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SKEWNESS AND KURTOSIS IN S&P 500 INDEX RETURNS IMPLIED BY OPTION PRICES

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Abstract

The Black-Scholes (1973) model frequently misprices deep-in-the-money and deep-out-of-the-money options. Practitioners popularly refer to these strike price biases as volatility smiles. In this paper we examine a method to extend the Black-Scholes model to account for biases induced by nonnormal skewness and kurtosis in stock return distributions. The method adapts a Gram-Charlier series expansion of the normal density function to provide skewness and kurtosis adjustment terms for the Black-Scholes formula. Using this method, we estimate option-implied coefficients of skewness and kurtosis in S&P 500 stock index returns. We find significant nonnormal skewness and kurtosis implied by option prices.

I. Introduction

The Black-Scholes (1973) option pricing model is commonly used to value a wide range of derivative securities. Despite its usefulness, however, the model has some well-known deficiencies. For example, the Black-Scholes model frequently misprices deep-in-the-money and deep-out-of-the-money options. These mispricing patterns are thought to result from the parsimonious assumptions used to derive the model. In particular, the Black-Scholes model assumes that stock log prices follow a constant variance diffusion process, where over any finite interval a log price is normally distributed. Early studies by Black and Scholes (1972) and Officer (1973) test and reject the validity of the constant variance assumption. Since then, a vast body of research, most notably the conditional heteroskedasticity literature originating with Engle (1982) and Bollerslev (1986), documents the volatile nature of stock return variances. Financial economists now know that stock return variances are stochastic and correlated with stock price

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levels. Consequently, stock return distributions are skewed and kurtotic relative to a normal distribution. Heston (1993), Hull and White (1987), Scott (1987), Stein and Stein (1991), and Wiggins (1987) show that under these conditions, the original Black-Scholes model yields option prices that are expected to differ systematically from observed option prices.

As Black (1975) points out, "One possible explanation for this [mispricing] pattern is that we have left something out of the formula." In this paper we derive a relatively simple method to extend the Black-Scholes formula to account for nonnormal skewness and kurtosis in stock return distributions. Our methodology is analogous to Jarrow and Rudd (1982). We begin by expanding the Black-Scholes formula to account for nonnormal skewness and kurtosis in stock return distributions. Then, based on this expanded Black-Scholes formula, we estimate coefficients of skewness and kurtosis implied by option prices. This methodology extends the widely used procedure of obtaining implied standard deviations to include obtaining implied coefficients of skewness and kurtosis.

II. Previous Studies of the Black-Scholes Model

Pricing biases associated with the Black-Scholes option pricing model are well documented. Black (1975), Emanuel and MacBeth (1982), MacBeth and Merville (1979), and Rubinstein (1985) report that the Black-Scholes model tends to systematically misprice in-the-money and out-of-the-money options. Early tests by Black (1975) find that the Black-Scholes model underprices deep-out-of-the-money stock options and overprices deep-in-the-money stock options. Later, MacBeth and Merville (1979) examine Chicago Board Options Exchange (CBOE) call options from December 1975 to December 1976. In contrast to Black (1975), they find that the Black-Scholes model overprices out-of-the-money call options and underprices in-the-money call options.

Rubinstein (1985) studies options price data for the thirty most actively traded option classes on the CBOE between August 1976 and August 1978. He divides the data into two subgroups. The first subgroup includes data from August 1976 to October 1977. In this period, Rubinstein reports a systematic mispricing pattern similar to that reported by MacBeth and Merville (1979), where the Black-Scholes model overprices out-of-the-money options and underprices in-the-money options. The second subgroup includes data from October 1977 to August 1978. During this period, he reports a systematic mispricing pattern similar to that reported by Black (1975), where the Black-Scholes model underprices out-of-the-money options and overprices in-the-money options. Rubinstein concludes that strike price biases for the Black-Scholes model are significant and that the direction of bias tends to be the same for most options at any point. However, the bias direction can change across different periods.

Jarrow and Rudd (1982) propose a semiparametric option pricing model to account for observed strike price biases in the Black-Scholes model. They derive an option pricing formula from an Edgeworth expansion of the lognormal probability density function to model the distribution of stock prices. In this paper we derive a semiparametric option pricing formula analogous to that of Jarrow and Rudd. The chief difference is that we use a Gram-Charlier series expansion of the normal probability density function to model the distribution of stock log prices. Stuart and Ord (1987, pp. 222–25) discuss the distinction between an Edgeworth expansion and a Gram-Charlier expansion. Operationally, the Jarrow and Rudd method accounts for skewness and kurtosis deviations from lognormality for stock prices, while the method developed here accounts for skewness and kurtosis deviations from normality for stock returns. As it turns out, both models are equally effective in providing accurate option price adjustment terms. In this paper we use a model based on skewness and kurtosis deviations from normality in stock returns because skewness and kurtosis coefficients for all normal distributions are 0 and 3, respectively (Stuart and Ord (1987, p. 183)). In contrast, skewness and kurtosis coefficients for lognormal distributions vary across different lognormal distributions (Aitchison and Brown (1963)). As a result, it is more convenient to report and interpret empirical results based on observed skewness and kurtosis deviations from a normal distribution since our reference points are constants.

