\end{aligned}$$

The above relationships between the steady-state probabilities are used in the derivations of the technical results.

Proof of Proposition 3.1:

We first consider the case when $EV\left(\mathbf{A}_4=[2,2,m]\right) \geq EV\left(\mathbf{A}_1=[1,1,m]\right)$, and thus, $\alpha^i_{1,2}=\underline{\alpha}^i_{1,2}$ for each state $i=1,2$. We derive the critical ratio for state 2 by equating the expected value of the following two policies:

$$\begin{aligned} EV\left(\mathbf{A}_3=[2,1,m]\right) &= EV\left(\mathbf{A}_4=[2,2,m]\right) \\ \frac{r_{12}(1-p^1_{22})+r_{21}p^2_{12}+r_{3m}(1-p^2_{11})(1-p^1_{22})}{\tau_2(1-p^1_{22})+\tau_1 p^2_{12}+\tau_m(1-p^2_{11})(1-p^1_{22})} &= \frac{r_{12}(1-p^2_{22})+r_{22}p^2_{12}+r_{3m}(1-p^2_{11})(1-p^2_{22})}{\tau_2(1-p^2_{22})+\tau_2 p^2_{12}+\tau_m(1-p^2_{11})(1-p^2_{22})} \end{aligned}$$

$$\frac{c\beta_{1,2}r_{12}(1-p^2_{22})+\underline{\alpha}^2_{1,2}r_{22}p^2_{12}+c\beta_{1,2}r_{3m}(1-p^2_{11})(1-p^2_{22})}{c\beta_{1,2}\tau_2(1-p^2_{22})+\beta_{1,2}\tau_2 p^2_{12}+c\beta_{1,2}\tau_m(1-p^2_{11})(1-p^1_{22})} = \\ \frac{r_{12}(1-p^2_{22})+r_{22}p^2_{12}+r_{3m}(1-p^2_{11})(1-p^2_{22})}{\tau_2(1-p^2_{22})+\tau_2 p^2_{12}+\tau_m(1-p^2_{11})(1-p^2_{22})}$$

$$\frac{c\beta_{1,2}r_{12}(1-p^2_{22})+\underline{\alpha}^2_{1,2}r_{22}p^2_{12}+c\beta_{1,2}r_{3m}(1-p^2_{11})(1-p^2_{22})}{c\beta_{1,2}\tau_2(1-p^2_{22})+c\beta_{1,2}\tau_2 p^2_{12}+c\beta_{1,2}\tau_m(1-p^2_{11})(1-p^1_{22})+\left(\beta_{1,2}-c\beta_{1,2}\right)\tau_2 p^2_{12}} \\ = \frac{r_{12}(1-p^2_{22})+r_{22}p^2_{12}+r_{3m}(1-p^2_{11})(1-p^2_{22})}{\tau_2(1-p^2_{22})+\tau_2 p^2_{12}+\tau_m(1-p^2_{11})(1-p^2_{22})}$$

$$\begin{aligned} \underline{\alpha}^2_{1,2}r_{22}p^2_{12} &= c\beta_{1,2}r_{22}p^2_{12}+\left(\beta_{1,2}-c\beta_{1,2}\right)\tau_2 p^2_{12}EV\left(\mathbf{A}_4=[2,2,m]\right) \\ \underline{\alpha}^2_{1,2} &= c\beta_{1,2}+\left(\beta_{1,2}-c\beta_{1,2}\right)\frac{\tau_2 EV\left(\mathbf{A}_4=[2,2,m]\right)}{r_{22}} \end{aligned}$$

Similarly, we derive the critical ratio for state 1 by equating the expected value of the following two policies:

$$\begin{aligned} EV\left(\mathbf{A}_2=[1,2,m]\right) &= EV\left(\mathbf{A}_4=[2,2,m]\right) \\ \frac{r_{11}(1-p^2_{22})+r_{22}p^1_{12}+r_{3m}(1-p^1_{11})(1-p^2_{22})}{\tau_1(1-p^2_{22})+\tau_2 p^1_{12}+\tau_m(1-p^1_{11})(1-p^2_{22})} &= \frac{r_{12}(1-p^2_{22})+r_{22}p^2_{12}+r_{3m}(1-p^2_{11})(1-p^2_{22})}{\tau_2(1-p^2_{22})+\tau_2 p^2_{12}+\tau_m(1-p^2_{11})(1-p^2_{22})} \end{aligned}$$

$$\frac{\underline{\alpha}^1_{1,2}r_{12}(1-p^2_{22})+r_{22}p^2_{12}+c\beta_{1,2}r_{3m}(1-p^2_{11})(1-p^2_{22})}{\beta_{1,2}\tau_2(1-p^2_{22})+c\beta_{1,2}\tau_2 p^2_{12}+c\beta_{1,2}\tau_m(1-p^2_{11})(1-p^2_{22})} = \\ \frac{r_{12}(1-p^2_{22})+r_{22}p^2_{12}+r_{3m}(1-p^2_{11})(1-p^2_{22})}{\tau_2(1-p^2_{22})+\tau_2 p^2_{12}+\tau_m(1-p^2_{11})(1-p^2_{22})}$$

$$\frac{\underline{\alpha}_{1,2}^1 r_{12}(1-p_{22}^2) + c\beta_{1,2} r_{22} p_{12}^2 + c\beta_{1,2} r_{3m}(1-p_{11}^2)(1-p_{22}^2)}{c\beta_{1,2}\tau_2(1-p_{22}^2) + c\beta_{1,2}\tau_2 p_{12}^2 + c\beta_{1,2}\tau_m(1-p_{11}^2)(1-p_{22}^1) + \left(\beta_{1,2} - c\beta_{1,2}\right)\tau_2(1-p_{22}^2)} = \frac{r_{12}(1-p_{22}^2) + r_{22}p_{12}^2 + r_{3m}(1-p_{11}^2)(1-p_{22}^2)}{\tau_2(1-p_{22}^2) + \tau_2 p_{12}^2 + \tau_m(1-p_{11}^2)(1-p_{22}^2)}$$

$$\begin{aligned}
\underline{\alpha}_{1,2}^1 r_{12}(1-p_{22}^2) &= c\beta_{1,2} r_{12}(1-p_{22}^2) + \left(\beta_{1,2} - c\beta_{1,2}\right)\tau_2(1-p_{22}^2) EV\left(\mathbf{A}_4 = [2,2,m]\right) \\
\underline{\alpha}_{1,2}^1 &= c\beta_{1,2} + \left(\beta_{1,2} - c\beta_{1,2}\right)\frac{\tau_2 EV\left(\mathbf{A}_4 = [2,2,m]\right)}{r_{12}}
\end{aligned}$$

a) State 1: The case when $RR_{1,2}^1 > \underline{\alpha}_{1,2}^1$ is proven by substituting $RR_{1,2}^1 r_{12}$ for r_{11} in the expected value expression of policy $\mathbf{A}_2 = [1,2,m]$.

$$\begin{aligned}
EV\left(\mathbf{A}_2 = [1,2,m]\right) &= \frac{RR_{1,2}^1 r_{12}(1-p_{22}^2) + r_{22}p_{12}^1 + r_{3m}(1-p_{11}^1)(1-p_{22}^2)}{\tau_1(1-p_{22}^2) + \tau_2 p_{12}^1 + \tau_m(1-p_{11}^1)(1-p_{22}^2)} \\
&> \frac{\underline{\alpha}_{1,2}^1 r_{12}(1-p_{22}^2) + r_{22}p_{12}^1 + r_{3m}(1-p_{11}^1)(1-p_{22}^2)}{\tau_1(1-p_{22}^2) + \tau_2 p_{12}^1 + \tau_m(1-p_{11}^1)(1-p_{22}^2)} = EV\left(\mathbf{A}_4 = [2,2,m]\right)
\end{aligned}$$

a) State 2: The case when $RR_{1,2}^2 > \underline{\alpha}_{1,2}^2$ is proven by substituting $RR_{1,2}^2 r_{22}$ for r_{21} in the expected value expression of policy $\mathbf{A}_3 = [2,1,m]$.

$$\begin{aligned}
EV\left(\mathbf{A}_3 = [2,1,m]\right) &= \frac{r_{12}(1-p_{22}^1) + RR_{1,2}^2 r_{22}p_{12}^2 + r_{3m}(1-p_{11}^2)(1-p_{22}^1)}{\tau_2(1-p_{22}^1) + \tau_1 p_{12}^2 + \tau_m(1-p_{11}^2)(1-p_{22}^1)} \\
&> \frac{r_{12}(1-p_{22}^1) + \underline{\alpha}_{1,2}^2 r_{22}p_{12}^2 + r_{3m}(1-p_{11}^2)(1-p_{22}^1)}{\tau_2(1-p_{22}^1) + \tau_1 p_{12}^2 + \tau_m(1-p_{11}^2)(1-p_{22}^1)} = EV\left(\mathbf{A}_4 = [2,2,m]\right)
\end{aligned}$$

b) State 1: The case when $RR_{1,2}^1 < \underline{\alpha}_{1,2}^1$ is proven by substituting $RR_{1,2}^1 r_{12}$ for r_{11} in the expected value expression of policy $\mathbf{A}_2 = [1,2,m]$.

$$\begin{aligned}
EV\left(\mathbf{A}_2 = [1,2,m]\right) &= \frac{RR_{1,2}^1 r_{12}(1-p_{22}^2) + r_{22}p_{12}^1 + r_{3m}(1-p_{11}^1)(1-p_{22}^2)}{\tau_1(1-p_{22}^2) + \tau_2 p_{12}^1 + \tau_m(1-p_{11}^1)(1-p_{22}^2)} \\
&< \frac{\underline{\alpha}_{1,2}^1 r_{12}(1-p_{22}^2) + r_{22}p_{12}^1 + r_{3m}(1-p_{11}^1)(1-p_{22}^2)}{\tau_1(1-p_{22}^2) + \tau_2 p_{12}^1 + \tau_m(1-p_{11}^1)(1-p_{22}^2)} = EV\left(\mathbf{A}_4 = [2,2,m]\right)
\end{aligned}$$

b) State 2: The case when $RR_{1,2}^2 < \underline{\alpha}_{1,2}^2$ is proven by substituting $RR_{1,2}^2 r_{22}$ for r_{21} in the expected value expression of policy $\mathbf{A}_3 = [2,1,m]$.

$$\begin{aligned}
EV\left(\mathbf{A}_3 = [2,1,m]\right) &= \frac{r_{12}(1-p_{22}^1) + RR_{1,2}^2 r_{22}p_{12}^2 + r_{3m}(1-p_{11}^2)(1-p_{22}^1)}{\tau_2(1-p_{22}^1) + \tau_1 p_{12}^2 + \tau_m(1-p_{11}^2)(1-p_{22}^1)} \\
&< \frac{r_{12}(1-p_{22}^1) + \underline{\alpha}_{1,2}^2 r_{22}p_{12}^2 + r_{3m}(1-p_{11}^2)(1-p_{22}^1)}{\tau_2(1-p_{22}^1) + \tau_1 p_{12}^2 + \tau_m(1-p_{11}^2)(1-p_{22}^1)} = EV\left(\mathbf{A}_4 = [2,2,m]\right)
\end{aligned}$$

c) State 1: The case when $RR^1_{1,2} = \underline{\alpha}^1_{1,2}$ is proven by substituting $RR^1_{1,2} r_{12}$ for r_{11} in the expected value expression of policy $\mathbf{A}_2 = [1, 2, m]$.

$$\begin{aligned} EV(\mathbf{A}_2 = [1,2,m]) &= \frac{RR^1_{1,2} r_{12}(1-p^2_{22}) + r_{22}p^1_{12} + r_{3m}(1-p^1_{11})(1-p^2_{22})}{\tau_1(1-p^2_{22}) + \tau_2 p^1_{12} + \tau_m(1-p^1_{11})(1-p^2_{22})} \\ &= \frac{\underline{\alpha}^1_{1,2} r_{12}(1-p^2_{22}) + r_{22}p^1_{12} + r_{3m}(1-p^1_{11})(1-p^2_{22})}{\tau_1(1-p^2_{22}) + \tau_2 p^1_{12} + \tau_m(1-p^1_{11})(1-p^2_{22})} = EV(\mathbf{A}_4 = [2,2,m]) \end{aligned}$$

c) State 2: The case when $RR^2_{1,2} = \underline{\alpha}^2_{1,2}$ is proven by substituting $RR^2_{1,2} r_{22}$ for r_{21} in the expected value expression of policy $\mathbf{A}_3 = [2, 1, m]$.

$$\begin{aligned} EV(\mathbf{A}_3 = [2,1,m]) &= \frac{r_{12}(1-p^1_{22}) + RR^2_{1,2} r_{22}p^2_{12} + r_{3m}(1-p^2_{11})(1-p^1_{22})}{\tau_2(1-p^1_{22}) + \tau_1 p^2_{12} + \tau_m(1-p^2_{11})(1-p^1_{22})} \\ &= \frac{r_{12}(1-p^1_{22}) + \underline{\alpha}^2_{1,2} r_{22}p^2_{12} + r_{3m}(1-p^2_{11})(1-p^1_{22})}{\tau_2(1-p^1_{22}) + \tau_1 p^2_{12} + \tau_m(1-p^2_{11})(1-p^1_{22})} = EV(\mathbf{A}_4 = [2,2,m]) \end{aligned}$$

We next consider the case when $EV(\mathbf{A}_4 = [1,1,m]) > EV(\mathbf{A}_4 = [2,2,m])$,and thus, $\alpha^i_{1,2} = \overline{\alpha}^i_{1,2}$ for each state $i = 1, 2$. We derive the critical ratio for state 2 by equating the expected value of the following two policies:

$$\begin{aligned} EV(\mathbf{A}_2 = [1,2,m]) &= EV(\mathbf{A}_1 = [1,1,m]) \\ \frac{r_{11}(1-p^2_{22}) + r_{22}p^1_{12} + r_{3m}(1-p^1_{11})(1-p^2_{22})}{\tau_1(1-p^2_{22}) + \tau_2 p^1_{12} + \tau_m(1-p^1_{11})(1-p^2_{22})} &= \frac{r_{11}(1-p^1_{22}) + r_{21}p^1_{12} + r_{3m}(1-p^1_{11})(1-p^1_{22})}{\tau_1(1-p^1_{22}) + \tau_1 p^1_{12} + \tau_m(1-p^1_{11})(1-p^1_{22})} \end{aligned}$$

$$\frac{\frac{1}{c\beta_{1,2}} r_{11}(1-p^1_{22}) + r_{22}p^1_{12} + \frac{1}{c\beta_{1,2}} r_{3m}(1-p^1_{11})(1-p^1_{22})}{\frac{1}{c\beta_{1,2}}\tau_1(1-p^1_{22}) + \frac{1}{\beta_{1,2}}\tau_1 p^1_{12} + \frac{1}{c\beta_{1,2}}\tau_m(1-p^1_{11})(1-p^1_{22})} = \frac{r_{11}(1-p^1_{22}) + r_{21}p^1_{12} + r_{3m}(1-p^1_{11})(1-p^1_{22})}{\tau_1(1-p^1_{22}) + \tau_1 p^1_{12} + \tau_m(1-p^1_{11})(1-p^1_{22})}$$

$$\frac{\frac{1}{c\beta_{1,2}} r_{11}(1-p^1_{22}) + r_{22}p^1_{12} + \frac{1}{c\beta_{1,2}} r_{3m}(1-p^1_{11})(1-p^1_{22})}{\frac{1}{c\beta_{1,2}}\tau_1(1-p^1_{22}) + \frac{1}{c\beta_{1,2}}\tau_1 p^1_{12} + \frac{1}{c\beta_{1,2}}\tau_m(1-p^1_{11})(1-p^1_{22}) + \left(\frac{1}{\beta_{1,2}} - \frac{1}{c\beta_{1,2}}\right)\tau_1 p^1_{12}} = \frac{r_{11}(1-p^1_{22}) + r_{21}p^1_{12} + r_{3m}(1-p^1_{11})(1-p^1_{22})}{\tau_1(1-p^1_{22}) + \tau_1 p^1_{12} + \tau_m(1-p^1_{11})(1-p^1_{22})}$$

$$
\begin{aligned}
r_{22}p^1_{12} &= \frac{1}{c\beta_{1,2}} r_{21} p^1_{12} + \left(\frac{1}{\beta_{1,2}} - \frac{1}{c\beta_{1,2}}\right) \tau_1 p^1_{12} EV\left(\mathbf{A}_1 = [1,1,m]\right) \\
r_{22} &= \frac{1}{c\beta_{1,2}} r_{21} + \left(\frac{1}{\beta_{1,2}} - \frac{1}{c\beta_{1,2}}\right) \beta_{1,2}\tau_2 EV\left(\mathbf{A}_1 = [1,1,m]\right) \\
c\beta_{1,2} r_{22} &= r_{21} + c\beta_{1,2}\left(\frac{1}{\beta_{1,2}} - \frac{1}{c\beta_{1,2}}\right) \beta_{1,2}\tau_2 EV\left(\mathbf{A}_1 = [1,1,m]\right) \\
r_{21} &= c\beta_{1,2} r_{22} - c\beta_{1,2}\left(\frac{1}{\beta_{1,2}} - \frac{1}{c\beta_{1,2}}\right) \beta_{1,2}\tau_2 EV\left(\mathbf{A}_1 = [1,1,m]\right) \\
\overline{\alpha}^2_{1,2} r_{22} &= c\beta_{1,2} r_{22} - (c-1)\,\beta_{1,2}\tau_2 EV\left(\mathbf{A}_1 = [1,1,m]\right) \\
\overline{\alpha}^2_{1,2} &= c\beta_{1,2} + \left(\beta_{1,2} - c\beta_{1,2}\right) \frac{\tau_2 EV\left(\mathbf{A}_1 = [1,1,m]\right)}{r_{22}}
\end{aligned}
$$

Similarly, we derive the critical ratio for state 1 by equating the expected value of the following two policies:

$$
\begin{aligned}
EV\left(\mathbf{A}_3 = [2,1,m]\right) &= EV(\mathbf{A}_1 = [1,1,m]) \\
\frac{r_{12}(1-p^1_{22}) + r_{21}p^2_{12} + r_{3m}(1-p^2_{11})(1-p^1_{22})}{\tau_2(1-p^1_{22}) + \tau_1 p^2_{12} + \tau_m(1-p^2_{11})(1-p^1_{22})} &= \frac{r_{11}(1-p^1_{22}) + r_{21}p^1_{12} + r_{3m}(1-p^1_{11})(1-p^1_{22})}{\tau_1(1-p^1_{22}) + \tau_1 p^1_{12} + \tau_m(1-p^1_{11})(1-p^1_{22})}
\end{aligned}
$$

$$
\frac{r_{12}(1-p^1_{22}) + \frac{1}{c\beta_{1,2}} r_{21}p^1_{12} + \frac{1}{c\beta_{1,2}} r_{3m}(1-p^1_{11})(1-p^1_{22})}{\frac{1}{\beta_{1,2}}\tau_1(1-p^1_{22}) + \frac{1}{c\beta_{1,2}}\tau_1 p^1_{12} + \frac{1}{c\beta_{1,2}}\tau_m(1-p^1_{11})(1-p^1_{22})} =
$$

$$
\frac{r_{11}(1-p^1_{22}) + r_{21}p^1_{12} + r_{3m}(1-p^1_{11})(1-p^1_{22})}{\tau_1(1-p^1_{22}) + \tau_1 p^1_{12} + \tau_m(1-p^1_{11})(1-p^1_{22})}
$$

$$
\frac{r_{12}(1-p^1_{22}) + \frac{1}{c\beta_{1,2}} r_{21}p^1_{12} + \frac{1}{c\beta_{1,2}} r_{3m}(1-p^1_{11})(1-p^1_{22})}{\frac{1}{c\beta_{1,2}}\tau_1(1-p^1_{22}) + \frac{1}{c\beta_{1,2}}\tau_1 p^1_{12} + \frac{1}{c\beta_{1,2}}\tau_m(1-p^1_{11})(1-p^1_{22}) + \left(\frac{1}{\beta_{1,2}} - \frac{1}{c\beta_{1,2}}\right)\tau_1(1-p^1_{22})} =
$$

$$
\frac{r_{11}(1-p^1_{22}) + r_{21}p^1_{12} + r_{3m}(1-p^1_{11})(1-p^1_{22})}{\tau_1(1-p^1_{22}) + \tau_1 p^1_{12} + \tau_m(1-p^1_{11})(1-p^1_{22})}
$$