III. Derivation of a Skewness- and Kurtosis-adjusted Black-Scholes Model

To incorporate option price adjustments for nonnormal skewness and kurtosis in an expanded Black-Scholes option pricing formula, we use a Gram-Charlier series expansion of a normal density function. Stuart and Ord (1987, pp. 222–23) discuss this approach, which is similar to a Taylor series expansion for analytic functions. From this expanded density, we obtain an option price formula that is the sum of a Black-Scholes price plus adjustment terms for nonnormal skewness and kurtosis. We derive this expanded formula below.

A Gram-Charlier series expansion (Type A) of the density function $f(x)$ is defined as

$$f(x) = \sum_{n=0}^{\infty} c_n H_n(x) \varphi(x)$$

where $\varphi(x)$ is a normal density function, $H_n(x)$ are Hermite polynomials derived from successively higher derivatives of $\varphi(x)$, and the coefficients c_n are determined by moments of the distribution function $F(x)$. This expansion is an infinite series. However, in actual applications the series is truncated to exclude terms beyond the fourth moment. The resulting truncated density provides an approximation that accounts for nonnormal skewness and kurtosis. Specifically, after standardizing to a mean zero and unit variance, a truncated series that accounts for skewness and kurtosis yields the following density function where μ_3 and μ_4 denote standardized coefficients of skewness and kurtosis, respectively:

$$g(z) = n(z)\left[1 + \frac{\mu_3}{3!}(z^3 - 3z) + \frac{\mu_4 - 3}{4!}(z^4 - 6z^2 + 3)\right] \quad (1)$$

where

$$n(z) = \frac{1}{\sqrt{2\pi}}\exp(-z^2/2);$$

$$z = \frac{\ln(S_t/S_0) - (r - \sigma^2/s)t}{\sigma\sqrt{t}};$$

S_0 = current stock price;
S_t = random stock price at time t;
r = risk-free interest rate;
t = time remaining until option maturity; and
σ = standard deviation of returns for the underlying stock.

An important property of the density function $g(z)$ in equation (1) is that it yields the following expected values: $E(z) = 0$, $E(z^2) = 1$, $E(z^3) = \mu_3$, $E(z^4) = \mu_4$. Thus, the coefficients of skewness and kurtosis for $g(z)$ are explicit parameters in its functional form. Under a normal specification we have the skewness and kurtosis coefficients $\mu_3 = 0$ and $\mu_4 = 3$, respectively, which upon substitution into $g(z)$ yield the special case of a standard normal density $n(z)$.

Assuming risk neutrality, we apply the density function $g(z)$ in equation (1) to derive a theoretical European call option price as the present value of an expected payoff at option expiration. Mathematically, this option price is derived from the following expression,

$$C_{GC} = e^{-rt}\int_k^{\infty}(S_t - K)g(z(S_t))dz(S_t) \quad (2)$$

where $z(S_t) = (\log S_t - \mu)/\sigma\sqrt{t}$, $\mu = \log S_0 + (r - \sigma^2/2)t$, and K is the option's strike price. We evaluate this integral in the Appendix. From it, we obtain the following option price formula based on a Gram-Charlier density expansion, denoted here by C_{GC}:

$$C_{GC} = C_{BS} + \mu_3 Q_3 + (\mu_4 - 3)Q_4 \tag{3}$$

where

$C_{BS} = S_0 N(d) - Ke^{-rt} N(d - \sigma\sqrt{t})$ is the Black-Scholes option pricing formula;

$$Q_3 = \frac{1}{3!} S_0 \sigma\sqrt{t} \left((2\sigma\sqrt{t} - d)n(d) - \sigma^2 t\, N(d)\right);$$

$$Q_4 = \frac{1}{4!} S_0 \sigma\sqrt{t} \left((d^2 - 1 - 3\sigma\sqrt{t}(d - \sigma\sqrt{t}))n(d) + \sigma^3 t^{3/2} N(d)\right); \text{ and}$$

$$d = \frac{\ln(S_0/K) + (r + \sigma^2/2)t}{\sigma\sqrt{t}}.$$

In equation (3), Q_3 and Q_4 represent the marginal effect of nonnormal skewness and kurtosis, respectively, on the option price C_{GC}. To graphically assess the expected effect of skewness and kurtosis on option prices, we plot Q_3 and Q_4 in Figure I in an example where $S_0 = 100$, $\sigma = 15$ percent, t = three months, $r = 0.04$, and K varies from 75 to 125. The horizontal axis measures option moneyness, defined as the percent difference between a stock price and a discounted strike price:

$$\text{Moneyness}(\%) = \frac{Ke^{-rt} - S_0}{Ke^{-rt}} \times 100.$$

The vertical axis measures dollar values for Q_3 and Q_4. The effects of skewness and kurtosis on option prices displayed in Figure I are qualitatively similar to Figures II and IV in Heston (1993). Most important, negative (positive) skewness causes the Black-Scholes model to overprice (underprice) out-of-the-money options and underprice (overprice) in-the-money options.

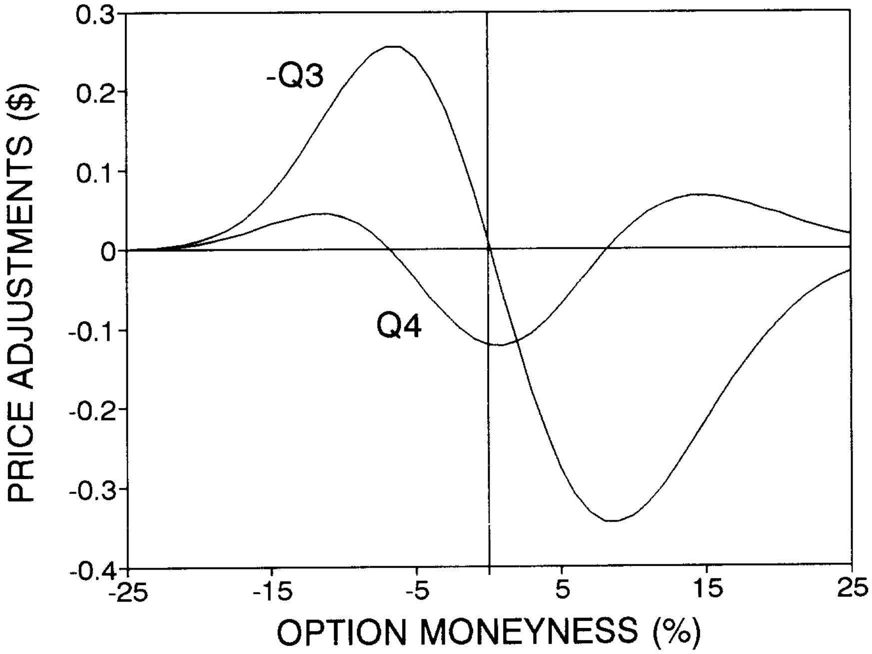

Figure I. Adjustments for Skewness and Kurtosis. Option price adjustments for negative skewness ($-Q_3$) and positive kurtosis (Q_4) deviations from normality.

Equation (3) is the fundamental option price formula of this study. If stock returns are normally distributed, then $\mu_3 = 0$ and $\mu_4 = 3$ and equation (3) collapses to the Black-Scholes option price formula. If $\mu_3 \neq 0$ and $\mu_4 \neq 3$, equation (3) is the sum of a Black-Scholes option price plus adjustment terms for nonnormal skewness and kurtosis.

IV. Data

Price data for this study come from the Berkeley Options Data Base of CBOE options. This database includes records of bid-ask price quotations and transactions time-stamped to the nearest second. We base this study on the CBOE market for S&P 500 index options. Rubinstein (1994) argues this market best approximates conditions required for Black-Scholes model accuracy. Option prices, index levels, strike prices, and option maturities come directly from the Berkeley database. To avoid bid-ask bounce problems in transaction data, we take option prices as midpoints of bid-ask price quotations. U.S. Treasury bill rates with maturities closest to option expirations state the risk-free interest rate. Interest rate information is culled from the *Wall Street Journal.* Since S&P 500 index options are European style, we use the method suggested by Black (1975) to adjust index levels by subtracting present values of future dividend payments before each option's expiration date. The S&P 500 Information Bulletin provides daily S&P 500 index dividends.

Following data screening procedures in Barone-Adesi and Whaley (1986), we delete all options with prices less than $0.125 and all option transactions occurring before 9 a.m. We also purge obvious outliers from the sample, including option prices that lie outside well-known no-arbitrage option price boundaries (Merton (1973)). We report results obtained from option price quotations for contracts traded in November 1990 and December 1993.

V. Estimation Procedures and Results

Our first set of estimation procedures assesses the performance of the Black-Scholes option pricing model. In these procedures we estimate implied standard deviations (ISD) on a daily basis for call options on the S&P 500 index. On a given day for a given option maturity class, we obtain a unique ISD from all bid-ask price midpoints using Whaley's (1982) simultaneous equations procedure. We use this ISD as an input to calculate theoretical Black-Scholes option prices for all price observations within the same maturity class. We then compare these theoretical Black-Scholes prices with corresponding market-observed prices.

Our second set of estimation procedures assesses the performance of the skewness- and kurtosis-adjusted Black-Scholes option pricing formula derived in the previous section. In these procedures we simultaneously estimate ISD, implied skewness (ISK), and implied kurtosis (IKT) parameters using all bid-ask midpoints on a given day for a given maturity class. We then compare these theoretical skewness- and kurtosis-adjusted Black-Scholes option prices with their corresponding market-observed prices.

The Black-Scholes Option Pricing Model

The Black-Scholes option price formula requires five inputs: a security price, a strike price, a risk-free interest rate, an option maturity, and a return standard deviation. Only the return standard deviation is not directly observable. Adopting Whaley's (1982) simultaneous equations procedure, we estimate a return standard deviation by minimizing the following sum of squares with respect to the argument BSISD, which denotes a Black-Scholes implied standard deviation.

TABLE 1. Comparison of Black-Scholes Prices and Observed Prices of S&P 500 Index Options (SPX) for November 1990.