$$
\begin{aligned}
r_{12}(1-p^1_{22}) &= \frac{1}{c\beta_{1,2}} r_{11}(1-p^1_{22}) + \left(\frac{1}{\beta_{1,2}} - \frac{1}{c\beta_{1,2}}\right)\tau_1(1-p^1_{22}) EV(\mathbf{A}_1 = [1,1,m]) \\
r_{12} &= \frac{1}{c\beta_{1,2}} r_{11} + \left(1 - \frac{1}{c}\right)\tau_2 EV\left(\mathbf{A}_1 = [1,1,m]\right) \\
r_{11} &= c\beta_{1,2} r_{12} + \left(\beta_{1,2} - c\beta_{1,2}\right)\tau_2 EV\left(\mathbf{A}_1 = [1,1,m]\right) \\
\overline{\alpha}^1_{1,2} r_{12} &= c\beta_{1,2} r_{12} + \left(\beta_{1,2} - c\beta_{1,2}\right)\tau_2 EV\left(\mathbf{A}_1 = [1,1,m]\right) \\
\overline{\alpha}^1_{1,2} &= c\beta_{1,2} + \left(\beta_{1,2} - c\beta_{1,2}\right)\frac{\tau_2 EV\left(\mathbf{A}_1 = [1,1,m]\right)}{r_{12}}
\end{aligned}
$$

a) State 1: The case when $RR^1_{1,2} > \overline{\alpha}^1_{1,2}$ can be proven by substituting $\frac{1}{RR^1_{1,2}} r_{11}$, where $\frac{1}{RR^1_{1,2}} r_{11} < \frac{1}{\overline{\alpha}^1_{1,2}} r_{11}$, for r_{12} in the expected value expression of policy $\mathbf{A}_3 = [2, 1, m]$.

$$\begin{aligned} EV(\mathbf{A}_3 = [2,1,m]) &= \frac{\frac{1}{RR^1_{1,2}} r_{11}(1-p^1_{22}) + r_{21}p^2_{12} + r_{3m}(1-p^2_{11})(1-p^1_{22})}{\tau_2(1-p^1_{22}) + \tau_1 p^2_{12} + \tau_m(1-p^2_{11})(1-p^1_{22})} \\ &< \frac{\frac{1}{\overline{\alpha}^1_{1,2}} r_{11}(1-p^1_{22}) + r_{21}p^2_{12} + r_{3m}(1-p^2_{11})(1-p^1_{22})}{\tau_2(1-p^1_{22}) + \tau_1 p^2_{12} + \tau_m(1-p^2_{11})(1-p^1_{22})} = EV(\mathbf{A}_1 = [1,1,m]) \end{aligned}$$

a) State 2: The case when $RR^2_{1,2} > \overline{\alpha}^2_{1,2}$ is proven by substituting $\frac{1}{RR^2_{1,2}} r_{21}$, where $\frac{1}{RR^2_{1,2}} r_{21} < \frac{1}{\overline{\alpha}^2_{1,2}} r_{21}$, for r_{22} in the expected value expression of policy $\mathbf{A}_2 = [1, 2, m]$.

$$\begin{aligned} EV(\mathbf{A}_2 = [1,2,m]) &= \frac{r_{11}(1-p^2_{22}) + \frac{1}{RR^2_{1,2}} r_{21}p^1_{12} + r_{3m}(1-p^1_{11})(1-p^2_{22})}{\tau_1(1-p^2_{22}) + \tau_2 p^1_{12} + \tau_m(1-p^1_{11})(1-p^2_{22})} \\ &< \frac{r_{11}(1-p^2_{22}) + \frac{1}{\overline{\alpha}^2_{1,2}} r_{21}p^1_{12} + r_{3m}(1-p^1_{11})(1-p^2_{22})}{\tau_1(1-p^2_{22}) + \tau_2 p^1_{12} + \tau_m(1-p^1_{11})(1-p^2_{22})} = EV(\mathbf{A}_1 = [1,1,m]) \end{aligned}$$

b) State 1: The case when $RR^1_{1,2} < \overline{\alpha}^1_{1,2}$ is proven by substituting $\frac{1}{RR^1_{1,2}} r_{11}$, where $\frac{1}{RR^1_{1,2}} r_{11} > \frac{1}{\overline{\alpha}^1_{1,2}} r_{11}$, for r_{12} in the expected value expression of policy $\mathbf{A}_3 = [2, 1, m]$.

$$\begin{aligned} EV(\mathbf{A}_3 = [2,1,m]) &= \frac{\frac{1}{RR^1_{1,2}} r_{11}(1-p^1_{22}) + r_{21}p^2_{12} + r_{3m}(1-p^2_{11})(1-p^1_{22})}{\tau_2(1-p^1_{22}) + \tau_1 p^2_{12} + \tau_m(1-p^2_{11})(1-p^1_{22})} \\ &> \frac{\frac{1}{\overline{\alpha}^1_{1,2}} r_{11}(1-p^1_{22}) + r_{21}p^2_{12} + r_{3m}(1-p^2_{11})(1-p^1_{22})}{\tau_2(1-p^1_{22}) + \tau_1 p^2_{12} + \tau_m(1-p^2_{11})(1-p^1_{22})} = EV(\mathbf{A}_1 = [1,1,m]) \end{aligned}$$

b) State 2: The case when $RR^2_{1,2} < \overline{\alpha}^2_{1,2}$ is proven by substituting $\frac{1}{RR^2_{1,2}} r_{21}$,where $\frac{1}{RR^2_{1,2}} r_{21} > \frac{1}{\overline{\alpha}^2_{1,2}} r_{21}$, for r_{22} in the expected value expression of policy $\mathbf{A}_2 = [1, 2, m]$.

$$\begin{aligned} EV(\mathbf{A}_2 = [1,2,m]) &= \frac{r_{11}(1-p^2_{22}) + \frac{1}{RR^2_{1,2}} r_{21}p^1_{12} + r_{3m}(1-p^1_{11})(1-p^2_{22})}{\tau_1(1-p^2_{22}) + \tau_2 p^1_{12} + \tau_m(1-p^1_{11})(1-p^2_{22})} \\ &> \frac{r_{11}(1-p^2_{22}) + \frac{1}{\overline{\alpha}^2_{1,2}} r_{21}p^1_{12} + r_{3m}(1-p^1_{11})(1-p^2_{22})}{\tau_1(1-p^2_{22}) + \tau_2 p^1_{12} + \tau_m(1-p^1_{11})(1-p^2_{22})} = EV(\mathbf{A}_1 = [1,1,m]) \end{aligned}$$

c) State 1: The case when $RR^1_{1,2} = \overline{\alpha}^1_{1,2}$ is proven by substituting $\frac{1}{RR^1_{1,2}} r_{11}$, where $\frac{1}{RR^1_{1,2}} r_{11} = \frac{1}{\overline{\alpha}^1_{1,2}} r_{11}$, for r_{12} in the expected value expression of policy $\mathbf{A}_3 = [2, 1, m]$.

$$\begin{aligned} EV(\mathbf{A}_3 = [2,1,m]) &= \frac{\frac{1}{RR^1_{1,2}} r_{11}(1-p^1_{22}) + r_{21}p^2_{12} + r_{3m}(1-p^2_{11})(1-p^1_{22})}{\tau_2(1-p^1_{22}) + \tau_1 p^2_{12} + \tau_m(1-p^2_{11})(1-p^1_{22})} \\ &= \frac{\frac{1}{\overline{\alpha}^1_{1,2}} r_{11}(1-p^1_{22}) + r_{21}p^2_{12} + r_{3m}(1-p^2_{11})(1-p^1_{22})}{\tau_2(1-p^1_{22}) + \tau_1 p^2_{12} + \tau_m(1-p^2_{11})(1-p^1_{22})} = EV(\mathbf{A}_1 = [1,1,m]) \end{aligned}$$

c) State 2: The case when $RR^2_{1,2} = \overline{\alpha}^2_{1,2}$ is proven by substituting $\frac{1}{RR^2_{1,2}} r_{21}$, where $\frac{1}{RR^2_{1,2}} r_{21} = \frac{1}{\overline{\alpha}^2_{1,2}} r_{21}$, for r_{22} in the expected value expression of policy $\mathbf{A}_2 = [1, 2, m]$.

$$\begin{aligned} EV\left(\mathbf{A}_2 = [1,2,m]\right) &= \frac{r_{11}(1-p^2_{22}) + \frac{1}{RR^2_{1,2}} r_{21} p^1_{12} + r_{3m}(1-p^1_{11})(1-p^2_{22})}{\tau_1(1-p^2_{22}) + \tau_2 p^1_{12} + \tau_m (1-p^1_{11})(1-p^2_{22})} \\ &= \frac{r_{11}(1-p^2_{22}) + \frac{1}{\overline{\alpha}^2_{1,2}} r_{21} p^1_{12} + r_{3m}(1-p^1_{11})(1-p^2_{22})}{\tau_1(1-p^2_{22}) + \tau_2 p^1_{12} + \tau_m (1-p^1_{11})(1-p^2_{22})} = EV\left(\mathbf{A}_1 = [1,1,m]\right) \end{aligned}$$

Proof of Proposition 3.2:

In both of the critical ratio expressions, the first term $c\beta_{1,2}$ is constant.

a) When $c \geq 1$, the second term in both critical ratio expressions, i.e., $\beta_{1,2}(1-c)\frac{\tau_2 EV(\mathbf{A}_4=[2,2,m])}{r_{i2}}$ and $\beta_{1,2}(1-c)\frac{\tau_2 EV(\mathbf{A}_1=[1,1,m])}{r_{i2}}$, are negative. Because $r_{i2} \geq r_{j2}$, where $i < j$, we get

$$\begin{aligned} \beta_{1,2}(1-c)\frac{\tau_2 EV(\mathbf{A}_4 = [2,2,m])}{r_{i2}} &\leq \beta_{1,2}(1-c)\frac{\tau_2 EV(\mathbf{A}_4 = [2,2,m])}{r_{j2}}, \text{ and} \\ \beta_{1,2}(1-c)\frac{\tau_2 EV(\mathbf{A}_1 = [1,1,m])}{r_{i2}} &\leq \beta_{1,2}(1-c)\frac{\tau_2 EV(\mathbf{A}_1 = [1,1,m])}{r_{j2}}. \end{aligned}$$