Date	No. of Price Obs.	Implied Std. Dev. (%)	Proportion of Theoretical Prices Outside Bid-Ask Spreads	Avg. Deviation of Theoretical Prices from Spread Boundaries ($)	Avg. Call Price ($)	Avg. Bid-Ask Spread ($)
11/2/90	1050	26.66	0.91	1.68	22.52	0.71
11/6/90	1107	25.70	0.91	1.58	22.78	0.70
11/8/90	2274	26.29	0.91	1.53	21.08	0.73
11/12/90	1367	24.21	0.91	1.51	22.85	0.69
11/14/90	1216	23.31	0.92	1.54	22.51	0.67
11/16/90	1898	22.59	0.91	1.47	21.52	0.78
11/20/90	1079	22.58	0.88	1.10	24.70	0.86
11/26/90	1084	24.45	0.83	1.17	24.62	0.86
11/28/90	672	24.70	0.87	1.18	26.04	0.82
11/30/90	1254	24.58	0.88	1.23	25.56	0.78
Average	1300	24.51	0.89	1.40	23.42	0.76

Notes: On each day indicated, a Black-Scholes implied standard deviation (BSISD) is estimated from current price observations. Theoretical Black-Scholes option prices are then calculated using BSISD. All observations correspond to call options traded in November 1990 and expiring in March 1991.

$$\min_{\text{BSISD}} \sum_{j=1}^{N} \left[C_{\text{OBS}} - C_{\text{BS},j(\text{BSISD})}\right]^2 \tag{4}$$

In equation (4) above, N is the total number of price quotations available on a given day for a given maturity class, C_{OBS} is a market-observed call price, and C_{BS}(BSISD) is a theoretical Black-Scholes call price calculated using the standard deviation parameter BSISD. Based on the value of BSISD that minimizes the sum of squared errors in equation (4), we calculate theoretical Black-Scholes option prices for all options on a given day within the same maturity class. We then compare these theoretical Black-Scholes option prices with their corresponding market-observed prices.

Table 1 summarizes our calculations for S&P 500 index call option prices observed during November 1990 for options maturing in March 1991. Column 1 lists sampling dates within the month. To maintain table compactness, we report results only for even-numbered dates within the month. Column 2 lists the number of price quotations available on each date. Black-Scholes implied standard deviations (BSISD) for each date are reported in column 3. To assess the economic significance of differences between theoretical and observed prices, we list the proportion of theoretical Black-Scholes option prices lying outside their bid-ask spreads, either below the bid price or above the ask price, in column 4.

Column 5 lists the average absolute deviation of theoretical prices from bid-ask boundaries for prices lying outside their bid-ask spreads. Specifically, for each theoretical option price lying outside its corresponding bid-ask spread, we calculate an absolute deviation according to the following formula:

$$\max(C_{\mathrm{BS}} - \mathrm{Ask}, \mathrm{Bid} - C_{\mathrm{BS}})$$

This absolute deviation statistic measures the economic significance of deviations of theoretical option prices from observed bid-ask spreads. Finally, column 6 lists day-by-day averages of observed call prices, and column 7 lists day-by-day averages of observed bid-ask spreads.

In Table 1, the bottom row lists column averages for all variables. For example, the average number of daily price observations is 1,300 (column 2), with an average option price of $23.42 (column 6) and an average bid-ask spread of $0.76 (column 7). The average implied standard deviation is 24.51 percent (column 3). The average proportion of theoretical Black-Scholes prices lying outside their corresponding bid-ask spreads is 89 percent (column 4), with an average deviation of $1.40 for observations lying outside a spread boundary. This average deviation is almost twice as large as the average bid-ask spread of $0.76.

Figure II displays deviations of observed call prices from theoretical Black-Scholes call prices for November 6, 1990. The horizontal axis measures option moneyness and the vertical axis measures price deviations in dollars. As previously defined, option moneyness is the percentage difference between a discounted strike price and a dividend-adjusted stock index level. A negative (positive) percentage corresponds to in-the-money (out-of-the-money) options with low (high) strike prices. Price deviations measured on the vertical axis are observed prices minus theoretical prices. So defined, the fixed horizontal zero axis corresponds to theoretical Black-Scholes prices, and the dots correspond to observed call prices relative to theoretical Black-Scholes prices.

Figure II reveals that the Black-Scholes model systematically overvalues out-of-the-money options and undervalues in-the-money options for this sample of S&P 500 index call options. Moreover, the mispricing is significant. For example, Figure II shows that for options more than 5 percent in-the-money or out-of-the-money, the typical deviation between observed prices and theoretical Black-Scholes prices is more than two dollars, or $200 per contract. By contrast, the average bid-ask spread is $0.76, or $76 per contract.

Skewness- and Kurtosis-adjusted Black-Scholes Model

In our second set of estimation procedures, on a given day within a given option maturity class, we simultaneously estimate return standard deviation, skewness, and kurtosis parameters by minimizing the following sum of squares with respect to the arguments ISD, ISK, and IKT:

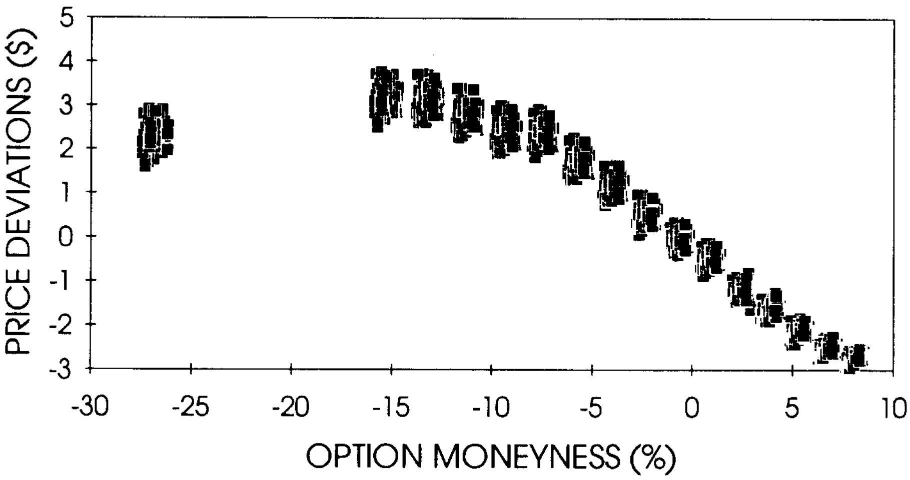

Figure II. Black-Scholes Formula (Four-month Options).

$$\min_{\text{ISD,ISK,IKT}} \sum_{j=1}^{N} \left[C_{\text{OBS},j} - (C_{\text{BS},j}(\text{ISD}) + \text{ISK}\, Q_3 + (\text{IKT} - 3)Q_4)\right]^2 \tag{5}$$

The resulting values for ISD, ISK, and IKT represent maximum likelihood estimates of implied standard deviation, implied skewness, and implied kurtosis parameters based on N price observations. Substituting ISD, ISK, and IKT estimates into equation (3), a skewness- and kurtosis-adjusted Black-Scholes option price is expressed as

$$C_{\text{GC}} = C_{\text{BS}}(\text{ISD}) + \text{ISK}\, Q_3 + (\text{IKT} - 3)Q_4 . \tag{6}$$

Equation (6) yields theoretical skewness- and kurtosis-adjusted Black-Scholes option prices from which we calculate deviations of theoretical prices from market-observed prices.

Table 2 summarizes our calculations for the same S&P 500 call option prices used to compile Table 1. Consequently, column 1 in Table 2 lists the same even-numbered dates and column 2 lists the same number of price quotations that are listed in Table 1. However, to assess the out-of-sample forecasting power of skewness and kurtosis adjustments, the simultaneously estimated ISD, ISK, and IKT coefficients are estimated from prices observed on trading days immediately before the dates in column 1. Thus, out-of-sample parameters ISD, ISK, and IKT reported in columns 3, 4, and 5, respectively, correspond to one-day lagged estimates. We use these one-day lagged values of ISD, ISK, and IKT to calculate theoretical skewness- and kurtosis-adjusted Black-Scholes option prices

TABLE 2. Comparison of Skewness- and Kurtosis-adjusted Black-Scholes Prices and Observed Prices of S&P 500 Index Options (SPX) for November 1990.

Date	No. of Price Obs.	Implied Std. Dev. (%)	Implied Skewness	Implied Kurtosis	Proportion of Theoretical Prices Outside Bid-Ask Spreads	Avg. Deviation of Theoretical Prices from Spread Boundaries ($)
11/2/90	1050	29.77	-1.66	3.25	0.81	0.48
11/6/90	1107	28.83	-1.88	3.82	0.86	0.33
11/8/90	2274	27.96	-1.59	3.23	0.55	0.22
11/12/90	1367	27.80	-1.78	4.01	0.98	0.96
11/14/90	1216	25.20	-1.81	3.61	0.68	0.38
11/16/90	1898	24.38	-1.87	3.37	0.54	0.39
11/20/90	1079	22.83	-1.83	3.82	0.14	0.16
11/26/90	1084	22.72	-1.52	3.01	0.95	0.61
11/28/90	672	24.97	-1.69	3.38	0.39	0.26
11/30/90	1254	24.19	-1.59	3.23	0.40	0.22
Average	1300	25.87	-1.72	3.47	0.63	0.40

Notes: On each day indicated, implied standard deviation (ISD), skewness (ISK), and kurtosis (IKT) parameters are estimated from one-day lagged price observations. Theoretical option prices are then calculated using these out-of-sample implied parameters. All observations correspond to call options traded in November 1990 and expiring in March 1991.

according to equation (6) for all price observations on the even-numbered dates listed in column 1. All skewness coefficients in column 4 are negative, with a column average of −1.72. All kurtosis coefficients in column 5 are greater than 3, with a column average of 3.47. Normal distribution skewness and kurtosis values are 0 and 3, respectively.

Column 6 of Table 2 lists the proportion of skewness- and kurtosis-adjusted prices lying outside their corresponding bid-ask spread boundaries. The column average proportion is 63 percent. Column 7 lists average absolute deviations of theoretical prices from bid-ask spread boundaries only for prices lying outside their bid-ask spreads. The column average price deviation is $0.40, or $40 per contract, which is close to half the size of the average bid-ask spread of $0.76, or $76 per contract, reported in Table 1.