Therefore, $\underline{\alpha}^i_{1,2} \geq \underline{\alpha}^j_{1,2}$, and $\overline{\alpha}^i_{1,2} \geq \overline{\alpha}^j_{1,2}$. Thus, $\underline{\alpha}^i_{1,2}$ and $\overline{\alpha}^i_{1,2}$ are non-increasing in i, and therefore, $\alpha^i_{1,2}$ is non-increasing in i.

b) When $\frac{1}{\beta_{1,2}} \leq c < 1$, the second term in both critical ratio expressions, i.e., $\beta_{1,2}(1-c)\frac{\tau_2 EV(\mathbf{A}_4=[2,2,m])}{r_{i2}}$ and $\beta_{1,2}(1-c)\frac{\tau_2 EV(\mathbf{A}_1=[1,1,m])}{r_{i2}}$, are positive. Because $r_{i2} \geq r_{j2}$, where $i < j$, we get

$$\begin{aligned} \beta_{1,2}(1-c)\frac{\tau_2 EV(\mathbf{A}_4 = [2,2,m])}{r_{i2}} &\geq \beta_{1,2}(1-c)\frac{\tau_2 EV(\mathbf{A}_4 = [2,2,m])}{r_{j2}}, \text{ and} \\ \beta_{1,2}(1-c)\frac{\tau_2 EV(\mathbf{A}_1 = [1,1,m])}{r_{i2}} &\geq \beta_{1,2}(1-c)\frac{\tau_2 EV(\mathbf{A}_1 = [1,1,m])}{r_{j2}}. \end{aligned}$$

Therefore, $\underline{\alpha}^i_{1,2} \leq \underline{\alpha}^j_{1,2}$, and $\overline{\alpha}^i_{1,2} \leq \overline{\alpha}^j_{1,2}$. Thus, $\underline{\alpha}^i_{1,2}$ and $\overline{\alpha}^i_{1,2}$ are non-decreasing in i, and therefore, $\alpha^i_{1,2}$ is non-decreasing in i.

Proof of Proposition 3.3:

a) When $c \geq 1$, we get $\beta_{1,2}(1-c) < 0$. Then, when $EV\left(\mathbf{A}_1 = [1,1,m]\right) > EV\left(\mathbf{A}_4 = [2,2,m]\right)$,

$$\underline{\alpha}^i_{1,2} - \overline{\alpha}^i_{1,2} = \beta_{1,2}(1-c)\frac{\tau_2}{r_{i2}}\left[EV(\mathbf{A}_4 = [2,2,m]) - EV(\mathbf{A}_1 = [1,1,m])\right] > 0.$$

Thus, $\underline{\alpha}^i_{1,2} > \overline{\alpha}^i_{1,2}$ for each state $i = 1, 2$.

b) When $c \geq 1$, we get $\beta_{1,2}(1-c) < 0$. Then, when $EV(\mathbf{A}_1 = [1,1,m]) \leq EV(\mathbf{A}_4 = [2,2,m])$

$$\underline{\alpha}^i_{1,2} - \overline{\alpha}^i_{1,2} = \beta_{1,2}(1-c)\frac{\tau_2}{r_{i2}}\left[EV(\mathbf{A}_4 = [2,2,m]) - EV(\mathbf{A}_1 = [1,1,m])\right] \leq 0.$$

Thus, $\underline{\alpha}^i_{1,2} \leq \overline{\alpha}^i_{1,2}$ for each state $i = 1,2$.

c) When $\frac{1}{\beta_{1,2}} \leq c < 1$, we get $\beta_{1,2}(1-c) > 0$. Then, when $EV(\mathbf{A}_1 = [1,1,m]) > EV(\mathbf{A}_4 = [2,2,m])$,

$$\underline{\alpha}^i_{1,2} - \overline{\alpha}^i_{1,2} = \beta_{1,2}(1-c)\frac{\tau_2}{r_{i2}}\left[EV(\mathbf{A}_4 = [2,2,m]) - EV(\mathbf{A}_1 = [1,1,m])\right] < 0.$$

Thus, $\underline{\alpha}^i_{1,2} < \overline{\alpha}^i_{1,2}$ for each state $i = 1,2$.

d) When $\frac{1}{\beta_{1,2}} \leq c < 1$, we get $\beta_{1,2}(1-c) > 0$. Then, when $EV(\mathbf{A}_1 = [1,1,m]) \leq EV(\mathbf{A}_4 = [2,2,m])$,

$$\underline{\alpha}^i_{1,2} - \overline{\alpha}^i_{1,2} = \beta_{1,2}(1-c)\frac{\tau_2}{r_{i2}}\left[EV(\mathbf{A}_4 = [2,2,m]) - EV(\mathbf{A}_1 = [1,1,m])\right] \geq 0.$$

Thus, $\underline{\alpha}^i_{1,2} \geq \overline{\alpha}^i_{1,2}$ for each state $i = 1,2$.

Proof of Theorem 3.1:

We provide the proof for the case when $c \geq 1$, and the proof for the case when $\frac{1}{\beta_{1,2}} \leq c < 1$ is similar.

a) Let us begin with the case when $EV(\mathbf{A}_1 = [1,1,m]) \geq EV(\mathbf{A}_4 = [2,2,m])$, and thus $\alpha^i_{12} = \overline{\alpha}^i_{12}$ for each $i = 1,2$. Under these conditions, we already know from Proposition 3.3.a) that $\underline{\alpha}^i_{1,2} \geq \overline{\alpha}^i_{12}$ for each $i = 1,2$. The value of the ratio of rewards can be in one of four possible scenarios:

Scenario 1: $\overline{\alpha}^1_{12} \leq RR^1_{1,2}$ and $\overline{\alpha}^2_{12} \leq RR^2_{1,2}$:

From Proposition 3.1, we know the following: $\overline{\alpha}^1_{12} \leq RR^1_{1,2}$ implies that $EV(\mathbf{A}_1 = [1,1,m]) \geq EV(\mathbf{A}_3 = [2,1,m])$, and $\overline{\alpha}^2_{12} \leq RR^2_{1,2}$ implies that $EV(\mathbf{A}_1 = [1,1,m]) \geq EV(\mathbf{A}_2 = [1,2,m])$. By the definition of this case, we already have $EV(\mathbf{A}_1 = [1,1,m]) \geq EV(\mathbf{A}_4 = [2,2,m])$. Therefore $EV(\mathbf{A}_1 = [1,1,m])$ is the highest expected reward collectively, and producing product 1 is optimal in both states, $i = 1,2$ (i.e., $a^*_1 = 1$, $a^*_2 = 1$).

Scenario 2: $\overline{\alpha}^1_{12} \leq RR^1_{1,2}$ and $\overline{\alpha}^2_{12} > RR^2_{1,2}$:

From Proposition 3.1, we know the following: $\overline{\alpha}^1_{12} \leq RR^1_{1,2}$ implies that $EV(\mathbf{A}_1 = [1,1,m]) \geq EV(\mathbf{A}_3 = [2,1,m])$, and $\overline{\alpha}^2_{12} > RR^2_{1,2}$ implies that $EV(\mathbf{A}_1 = [1,1,m]) < EV(\mathbf{A}_2 = [1,2,m])$. By the definition of this case, we already have $EV(\mathbf{A}_1 = [1,1,m]) \geq EV(\mathbf{A}_4 = [2,2,m])$. Therefore, the expected values are in the following order: $EV(\mathbf{A}_2 = [1,2,m]) > EV(\mathbf{A}_1 = [1,1,m]) \geq \left\{ \begin{smallmatrix} EV(\mathbf{A}_3=[2,1,m]) \\ EV(\mathbf{A}_4=[2,2,m]) \end{smallmatrix} \right\}$. Thus, policy $\mathbf{A}_2 = [1,2,m]$ is optimal. This leads to the optimal choices of product 1 in state 1 and product 2 in state 2, i.e., $a_1^* = 1$, $a_2^* = 2$.

Scenario 3: $\overline{\alpha}_{12}^1 > RR_{1,2}^1$ and $\overline{\alpha}_{12}^2 \leq RR_{1,2}^2$:

From Proposition 3.1, we know the following: $\overline{\alpha}_{12}^1 > RR_{1,2}^1$ implies that $EV(\mathbf{A}_1 = [1,1,m]) < EV(\mathbf{A}_3 = [2,1,m])$ and $\overline{\alpha}_{12}^2 \leq RR_{1,2}^2$ implies that $EV(\mathbf{A}_1 = [1,1,m]) \geq EV(\mathbf{A}_2 = [1,2,m])$. By the definition of this case, we already have $EV(\mathbf{A}_1 = [1,1,m]) \geq EV(\mathbf{A}_4 = [2,2,m])$. Therefore, the expected values are in the following order: $EV(\mathbf{A}_3 = [2,1,m]) > EV(\mathbf{A}_1 = [1,1,m]) \geq \left\{ \begin{smallmatrix} EV(\mathbf{A}_2=[1,2,m]) \\ EV(\mathbf{A}_4=[2,2,m]) \end{smallmatrix} \right\}$. Thus, policy $EV(\mathbf{A}_3 = [2,1,m])$ is optimal. This leads to the optimal choices of product 2 in state 1 and product 1 in state 2, i.e., $a_1^* = 2$, $a_2^* = 1$.

Scenario 4: $\overline{\alpha}_{12}^1 > RR_{1,2}^1$ and $\overline{\alpha}_{12}^2 > RR_{1,2}^2$:

From Proposition 3.1, we know the following: $\overline{\alpha}_{12}^1 > RR_{1,2}^1$ implies that $EV(\mathbf{A}_1 = [1,1,m]) < EV(\mathbf{A}_3 = [2,1,m])$, and $\overline{\alpha}_{12}^2 > RR_{1,2}^2$ implies that $EV(\mathbf{A}_1 = [1,1,m]) < EV(\mathbf{A}_2 = [1,2,m])$. Thus, $\left\{ \begin{smallmatrix} EV(\mathbf{A}_2=[1,2,m]) \\ EV(\mathbf{A}_3=[2,1,m]) \end{smallmatrix} \right\} > EV(\mathbf{A}_1 = [1,1,m])$.

However, we already know from Proposition 3.3.a) that $\underline{\alpha}_{1,2}^i \geq \overline{\alpha}_{12}^i$ for each $i = 1,2$. Therefore in this scenario, we have $\underline{\alpha}_{1,2}^1 \geq \overline{\alpha}_{12}^1 > RR_{1,2}^1$ and $\underline{\alpha}_{1,2}^2 \geq \overline{\alpha}_{12}^2 > RR_{1,2}^2$. From Proposition 3.1, we know that $\underline{\alpha}_{1,2}^1 > RR_{1,2}^1$ implies $EV(\mathbf{A}_4 = [2,2,m]) > EV(\mathbf{A}_2 = [1,2,m])$ and that $\underline{\alpha}_{1,2}^2 > RR_{1,2}^2$ implies $EV(\mathbf{A}_4 = [2,2,m]) > EV(\mathbf{A}_3 = [2,1,m])$. Collectively, we get $EV(\mathbf{A}_4 = [2,2,m]) > \left\{ \begin{smallmatrix} EV(\mathbf{A}_2=[1,2,m]) \\ EV(\mathbf{A}_3=[2,1,m]) \end{smallmatrix} \right\}$. When these are combined with the earlier comparisons, we get $EV(\mathbf{A}_4 = [2,2,m]) > \left\{ \begin{smallmatrix} EV(\mathbf{A}_2=[1,2,m]) \\ EV(\mathbf{A}_3=[2,1,m]) \end{smallmatrix} \right\} > EV(\mathbf{A}_1 = [1,1,m])$, contradicting the motivating case of $EV(\mathbf{A}_1 = [1,1,m]) \geq EV(\mathbf{A}_4 = [2,2,m])$. As a result, this scenario is never encountered when $EV(\mathbf{A}_1 = [1,1,m]) \geq EV(\mathbf{A}_4 = [2,2,m])$, proving part c) of the theorem.

Scenarios 1, 2 and 3 collectively prove that when $\overline{\alpha}_{12}^i \leq RR_{1,2}^i$, then the optimal production decision is $a_i^* = 1$, and when $\overline{\alpha}_{12}^i > RR_{1,2}^i$, then the optimal production decision is $a_i^* = 2$. This completes parts a) and b) of the proof of the theorem.

b) The case when $EV(\mathbf{A}_1 = [1,1,m]) < EV(\mathbf{A}_4 = [2,2,m])$:

Under these conditions, we already know from Proposition 3.4.b) that $\underline{\alpha}_{1,2}^i < \overline{\alpha}_{12}^i$ for each $i = 1,2$. The value of the ratio of rewards can be in one of four possible scenarios:

Scenario: 1: $\underline{\alpha}_{12}^1 > RR_{1,2}^1$ and $\underline{\alpha}_{12}^2 > RR_{1,2}^2$:

From Proposition 3.1, we know the following: $\underline{\alpha}_{12}^1 > RR_{1,2}^1$ implies that $EV(\mathbf{A}_2 = [1,2,m]) < EV(\mathbf{A}_4 = [2,2,m])$ and $\underline{\alpha}_{12}^2 > RR_{1,2}^2$ implies that $EV(\mathbf{A}_3 = [2,1,m]) < EV(\mathbf{A}_4 = [2,2,m])$. As a result, we have $EV(\mathbf{A}_4 = [2,2,m]) > \left\{\begin{array}{c} EV(\mathbf{A}_1 = [1,1,m]) \\ EV(\mathbf{A}_2 = [1,2,m]) \\ EV(\mathbf{A}_3 = [2,1,m]) \end{array}\right\}$. Thus, the optimal policy is $\mathbf{A}_4 = [2,2,m]$ and the optimal production choice is product 2 in both states, i.e., $a_1^* = 2$, $a_2^* = 2$.

Scenario 2: $\underline{\alpha}_{12}^1 > RR_{1,2}^1$ and $\underline{\alpha}_{12}^2 \leq RR_{1,2}^2$:

From Proposition 3.1, we know the following: $\underline{\alpha}_{12}^1 > RR_{1,2}^1$ implies that $EV(\mathbf{A}_2 = [1,2,m]) < EV(\mathbf{A}_4 = [2,2,m])$, and $\underline{\alpha}_{12}^2 \leq RR_{1,2}^2$ implies that $EV(\mathbf{A}_3 = [2,1,m]) \geq EV(\mathbf{A}_4 = [2,2,m])$. By the definition of this case, we already have $EV(\mathbf{A}_1 = [1,1,m]) < EV(\mathbf{A}_4 = [2,2,m])$. Therefore, the expected values are in the following order: $EV(\mathbf{A}_3 = [2,1,m]) \geq EV(\mathbf{A}_4 = [2,2,m]) > \left\{\begin{smallmatrix} EV(\mathbf{A}_1 = [1,1,m]) \\ EV(\mathbf{A}_2 = [1,2,m]) \end{smallmatrix}\right\}$. Thus, policy $\mathbf{A}_3 = [2,1,m]$ is optimal. This leads to the optimal choices of product 2 in state 1 and product 1 in state 2, i.e., $a_1^* = 2$, $a_2^* = 1$.

Scenario 3: $\underline{\alpha}_{12}^1 \leq RR_{1,2}^1$ and $\underline{\alpha}_{12}^2 > RR_{1,2}^2$:

From Proposition 3.1, we know the following: $\underline{\alpha}_{12}^1 \leq RR_{1,2}^1$ implies that $EV(\mathbf{A}_2 = [1,2,m]) \geq EV(\mathbf{A}_4 = [2,2,m])$, and $\underline{\alpha}_{12}^2 > RR_{1,2}^2$ implies that $EV(\mathbf{A}_3 = [2,1,m]) < EV(\mathbf{A}_4 = [2,2,m])$. By the definition of this case, we already have $EV(\mathbf{A}_1 = [1,1,m]) < EV(\mathbf{A}_4 = [2,2,m])$. Therefore, the expected values are in the following order: $EV(\mathbf{A}_2 = [1,2,m]) \geq EV(\mathbf{A}_4 = [2,2,m]) > \left\{\begin{smallmatrix} EV(\mathbf{A}_1 = [1,1,m]) \\ EV(\mathbf{A}_3 = [2,1,m]) \end{smallmatrix}\right\}$. Thus, policy $\mathbf{A}_2 = [1,2,m]$ is optimal. This leads to the optimal choices of product 1 in state 1 and product 2 in state 2, i.e., $a_1^* = 1$, $a_2^* = 2$.

Scenario 4: $\underline{\alpha}_{12}^1 > RR_{1,2}^1$ and $\underline{\alpha}_{12}^2 > RR_{1,2}^2$:

It can easily be seen that when both $\underline{\alpha}_{12}^1 > RR_{1,2}^1$ and $\underline{\alpha}_{12}^2 > RR_{1,2}^2$, we get $EV(\mathbf{A}_1 = [1,1,m]) > EV(\mathbf{A}_4 = [2,2,m])$, and this contradicts the original case, proving part c) of the theorem.

Scenarios 1, 2 and 3 collectively prove that when $\underline{\alpha}_{12}^i \leq RR_{1,2}^i$, then the optimal production decision is $a_i^* = 1$, and when $\underline{\alpha}_{12}^i > RR_{1,2}^i$, then the optimal production decision is $a_i^* = 2$. This completes parts a) and b) of the proof of the theorem.

Proof of Proposition 3.4:

The proof follows from Proposition 3.3.

a) When $c > 1$, both critical ratios are non-increasing in i. Therefore, in the case of constant ratio of rewards, i.e., $RR_{1,2}^{1}=RR_{1,2}^{2}=RR_{1,2}^{i}=RR_{1,2}$ for all $i=1,...,N$, we get $RR_{1,2}-\underline{\alpha}_{1,2}^{i}\leq RR_{1,2}-\underline{\alpha}_{1,2}^{j}$ for all states $i<j$ as well as $RR_{1,2}-\overline{\alpha}_{1,2}^{i}\leq RR_{1,2}-\overline{\alpha}_{1,2}^{j}$ for all states $i<j$; and, their signs can switch from negative to positive only once.

b) When $\frac{1}{\beta_{1,2}}\leq c<1$, both critical ratios are non-decreasing in i. Therefore, in the case of constant ratio of rewards, i.e., $RR_{1,2}^{1}=RR_{1,2}^{2}=RR_{1,2}^{i}=RR_{1,2}$, we get $RR_{1,2}-\underline{\alpha}_{1,2}^{i}\geq RR_{1,2}-\underline{\alpha}_{1,2}^{j}$ for all states $i<j$ as well as $RR_{1,2}^{i}-\overline{\alpha}_{1,2}^{1}\geq RR_{1,2}^{i}-\overline{\alpha}_{1,2}^{2}$for all states $i<j$; and, their signs can switch from positive to negative only once.

Proof of Corollary 3.1:

The proof follows directly from Theorem 3.1.

The Description of the Solution Approach for Three and More Products:

The solution approach presented for the two-product problem can be extended to problem settings with three or more products. For example, when there are three products, i.e., $k=1,2$, and 3, the solution technique features two, at most three, pairwise comparisons in order to determine the optimal production decision in each state. First, the expected values for the three single-product policies, $EV\left(\mathbf{A}_{1}=[1,...,1,m]\right)$, $EV\left(\mathbf{A}_{2}=[2,...,2,m]\right)$ and $EV\left(\mathbf{A}_{3}=[3,...,3,m]\right)$, are calculated, and the policy that has the highest expected value is selected as the reference policy and its corresponding product as the reference product. Suppose, for example, the reference product is $k=3$. Then the firm can use the following two pairwise comparisons: product 1 vs. product 3 constitutes the first pairwise comparison, and product 2 vs. product 3 constitutes the second. For a given state, if the comparison of products 1 and 3 leads to the preference of product 1 while the comparison of products 2 and 3 results in the preference of product 3, then the optimal production decision is to manufacture product 1. Alternatively, if the comparison of products 1 and 3 leads to the preference of product 3 while the comparison of products 2 and 3 results in the preference of product 2, then the optimal production decision is to manufacture product 2. In another scenario, if both comparisons recommend the manufacturing of product 3, then the optimal production choice is product 3. However, when the comparison of products 1 and 3 leads to the preference of product 1 and the comparison of products 2 and 3 results in the preference of product 2, then a third comparison between products 1 and 2 becomes necessary in order to determine the optimal production choice for the state in question. This algorithm makes two, at most three comparisons for each state $i = 1, ..., N-1$, leading to a worst case scenario of $3\times(N-1)$ comparisons in order to determine the optimal policy that captures the best production choice in each machine state for a problem setting that has N machine-states. It should be noted here that we already know from Theorem 3.1 that the total number of comparisons will strictly be less than $3 \times (N-1)$. This effort is significantly less than the total enumeration of all possible production policies. The optimal solution approach can be generalized to a problem setting that features K products. However, considering pairwise comparisons between the reference product and products 1 through K, resulting in $K-1$ pairwise comparisons, may not be the most practical approach as the total number of comparisons grows exponentially. However, such pairwise comparisons lead to a significant reduction in the feasible set of potentially optimal products in each state. Therefore, the set of potentially optimal policies can be reduced by using the pairwise comparison that utilize the proposed critical ratios. We believe that the critical ratios can be used for heuristics (to be developed in the future) in order to obtain feasible solutions with a high degree of solution quality.