We assess the statistical significance of the improvement in performance from out-of-sample adjustments for skewness and kurtosis using the following Z-statistic for the difference between two proportions (Hoel (1984)):

$$Z = \frac{p_1 - p_2}{\sqrt{p_1(1 - p_1)/N_1 + p_2(1 - p_2)/N_2}}$$

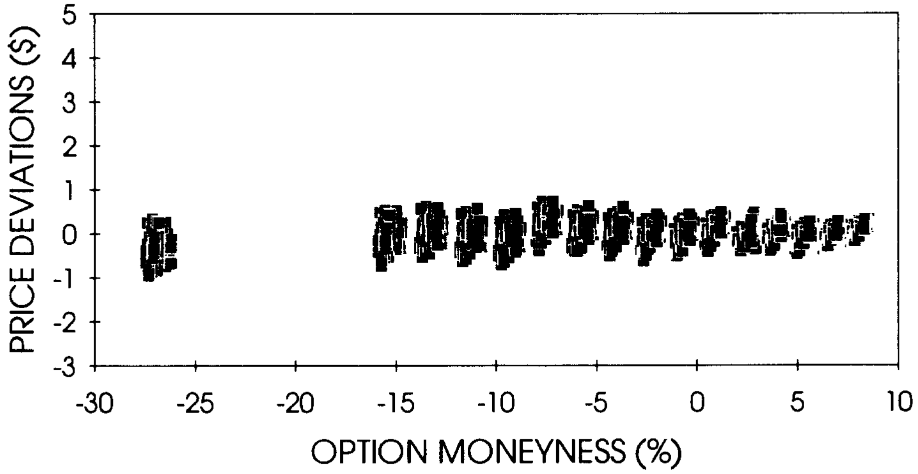

Figure III. Skewness- and Kurtosis-adjusted Formula (Four-month Options).

In this statistic, p_1 and p_2 are sample proportions, and N_1 and N_2 are sample sizes corresponding to these proportions. For example, from Table 1 we get the volume-weighted average proportion $p_1 = .90$ and from Table 2 we get the volume-weighted average proportion $p_2 = .63$. Both of these proportions are based on a total sample size of $N = 13{,}001$ for all even-numbered days in the month. A quick computation yields a Z-statistic of 54.2, which is statistically significant at more than a 99.99 percent confidence level. We conclude that out-of-sample adjustments for skewness and kurtosis significantly reduce the proportion of theoretical prices lying outside their corresponding observed bid-asked spreads.

Figure III presents deviations of market-observed option prices from theoretical skewness- and kurtosis-adjusted Black-Scholes option prices for November 6, 1990. As with Figure II, Figure III measures option moneyness on the horizontal axis, where negative (positive) moneyness corresponds to in-the-money (out-of-the-money) options with low (high) strike prices. Dollar price deviations measured on the vertical axis are calculated as market-observed prices less theoretical prices.

Figure III reveals that out-of-sample adjustments for nonnormal skewness and kurtosis remove almost all strike price biases of the Black-Scholes model for this sample of S&P 500 index options. Furthermore, all deviations of skewness- and kurtosis-adjusted Black-Scholes prices from observed prices are less than one dollar in magnitude.

TABLE 3. Comparison of Black-Scholes Prices and Observed Prices of S&P 500 Index Options (SPX) for December 1993.

Date	No. of Price Obs.	Implied Std. Dev. (%)	Proportion of Theoretical Prices Outside Bid-Ask Spreads	Avg. Deviation of Theoretical Prices from Spread Boundaries ($)	Avg. Call Price ($)	Avg. Bid-Ask Spread ($)
12/2/93	2062	15.35	0.77	0.54	30.30	0.72
12/6/93	2018	14.90	0.65	0.44	29.59	0.69
12/8/93	1346	14.92	0.71	0.50	30.93	0.65
12/10/93	3221	14.64	0.75	0.54	30.37	0.62
12/14/93	3546	15.12	0.69	0.53	29.21	0.70
12/16/93	2303	15.62	0.76	0.59	31.06	0.67
12/20/93	1305	14.79	0.75	0.78	29.93	0.59
12/22/93	1046	14.50	0.73	0.63	27.32	0.64
12/28/93	778	13.78	0.56	0.75	25.84	0.64
12/30/93	544	13.33	0.47	0.56	23.65	0.63
Average	1816	14.70	0.68	0.59	28.82	0.65

Notes: On each day indicated, a Black-Scholes implied standard deviation (BSISD) is estimated from current price observations. Theoretical Black-Scholes option prices are then calculated using BSISD. All observations correspond to call options traded in December 1993 and expiring in March 1994.

Further Empirical Results

To corroborate our empirical results, we apply all procedures leading to Tables 1 and 2 and Figures II and III to a data set representing options traded in December 1993 and maturing in March 1994. Tables 3 and 4 and Figures IV and V below present empirical results obtained from these data.

For this second sample of S&P 500 index options observed in December 1993, the average Black-Scholes implied standard deviation reported in Table 3 is 14.70 percent (column 3). This is less than the average standard deviation of 24.51 percent reported in Table 1, indicating that market volatility was much lower at year-end 1993 than at year-end 1990. The average number of daily price observations is 1,816 (column 2), with an average option price of $28.82 (column 6) and an average bid-ask spread of $0.65 (column 7). The average proportion of theoretical prices lying outside their corresponding bid-ask spreads is 68 percent (column 4), with an average deviation of $0.59 for observations lying outside a spread boundary. This average deviation of $59 per contract is almost as large as the average bid-ask spread of $65 per contract.