Proof of Lemma 3.1:

The proof follows from Theorem 3 of Kao (1973) and the fact that the reward structure is non-increasing in the machine state, i.e., $r_{ia} \geq r_{ja}$ for each action a and for all states $1 \leq i < j \leq N$.

Proof of Proposition 3.5:

When maintenance is performed, the machine returns to its best state with probability $p_{i1}^m = 1$ for all $i = 2, ..., N$. First, consider the problem setting with N states. The steady-state probability for states where production takes place, i.e., $i = 1, ..., M-1$, using any policy $\mathbf{A}_N = [a_1, ..., a_{M-1}, a_M = m, ..., a_{N-1} = m, a_N = m]$ generates $\pi_i(\mathbf{A}_N = [a_1, ..., a_{M-1}, a_M = m, ..., a_{N-1} = m, a_N = m])$. Now, consider the problem setting with $N-1$ states, and a policy that uses the same sequence of actions in all states from $i = 1$ to $i = N-1$, denoted by $\mathbf{A}_{N-1}$. The steady-state probability for this policy in the states that production takes place, i.e., $i = 1, ..., M-1$, is denoted by

$\pi_i(\mathbf{A}_{N-1} = [a_1, ..., a_{M-1}, a_M = m, ..., a_{N-1} = m])$. It should be observed that

$$\pi_i(\mathbf{A}_N = [a_1, ..., a_{M-1}, a_M = m, ..., a_{N-1} = m, a_N = m]) =$$
$$\pi_i(\mathbf{A}_{N-1} = [a_1, ..., a_{M-1}, a_M = m, ..., a_{N-1} = m]) \times p^m_{N1}.$$

Because $p^m_{N1} = 1$, we get

$$\begin{aligned} \pi_i(\mathbf{A}_N &= [a_1, ..., a_{M-1}, a_M = m, ..., a_{N-1} = m, a_N = m]) \\ &= \pi_i(\mathbf{A}_{N-1} = [a_1, ..., a_{M-1}, a_M = m, ..., a_{N-1} = m]). \end{aligned}$$

Therefore, the expected values of these two policies, despite the fact that they are in two different settings, are equal. Thus, $EV(\mathbf{A}_N) = EV(\mathbf{A}_{N-1})$. By induction, we get

$$\begin{aligned} \pi_i(\mathbf{A}_N &= [a_1, ..., a_{M-1}, a_M = m, ..., a_{N-1} = m, a_N = m]) \\ &= \pi_i(\mathbf{A}_{N-1} = [a_1, ..., a_{M-1}, a_M = m, ..., a_{N-1} = m]) \\ &= ... \\ &= \pi_i(\mathbf{A}_M = [a_1, ..., a_{M-1}, a_M = m]). \end{aligned}$$

Moreover, we get $EV(\mathbf{A}_N) = EV(\mathbf{A}_{N-1}) = ... = EV(\mathbf{A}_M)$. Let us denote $\mathbf{A}_N(1)$ and $\mathbf{A}_N(2)$ as the single-product policies of manufacturing products 1 and 2, respectively, in a problem setting that features N machine states. We next show that the critical ratios for problem settings that have production actions in the first $M-1$ states and maintenance in the following (worse) states are equal for all problem settings that feature M or more machine states. Thus,

$$\begin{aligned} \underline{\alpha}^i_{k,l}(N, M) &= c\beta_{1,2} + \beta_{1,2}(1-c)\frac{\tau_2 EV(\mathbf{A}_N(2))}{r_{i2}} \\ &= \underline{\alpha}^i_{k,l}(N-1, M) = c\beta_{1,2} + \beta_{1,2}(1-c)\frac{\tau_2 EV(\mathbf{A}_{N-1}(2))}{r_{i2}} \\ &= ... = \underline{\alpha}^i_{k,l}(M, M) = c\beta_{1,2} + \beta_{1,2}(1-c)\frac{\tau_2 EV(\mathbf{A}_M(2))}{r_{i2}} \text{ for all } i = 1, ..., M-1, \text{ and} \\ \overline{\alpha}^i_{k,l}(N, M) &= c\beta_{1,2} + \beta_{1,2}(1-c)\frac{\tau_2 EV(\mathbf{A}_N(1))}{r_{i2}} \\ &= \overline{\alpha}^i_{k,l}(N-1, M) = c\beta_{1,2} + \beta_{1,2}(1-c)\frac{\tau_2 EV(\mathbf{A}_{N-1}(1))}{r_{i2}} \\ &= ... = \overline{\alpha}^i_{k,l}(M, M) = c\beta_{1,2} + \beta_{1,2}(1-c)\frac{\tau_2 EV(\mathbf{A}_M(1))}{r_{i2}} \text{ for all } i = 1, ..., M-1. \end{aligned}$$

As a result, we have $\alpha^i_{k,l}(N, M) = \alpha^i_{k,l}(N-1, M) = ... = \alpha^i_{k,l}(M, M)$ for all states $i = 1, ..., M-1$.

Proof of Proposition 4.1:

Products 1 and 2 have the same mean processing time, however, the variance of processing time is higher for product 1 than for product 2. Therefore, we have $\tau_1 = \tau_2$, and $\sigma_1^2 > \sigma_2^2$.

We first provide the proof for the case when $\alpha_{1,2}^i = \overline{\alpha}_{1,2}^i$. Let us determine the critical ratios defined by $\overline{\alpha}_{1,2}^i$ for each state $i = 1, 2$. a) For state 1, we have:

$$\begin{aligned} EV[1,1,m] &= EV[2,1,m] \\ \frac{r_{11}(1-p_{22}^1) + r_{21}p_{12}^1 + r_{3m}(1-p_{11}^1)(1-p_{22}^1)}{\tau_2(1-p_{22}^1) + \tau_2 p_{12}^1 + \tau_m(1-p_{11}^1)(1-p_{22}^1)} &= \frac{r_{12}(1-p_{22}^1) + r_{21}p_{12}^2 + r_{3m}(1-p_{11}^2)(1-p_{22}^1)}{\tau_2(1-p_{22}^1) + \tau_2 p_{12}^2 + \tau_m(1-p_{11}^2)(1-p_{22}^1)} \end{aligned}$$

Substituting $p_{12}^1 = p_{12}^2 + \varepsilon_{12}^{12}\left(\sigma_1^2, \sigma_2^2\right)$ and $\left(1-p_{11}^1\right) = \left(1-p_{11}^2\right) - \varepsilon_{11}^{12}\left(\sigma_1^2, \sigma_2^2\right)$ into $EV[2,1,m]$ provides the following:

$$\begin{aligned} r_{11}(1-p_{22}^1) &= r_{12}(1-p_{22}^1) - \varepsilon_{12}^{12}\left(\sigma_1^2, \sigma_2^2\right)(r_{21} - \tau_2 EV[1,1,m]) \\ &\quad + \varepsilon_{11}^{12}\left(\sigma_1^2, \sigma_2^2\right)(1-p_{22}^1)(r_{3m} - \tau_m EV[1,1,m]) \\ \frac{r_{11}}{r_{12}} &= \overline{\alpha}_{1,2}^1 = 1 - \frac{\varepsilon_{12}^{12}\left(\sigma_1^2, \sigma_2^2\right)}{r_{12}(1-p_{22}^1)}(r_{21} - \tau_2 EV[1,1,m]) + \frac{\varepsilon_{11}^{12}\left(\sigma_1^2, \sigma_2^2\right)}{r_{12}}(r_{3m} - \tau_m EV[1,1,m]). \end{aligned}$$

The critical ratio is greater than 1 when

$$\varepsilon_{11}^{12}\left(\sigma_1^2, \sigma_2^2\right)\left(1-p_{22}^1\right)(r_{3m} - \tau_m EV[1,1,m]) > \varepsilon_{12}^{12}\left(\sigma_1^2, \sigma_2^2\right)(r_{21} - \tau_2 EV[1,1,m]).$$

Therefore when this condition is satisfied, $\overline{\alpha}_{1,2}^1$ is increasing in variance; otherwise, it is decreasing in the variance of processing times. b) Similarly, for state 2, we have:

$$\begin{aligned} EV[1,1,m] &= EV[1,2,m] \\ \frac{r_{11}(1-p_{22}^1) + r_{21}p_{12}^1 + r_{3m}(1-p_{11}^1)(1-p_{22}^1)}{\tau_2(1-p_{22}^1) + \tau_2 p_{12}^1 + \tau_m(1-p_{11}^1)(1-p_{22}^1)} &= \frac{r_{11}(1-p_{22}^2) + r_{22}p_{12}^1 + r_{3m}(1-p_{11}^1)(1-p_{22}^2)}{\tau_2(1-p_{22}^2) + \tau_2 p_{12}^1 + \tau_m(1-p_{11}^1)(1-p_{22}^2)} \end{aligned}$$

Substituting $\left(1-p_{11}^1\right) = \left(1-p_{11}^2\right) - \varepsilon_{11}^{12}\left(\sigma_1^2, \sigma_2^2\right)$ into $EV[1,2,m]$ provides the following:

$$\begin{aligned} r_{21}p_{12}^1 &= r_{22}p_{12}^1 + \varepsilon_{11}^{12}\left(\sigma_1^2, \sigma_2^2\right)(r_{21} - \tau_2 EV[1,1,m]) + \varepsilon_{11}^{12}\left(\sigma_1^2, \sigma_2^2\right)(1-p_{11}^1)(r_{3m} - \tau_m EV[1,1,m]) \\ \frac{r_{21}}{r_{22}} &= \overline{\alpha}_{1,2}^2 = 1 + \frac{\varepsilon_{11}^{12}\left(\sigma_1^2, \sigma_2^2\right)}{r_{22}p_{12}^1}(r_{11} - \tau_2 EV[1,1,m]) + \frac{\varepsilon_{11}^{12}\left(\sigma_1^2, \sigma_2^2\right)}{r_{22}p_{12}^1}\left(1-p_{11}^1\right)(r_{3m} - \tau_m EV[1,1,m]) \end{aligned}$$

The critical ratio is greater than 1 when

$$r_{11} - \tau_2 EV[1,1,m] < -\left(1-p_{11}^1\right)(r_{3m} - \tau_m EV[1,1,m]).$$

Therefore when this condition is satisfied, $\overline{\alpha}^2_{1,2}$ is increasing in variance; otherwise, it is decreasing in the variance of processing times.

We next provide the proof for the case when $\alpha^i_{1,2} = \underline{\alpha}^i_{1,2}$. Let us develop the critical ratios defined by $\underline{\alpha}^i_{1,2}$ for each state $i = 1, 2$. a) For state 1, we have:

$$\begin{aligned} EV[1,2,m] &= EV[2,2,m] \\ \frac{r_{11}(1-p^2_{22}) + r_{22}p^1_{12} + r_{3m}(1-p^1_{11})(1-p^2_{22})}{\tau_2(1-p^2_{22}) + \tau_2 p^1_{12} + \tau_m(1-p^1_{11})(1-p^2_{22})} &= \frac{r_{12}(1-p^2_{22}) + r_{22}p^2_{12} + r_{3m}(1-p^2_{11})(1-p^2_{22})}{\tau_2(1-p^2_{22}) + \tau_2 p^2_{12} + \tau_m(1-p^2_{11})(1-p^2_{22})} \end{aligned}$$

Substituting $p^1_{12} = p^2_{12} + \varepsilon^{12}_{12}\left(\sigma^2_1, \sigma^2_2\right)$ and $\left(1 - p^1_{11}\right) = \left(1 - p^2_{11}\right) - \varepsilon^{12}_{11}\left(\sigma^2_1, \sigma^2_2\right)$ into $EV[1,2,m]$ provides the following:

$$\begin{aligned} r_{11}(1-p^2_{22}) &= r_{12}(1-p^2_{22}) - \varepsilon^{12}_{12}\left(\sigma^2_1,\sigma^2_2\right)(r_{22} - \tau_2 EV[2,2,m]) \\ &\quad + \varepsilon^{12}_{11}\left(\sigma^2_1,\sigma^2_2\right)(1-p^2_{22})(r_{3m} - \tau_m EV[2,2,m]) \\ \frac{r_{11}}{r_{12}} &= \underline{\alpha}^1_{1,2} = 1 - \frac{\varepsilon^{12}_{12}\left(\sigma^2_1,\sigma^2_2\right)}{r_{12}(1-p^2_{22})}(r_{22} - \tau_2 EV[2,2,m]) + \frac{\varepsilon^{12}_{11}\left(\sigma^2_1,\sigma^2_2\right)}{r_{12}}(r_{3m} - \tau_m EV[2,2,m]). \end{aligned}$$

The critical ratio is greater than 1 when

$$\varepsilon^{12}_{11}\left(\sigma^2_1,\sigma^2_2\right)\left(1-p^2_{22}\right)(r_{3m} - \tau_m EV[2,2,m]) > \varepsilon^{12}_{12}\left(\sigma^2_1,\sigma^2_2\right)(r_{22} - \tau_2 EV[2,2,m]).$$

Therefore when this condition is satisfied, $\underline{\alpha}^1_{1,2}$ is increasing in variance; otherwise, it is decreasing in the variance of processing times. b) Similarly, for state 2, we have:

$$\begin{aligned} EV[2,1,m] &= EV[2,2,m] \\ \frac{r_{12}(1-p^1_{22}) + r_{21}p^2_{12} + r_{3m}(1-p^1_{11})(1-p^2_{22})}{\tau_2(1-p^1_{22}) + \tau_2 p^2_{12} + \tau_m(1-p^1_{11})(1-p^2_{22})} &= \frac{r_{12}(1-p^2_{22}) + r_{22}p^2_{12} + r_{3m}(1-p^2_{11})(1-p^2_{22})}{\tau_2(1-p^2_{22}) + \tau_2 p^2_{12} + \tau_m(1-p^2_{11})(1-p^2_{22})} \end{aligned}$$

Substituting $\left(1 - p^1_{11}\right) = \left(1 - p^2_{11}\right) - \varepsilon^{12}_{11}\left(\sigma^2_1, \sigma^2_2\right)$ into $EV[2,1,m]$ provides the following:

$$\begin{aligned} r_{21}p^2_{12} &= r_{22}p^2_{12} + \varepsilon^{12}_{11}\left(\sigma^2_1,\sigma^2_2\right)(r_{12} - \tau_2 EV[2,2,m]) + \varepsilon^{12}_{11}\left(\sigma^2_1,\sigma^2_2\right)\left(1-p^2_{11}\right)(r_{3m} - \tau_m EV[2,2,m]) \\ \frac{r_{21}}{r_{22}} &= \underline{\alpha}^2_{1,2} = 1 + \frac{\varepsilon^{12}_{11}\left(\sigma^2_1,\sigma^2_2\right)}{r_{22}p^2_{12}}(r_{12} - \tau_2 EV[2,2,m]) + \frac{\varepsilon^{12}_{11}\left(\sigma^2_1,\sigma^2_2\right)}{r_{22}p^2_{12}}\left(1-p^2_{11}\right)(r_{3m} - \tau_m EV[2,2,m]). \end{aligned}$$

Because $\varepsilon^{12}_{11}\left(\sigma^2_1, \sigma^2_2\right) < 0$, the critical ratio is greater than 1 when

$$(r_{12} - \tau_2 EV[2,2,m]) < -\left(1-p^2_{11}\right)(r_{3m} - \tau_m EV[2,2,m]).$$

Therefore when this condition is satisfied, $\underline{\alpha}^2_{1,2}$ is increasing in variance; otherwise, it is decreasing in the variance of processing times.

Proof of Proposition 4.2:

From Proposition 3.1, we already know the critical ratios when products 2 and 3 are compared. These are:

$$\begin{aligned}
\underline{\alpha}^1_{2,3} &= c\beta_{2,3} - \left(c\beta_{2,3} - \beta_{2,3}\right)\frac{\tau_3 EV\left[3,3,m\right]}{r_{13}} \\
\underline{\alpha}^2_{2,3} &= c\beta_{2,3} - \left(c\beta_{2,3} - \beta_{2,3}\right)\frac{\tau_3 EV\left[3,3,m\right]}{r_{23}} \\
\overline{\alpha}^1_{2,3} &= c\beta_{2,3} - \left(c\beta_{2,3} - \beta_{2,3}\right)\frac{\tau_3 EV\left[2,2,m\right]}{r_{13}} \\
\overline{\alpha}^2_{2,3} &= c\beta_{2,3} - \left(c\beta_{2,3} - \beta_{2,3}\right)\frac{\tau_3 EV\left[2,2,m\right]}{r_{23}}
\end{aligned}$$

We next compare products 1 and 3 in order to develop the critical ratios that capture the combined effect of mean and variance in processing times. Product 1 has a higher mean and variance in its processing time than product 3. Thus, $\tau_1 = \tau_2 > \tau_3$ and $\sigma_1^2 > \sigma_2^2 = \sigma_3^2$. We first develop the critical ratios defined by $\underline{\alpha}^i_{1,3}$ for each state $i = 1, 2$. a) We can determine the critical ratio for state 1 by equating $EV\left[1,3,m\right] = EV\left[3,3,m\right]$.

$$\begin{aligned}
EV\left[1,3,m\right] &= EV\left[3,3,m\right] \\
\frac{r_{11}(1-p^3_{22}) + r_{23}p^1_{12} + r_{3m}(1-p^1_{11})(1-p^3_{22})}{\tau_2(1-p^3_{22}) + \tau_3 p^1_{12} + \tau_m(1-p^1_{11})(1-p^3_{22})} &= \frac{r_{13}(1-p^3_{22}) + r_{23}p^3_{12} + r_{3m}(1-p^3_{11})(1-p^3_{22})}{\tau_3(1-p^3_{22}) + \tau_3 p^3_{12} + \tau_m(1-p^3_{11})(1-p^3_{22})}
\end{aligned}$$

Note that $p^1_{12} = c\beta_{2,3}p^3_{12} + \varepsilon^{12}_{12}\left(\sigma_1^2, \sigma_2^2\right)$ and $(1-p^1_{11})(1-p^3_{22}) = c\beta_{2,3}(1-p^3_{11})(1-p^3_{22}) - \varepsilon^{12}_{11}\left(\sigma_1^2, \sigma_2^2\right)(1-p^3_{11})$. Substituting these two expressions in $EV\left[1,3,m\right]$, we get the following:

$$\begin{aligned}
\frac{r_{11}}{r_{13}} &= \underline{\alpha}^1_{1,3} = c\beta_{2,3} - \left(c\beta_{2,3} - \beta_{2,3}\right)\frac{\tau_3 EV\left[3,3,m\right]}{r_{13}} \\
&\quad - \frac{\varepsilon^{12}_{12}\left(\sigma_1^2, \sigma_2^2\right)}{r_{13}\left(1-p^3_{22}\right)}\left(r_{23} - \tau_3 EV\left[3,3,m\right]\right) + \frac{\varepsilon^{12}_{11}\left(\sigma_1^2, \sigma_2^2\right)}{r_{13}}\left(r_{3m} - \tau_m EV\left[3,3,m\right]\right) \\
\underline{\alpha}^1_{1,3} &= \underline{\alpha}^1_{2,3} - \frac{\varepsilon^{12}_{12}\left(\sigma_1^2, \sigma_2^2\right)}{r_{13}\left(1-p^3_{22}\right)}\left(r_{23} - \tau_3 EV\left[3,3,m\right]\right) + \frac{\varepsilon^{12}_{11}\left(\sigma_1^2, \sigma_2^2\right)}{r_{13}}\left(r_{3m} - \tau_m EV\left[3,3,m\right]\right)
\end{aligned}$$