Figure IV displays deviations of observed call prices from theoretical Black-Scholes call prices for December 8, 1993. Option moneyness on the horizontal axis is the percentage difference between a discounted strike price and a dividend-adjusted stock index level, where a negative (positive) percentage corresponds to in-the-money (out-of-the-money) options with low (high) strike

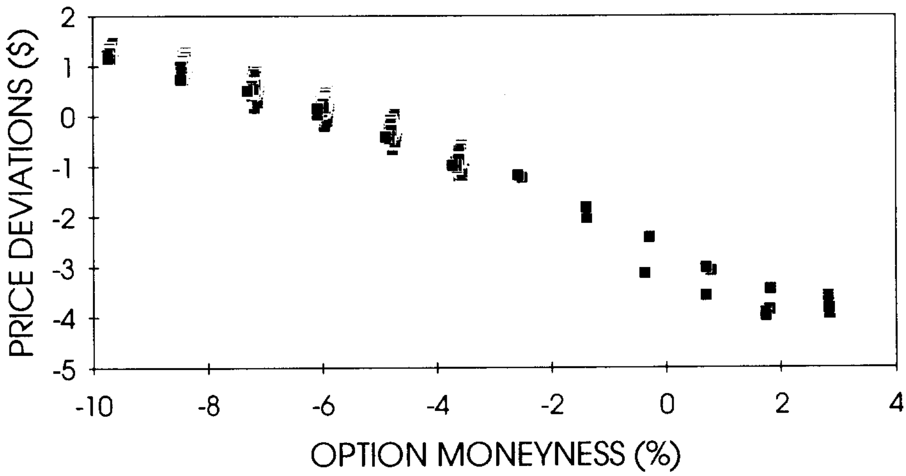

Figure IV. Black-Scholes Formula (Three-month Options).

prices. Price deviations measured on the vertical axis are market-observed prices minus theoretical prices.

Figure IV reveals the same general pattern of strike price biases shown in Figure II, where the Black-Scholes model underprices in-the-money options and overprices out-of-the-money options. However, because of lower intraday index volatility, considerably more observations overlap in Figure IV. Figure IV is actually constructed from 1,346 price observations, although only a much smaller number of dots are visually distinguishable.

Table 4 summarizes our calculations for the same S&P 500 call option prices used to compile Table 3, where out-of-sample parameters ISD, ISK, and IKT are one-day lagged estimates used to calculate theoretical skewness- and kurtosis-adjusted Black-Scholes option prices. Again, all skewness coefficients (column 4) are negative, with a column average of −1.66, and all kurtosis coefficients (column 5) are greater than 3, with a column average of 5.75.

Column 6 of Table 4 lists the proportion of skewness- and kurtosis-adjusted prices lying outside bid-ask spread boundaries, where the column average proportion is 18 percent. Column 7 lists average absolute deviations of theoretical prices from bid-ask spread boundaries. The column average deviation is $0.12, or $12 per contract, which is less than one-fifth the average bid-ask spread of $0.65, or $65 per contract.

We assess the statistical significance of the improvement in performance from out-of-sample adjustments for skewness and kurtosis using the same Z-statistic for the difference between two proportions described above. From Table 3 we get the average proportion p_1 = .68, and from Table 4 we get the average proportion p_2 = .18. Using the sample size of N = 18,160 for all

TABLE 4. Comparison of Skewness- and Kurtosis-adjusted Black-Scholes Prices and Observed Prices of S&P 500 Index Options (SPX) for December 1993.

Date	No. of Price Obs.	Implied Std. Dev. (%)	Implied Skewness	Implied Kurtosis	Proportion of Theoretical Prices Outside Bid-Ask Spreads	Avg. Deviation of Theoretical Prices from Spread Boundaries ($)
12/2/93	2062	14.14	-1.45	5.35	0.07	0.07
12/6/93	2018	13.30	-1.54	4.83	0.09	0.10
12/8/93	1346	13.32	-1.65	5.77	0.05	0.08
12/10/93	3221	13.42	-1.49	5.59	0.15	0.10
12/14/93	3546	13.77	-1.45	6.06	0.11	0.10
12/16/93	2303	13.69	-1.64	5.48	0.16	0.15
12/20/93	1305	13.06	-1.81	5.98	0.24	0.20
12/22/93	1046	12.83	-1.90	6.18	0.21	0.10
12/28/93	778	12.40	-1.92	6.37	0.57	0.18
12/30/93	544	11.56	-1.74	5.90	0.12	0.12
Average	1816	13.15	-1.66	5.75	0.18	0.12

Notes: On each day indicated, implied standard deviation (ISD), skewness (ISK), and kurtosis (IKT) parameters are estimated from one-day lagged price observations. Theoretical option prices are then calculated using these out-of-sample implied parameters. All observations correspond to call options traded in December 1993 and expiring in March 1994.

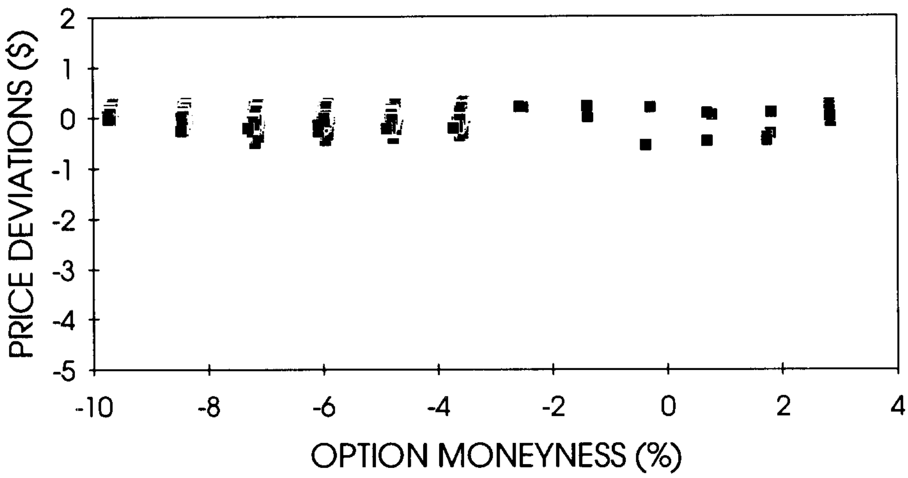

Figure V. Skewness and Kurtosis-adjusted Formula (Three-month Options).

even-numbered days in the month yields a Z-statistic of 111.5, which supports the conclusion that out-of-sample adjustments for skewness and kurtosis significantly reduce the proportion of theoretical prices lying outside their corresponding observed bid-asked spreads.

Figure V displays deviations of observed option prices from theoretical skewness- and kurtosis-adjusted Black-Scholes option prices for prices observed on December 8, 1993. Figure V reveals that out-of-sample adjustments for nonnormal skewness and kurtosis remove almost all strike price biases of the Black-Scholes model for this sample of S&P 500 index options. Overall, this second data set supports all conclusions reached with the first data set.

In addition to results for the two months of data reported in this paper, all empirical procedures were performed using several other months of data. Empirical results obtained from these other months of data are essentially identical to results reported here. Furthermore, we applied all procedures to all data using Jarrow and Rudd's (1982) option price formula with qualitatively similar results. These results are available from the authors upon request.

VI. Summary and Conclusion

We derive and empirically test a European option pricing model that extends the Black-Scholes (1973) model to account for nonnormal skewness and kurtosis in the distribution of stock returns. We find that adjustments for skewness and kurtosis are effective in removing systematic strike price biases from the Black-Scholes model for S&P 500 index options. The model is simple to implement, because it is specified as a sum of three parts: a Black-Scholes option price, plus separate adjustments for nonnormal skewness and kurtosis.

We apply the expanded model to estimate coefficients of skewness and kurtosis implied by market-observed S&P 500 index option prices. These skewness and kurtosis coefficients are simultaneously estimated with an implied standard deviation. This method for simultaneously calculating implied standard deviations and implied coefficients of skewness and kurtosis extends the widely used procedure of only obtaining implied standard deviations. The model has implications for researchers and practitioners concerned with the often-observed strike price biases associated with the Black-Scholes option pricing formula.

Appendix

We outline a derivation of the expanded option price formula appearing in equation (3). First, define $\mu = \ln S_0 + (r - \sigma^2/2)t$ and $d_2 = (\mu - \ln K)/\sigma\sqrt{t}$, then apply the change of variable $z = (\ln S_t - \mu)/\sigma\sqrt{t}$ to equation (2). The option price formula based on a Gram-Charlier expansion can then be expressed as

$$C_{GC} = e^{-rt} \int_{-d_2}^{\infty} \left(e^{\mu + \sigma\sqrt{t}z} - K\right)\left(1 + \frac{\mu_3}{3!} H_3(z) + \frac{\mu_4 - 3}{4!} H_4(z)\right) n(z)dz,$$

where Hermite polynomials are defined by the relation $H_n(z)n(z) = d^n n(z)/dz^n$, $H_0(z) = 1$ (Stuart and Ord (1987)). The first component of this expanded option price formula is the Black-Scholes formula, which is derived in Stoll and Whaley (1993). The second and third components are based on Hermite polynomials of order $n = 3$ and $n = 4$, respectively. For these components, integration by parts yields this equality:

$$\int_{-d_2}^{\infty} \left(e^{\mu + \sigma\sqrt{t}z} - K\right) H_n(z) n(z)dz = -\sigma\sqrt{t}\, e^{\mu} \int_{-d_2}^{\infty} e^{\sigma\sqrt{t}z} H_{n-1}(z) n(z)dz$$

Successive integration by parts transforms the right-hand side above into this expression:

$$-Kn(d_2) \sum_{j=1}^{n-1} \left(-\sigma\sqrt{t}\right)^j H_{n-1-j}(-d_2) + \left(-\sigma\sqrt{t}\right)^n S_0 e^{rt} N(d_1)$$

Using the well-known equality $Kn(d_2) = S_0 e^{rt} n(d_1)$, where $d_2 = d_1 - \sigma\sqrt{t}$ yields this expression:

$$S_0 e^{rt}\left(-\sum_{j=1}^{n-1}\left(-\sigma\sqrt{t}\right)^j H_{n-1-j}(-d_2) n(d_1) + \left(-\sigma\sqrt{t}\right)^n N(d_2)\right)$$

Substituting the Hermite polynomials $H_1(z) = -z$ and $H_2(z) = (z^2 - 1)$ yields the expressions needed to explicitly state the terms Q_3 and Q_4 in equation (3).

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