The critical ratio $\underline{\alpha}^1_{1,3}$ is greater than $\underline{\alpha}^1_{2,3}$ when

$$\varepsilon^{12}_{11}\left(\sigma_1^2, \sigma_2^2\right)\left(1-p^3_{22}\right)\left(r_{3m} - \tau_m EV\left[3,3,m\right]\right) > \varepsilon^{12}_{12}\left(\sigma_1^2, \sigma_2^2\right)\left(r_{23} - \tau_3 EV\left[3,3,m\right]\right).$$

Therefore, when this condition is satisfied, the firm needs to earn a higher reward to switch from product 3 to product 1 than from product 3 to product 2 in state 1. b) Similarly, we can obtain the critical ratio for state 2 as follows:

$$\begin{aligned} EV[3,1,m] &= EV[3,3,m] \\ \frac{r_{13}(1-p_{22}^1)+r_{21}p_{12}^3+r_{3m}(1-p_{11}^3)(1-p_{22}^1)}{\tau_3(1-p_{22}^1)+\tau_2 p_{12}^3+\tau_m(1-p_{11}^3)(1-p_{22}^1)} &= \frac{r_{13}(1-p_{22}^3)+r_{23}p_{12}^3+r_{3m}(1-p_{11}^3)(1-p_{22}^3)}{\tau_3(1-p_{22}^3)+\tau_3 p_{12}^3+\tau_m(1-p_{11}^3)(1-p_{22}^3)} \end{aligned}$$

Note that $(1-p_{22}^1)=c\beta_{2,3}(1-p_{22}^3)-\varepsilon_{11}^{12}\left(\sigma_1^2,\sigma_2^2\right)$ and $(1-p_{11}^1)(1-p_{22}^3)=c\beta_{2,3}(1-p_{11}^3)(1-p_{22}^3)-\varepsilon_{11}^{12}\left(\sigma_1^2,\sigma_2^2\right)(1-p_{11}^3)$. Substituting these two expressions in $EV[3,1,m]$, we get the following:

$$\begin{aligned} \frac{r_{21}}{r_{23}} &= \underline{\alpha}_{1,3}^2 = c\beta_{2,3}-\left(c\beta_{2,3}-\beta_{2,3}\right)\frac{\tau_3 EV[3,3,m]}{r_{23}} \\ &\quad +\frac{\varepsilon_{11}^{12}\left(\sigma_1^2,\sigma_2^2\right)}{r_{23}p_{12}^3}\left(r_{13}-\tau_3 EV[3,3,m]\right)+\frac{\varepsilon_{11}^{12}\left(\sigma_1^2,\sigma_2^2\right)}{r_{23}p_{12}^3}\left(1-p_{11}^3\right)\left(r_{3m}-\tau_m EV[3,3,m]\right) \\ \underline{\alpha}_{1,3}^2 &= \underline{\alpha}_{2,3}^2+\frac{\varepsilon_{11}^{12}\left(\sigma_1^2,\sigma_2^2\right)}{r_{23}p_{12}^3}\left[\left(r_{13}-\tau_3 EV[3,3,m]\right)+\left(1-p_{11}^3\right)\left(r_{3m}-\tau_m EV[3,3,m]\right)\right] \end{aligned}$$

The critical ratio $\underline{\alpha}_{1,3}^2$ is greater than $\underline{\alpha}_{2,3}^2$ when

$$r_{13}-\tau_3 EV[3,3,m] < -\left(1-p_{11}^3\right)\left(r_{3m}-\tau_m EV[3,3,m]\right).$$

Therefore, when this condition is satisfied, the firm needs to earn a higher reward to switch from product 3 to product 1 than from product 3 to product 2 in state 2.

Using product 2 as the reference product and $EV[2,2,m]$ as the calibrating reference policy, we next develop the critical ratios defined by $\overline{\alpha}_{1,3}^i$ for each state $i=1,2$. c) For state 1, we determine the critical ratio as follows:

$$\begin{aligned} EV[3,1,m] &= EV[2,2,m] \\ \frac{r_{13}(1-p_{22}^1)+r_{21}p_{12}^3+r_{3m}(1-p_{11}^3)(1-p_{22}^1)}{\tau_3(1-p_{22}^1)+\tau_2 p_{12}^3+\tau_m(1-p_{11}^3)(1-p_{22}^1)} &= \frac{r_{12}(1-p_{22}^2)+r_{22}p_{12}^2+r_{3m}(1-p_{11}^2)(1-p_{22}^2)}{\tau_2(1-p_{22}^2)+\tau_2 p_{12}^2+\tau_m(1-p_{11}^2)(1-p_{22}^2)} \end{aligned}$$

Note that $(1-p_{22}^1)=(1-p_{22}^2)-\varepsilon_{11}^{12}\left(\sigma_1^2,\sigma_2^2\right)$, and $p_{12}^3=\frac{1}{c\beta_{2,3}}p_{12}^2$, and $(1-p_{11}^3)(1-p_{22}^1)=\frac{1}{c\beta_{2,3}}(1-p_{11}^2)(1-p_{22}^2)-\frac{\varepsilon_{11}^{12}\left(\sigma_1^2,\sigma_2^2\right)(1-p_{11}^2)}{c\beta_{2,3}}$. Substituting these three expressions in $EV[3,1,m]$, we get the following:

$$\begin{aligned} \overline{\alpha}_{1,3}^1 &= c\beta_{2,3}-\left(c\beta_{2,3}-\beta_{2,3}\right)\frac{\tau_2 EV[2,2,m]}{r_{13}} \\ &\quad -\frac{\varepsilon_{11}^{12}\left(\sigma_1^2,\sigma_2^2\right)}{r_{13}\left(1-p_{11}^2\right)}c\beta_{23}\left(r_{13}-\tau_2 EV[2,2,m]\right)-\frac{\varepsilon_{11}^{12}\left(\sigma_1^2,\sigma_2^2\right)}{r_{13}}\left(r_{3m}-\tau_m EV[2,2,m]\right) \\ \overline{\alpha}_{13}^1 &= \overline{\alpha}_{23}^1-\frac{\varepsilon_{11}^{12}\left(\sigma_1^2,\sigma_2^2\right)}{r_{13}\left(1-p_{11}^2\right)}\left[c\beta_{23}\left(r_{13}-\tau_2 EV[2,2,m]\right)+\left(1-p_{11}^2\right)\left(r_{3m}-\tau_m EV[2,2,m]\right)\right] \end{aligned}$$

Because $\varepsilon_{11}^{12}\left(\sigma_1^2,\sigma_2^2\right)<0$, the critical ratio $\overline{\alpha}_{1,3}^1$ is greater than $\overline{\alpha}_{2,3}^1$ when

$$c\beta_{23}\left(r_{13}-\tau_2 EV\left[2,2,m\right]\right)>-\left(1-p_{22}^2\right)\left(r_{3m}-\tau_m EV\left[2,2,m\right]\right).$$

Therefore, when this condition is satisfied, the firm needs to earn a higher reward to switch from product 1 to product 3 than from product 2 to 3 in state 1. d) For state 2, we determine the critical ratio as follows:

$$\begin{aligned} EV\left[1,3,m\right] &= EV\left[2,2,m\right] \\ \frac{r_{11}(1-p_{22}^3)+r_{23}p_{12}^1+r_{3m}(1-p_{11}^1)(1-p_{22}^3)}{\tau_2(1-p_{22}^3)+\tau_3 p_{12}^1+\tau_m(1-p_{11}^1)(1-p_{22}^3)} &= \frac{r_{12}(1-p_{22}^2)+r_{22}p_{12}^2+r_{3m}(1-p_{11}^2)(1-p_{22}^2)}{\tau_2(1-p_{22}^2)+\tau_2 p_{12}^2+\tau_m(1-p_{11}^2)(1-p_{22}^2)} \end{aligned}$$

Note that $(1-p_{22}^3)=\frac{1}{c\beta_{2,3}}(1-p_{22}^2)$, and $p_{12}^1=p_{12}^2+\varepsilon_{12}^{12}\left(\sigma_1^2,\sigma_2^2\right)$, and $(1-p_{11}^1)(1-p_{22}^3)=\frac{1}{c\beta_{2,3}}(1-p_{11}^2)(1-p_{22}^2)-\frac{\varepsilon_{11}^{12}\left(\sigma_1^2,\sigma_2^2\right)(1-p_{22}^2)}{c\beta_{2,3}}$. Substituting these three expressions in $EV\left[1,3,m\right]$, we get the following:

$$\begin{aligned} \overline{\alpha}_{1,3}^2 &= c\beta_{2,3}-\left(c\beta_{2,3}-\beta_{2,3}\right)\frac{\tau_2 EV\left[2,2,m\right]}{r_{23}} \\ &\quad+\frac{\varepsilon_{12}^{12}\left(\sigma_1^2,\sigma_2^2\right)}{r_{23}p_{12}^2}c\beta_{23}\left(r_{23}-\tau_2 EV\left[2,2,m\right]\right)-\frac{\varepsilon_{11}^{12}\left(\sigma_1^2,\sigma_2^2\right)\left(1-p_{11}^2\right)}{r_{23}p_{12}^3}\left(r_{3m}-\tau_m EV\left[2,2,m\right]\right) \\ \overline{\alpha}_{13}^2 &= \overline{\alpha}_{23}^2+\frac{\varepsilon_{12}^{12}\left(\sigma_1^2,\sigma_2^2\right)}{r_{23}p_{12}^2}c\beta_{23}\left(r_{23}-\tau_2 EV\left[2,2,m\right]\right)-\frac{\varepsilon_{11}^{12}\left(\sigma_1^2,\sigma_2^2\right)\left(1-p_{11}^2\right)}{r_{23}p_{12}^3}\left(r_{3m}-\tau_m EV\left[2,2,m\right]\right) \end{aligned}$$

The critical ratio $\overline{\alpha}_{1,3}^2$ is greater than $\overline{\alpha}_{2,3}^2$ when

$$\varepsilon_{12}^{12}\left(\sigma_1^2,\sigma_2^2\right)c\beta_{23}\left(r_{23}-\tau_2 EV\left[2,2,m\right]\right)>\varepsilon_{11}^{12}\left(\sigma_1^2,\sigma_2^2\right)\left(r_{3m}-\tau_m EV\left[2,2,m\right]\right).$$

Therefore, when this condition is satisfied, the firm needs to earn a higher reward to switch from product 1 to product 3 than from product 2 to 3 in state 2.

Proof of Corollary 4.1:

The proof follows directly from Proposition 3.1.