The Seasonality of Banking Failures During The Late National Banking Era

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Abstract

In this paper, we consider a model in which spatial separation and limited communication create a transactions role for currency, and in which aggregate uncertainty associated with agents’ patterns of movement, creates a role for banks and allows the possibility of banking failures. We show that banks provide complete insurance in states of the world where there are no banking failures and that not all banking failures have associated large output losses. When bank failures occur currency premiums are observed. Furthermore, the scope for a fragile banking system decreases as the nominal rates of interest increase. However, a policy of letting nominal interest rate equal to zero is not an optimal policy in all states of the world. Furthermore, when income, real rate of return and the return on scrapping fluctuate seasonally, the bank’s optimal return schedule, optimal reserve to deposit ratio, currency premiums, and the probability of banking failures will also fluctuate seasonally. Finally, when seasonals are present, the operating procedure for monetary policy of affecting the supply of reserves is indeed a desirable policy.

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†While working on this paper Bruce Smith passed away, I would like to dedicate this finished paper to him, a mentor, a friend and an inspirational figure in my life.
1 Introduction

During the late National Banking era, 1880 to 1913, the U.S. economy was beset by recurring bank failures. As noted by Miron (1986) [14], Champ, Smith, and Williamson (1996) [3], and many contemporary observers,¹ bank failures were closely associated with seasonal financial stringencies. Indeed, bank failures tended to occur during the fall and spring because of the simultaneous high level of currency demand to make payments,² and a high level of credit demand as workers had to be paid at harvest, and crops shipped before they could be sold and revenue could be realized.

Accompanying these seasonal strains, there were large seasonal movements in the nominal rates of interest.³ Miron (1986) [14] finds that the standard deviation of the seasonal cycle on financial market variables was 130 basis points before 1914, but only 46 basis points afterwards. Contemporaries, such as Laurence Laughlin (1912) [11], took the view that bank failures were but one—and perhaps a relatively minor—manifestation of the costs of large seasonal variation in nominal rates of interest. In particular, Miron (1986) finds that seasonal movements in the loan to reserve ratio are large, with a standard deviation of the seasonal cycle being 3.5 percent and an amplitude of 16.2 percent. In the views of Sprague (1910) [22], Andrew (1908) [1], and Goodhart (1969) [8], when these strains occurred it took only small additional pressure on the banking system to push it into a crisis. It is not too surprising then that Kemmerer (1910) finds that for the period 1890-1908, if minor failures are considered, the frequency of banking crises was slightly more than one per year [10].⁴

While the literature on the appropriate response of monetary policy to business cycle fluctuations is enormous, there is—relatively speaking—little modern literature on the appropriate treatment of seasonality within the context of monetary policy.⁵ In this paper, keeping with the views of Sprague (1910) [22], Andrew (1908) [1], Goodhart (1969) [8], and Friedman and Schwartz (1963) [6]—we proceed based on the notion that seasonal fluctuations exert much of

¹See, for instance, Sprague (1910) [22] and Andrew (1908) [1].
²In other words, the currency-deposit ratio was high when crops were being sent to market.
³An important source for these seasonal fluctuations arises from the seasonal returns from farming activities.
⁴According to Kemmerer’s classification, major failures necessarily involved a suspension of convertibility of deposits into currency. The dates of these events are determined by the starting dates recorded by the Commercial and Financial Chronicle and the Financial Review. These sources specify the type of crises, they distinguish between major or minor disruptions in the banking system.
⁵Important exceptions to this statement include Sargent and Wallace (1982) [18], Mankiw and Miron (1991) [13], Champ, Smith and Williamson (1996) [3], Chatterjee (1997) [4], Braun and Evans (1998) [2], Liu (2000) [12], and Gomis-Porqueras and Smith (2002) [7].
their effect in the economy through the banking system. Thus we base our analysis on a model that brings banking and credit market conditions to the forefront, allowing bank failures as possible equilibrium outcomes.

More specifically, we consider a pure exchange closed economy in which spatial separation and limited communication create a transactions role for currency, and in which aggregate uncertainty associated with agents’ patterns of movement, creates a role for banks as well as the possibility of banking failures. In this context, we also introduce deterministic seasonal fluctuations. We consider three possible sources of seasonality. One of these sources is seasonal variability in income (here endowments). A second is seasonal variability in real rates of return. And the third source of seasonality arises from the costs associated with the scrapping value of investments. Within this context, we study how the banking system during the later National Banking era is affected by seasonal fluctuations.

The results we obtain are as follows. First, banks provide complete insurance in states of the world where there are no banking failures and that not all banking failures have associated large output losses. When bank failures occur currency premiums are observed. Furthermore, the scope for a fragile banking system decreases as the nominal rates of interest increase. However, a policy of letting nominal interest rate equal to zero is not an optimal policy in all states of the world. Furthermore, when income, real rate of return and the return on scrapping fluctuate seasonally, the bank’s optimal return schedule, optimal reserve to deposit ratio, currency premiums, and the probability of banking failures will also fluctuate seasonally. Finally, when seasonals are present, the operating procedure for monetary policy of affecting the supply of reserves is indeed a desirable policy.

The remainder of the paper proceeds as follows. Section 2 describes the economic environment. Section 3 discusses the general equilibrium properties of this economy. Finally, section 4 offers some concluding remarks.

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6 Such variability could arise, for example, as a result of seasonal fluctuations in credit demand, as modeled by Sargent and Wallace (1982) [18]. Here it arises from seasonal fluctuations in a technology parameter that affects the returns to farming.

7 During the later National Banking era the monetary authority, the Treasury, allowed the nominal rates of interest to fluctuate seasonally.
2 The model

A. Environment

We consider a discrete time economy populated by an infinite sequence of two period lived overlapping generations with unit mass. Let \( t=1,2... \) index time. In each period economic activity takes place in two physically separate locations, denoted locations 1 and 2. These locations are symmetric, so that there is no need to index variables separately by location. Furthermore, at each date \( t \), a new young generation appears in each location.

At date \( t \) young agents have a (before-tax) endowment of \( w_t > 0 \) units of the good. Agents have no endowment when old. For simplicity, we assume that agents derive utility from consuming goods in the second period of life. As a result, young agents save their entire after tax endowment. Thus, if \( c \) denotes old consumption, young agents have the lifetime utility \( u(c) = \ln(c) \).

There are two (primary) assets in this economy, money and storage. In particular, all agents have access to a constant return to scale technology for storing the good whereby one unit stored at \( t \) yields \( R_t \) units of consumption at time \( t+1 \). \( R_t \) can be interpreted as returns to farming activities. Furthermore, investments in this linear technology can be “scrapped” after their initiation in period \( t \), which could be interpreted as the return of harvesting before the crops are due. A storage investment that is scrapped generates \( r_t < 1 \) units of consumption.\(^8\) We further assume that \( R_t > 1 \) holds, so that the use of the storage technology is socially efficient.

B. Spatial Separation, Limited Communication and Aggregate Uncertainty

Following Townsend (1987) [23], we use spatial separation and limited communication to motivate a transactions role for currency. And, following Champ, Smith and Williamson (1996) [3] and Schreft and Smith (1997) [19], we use stochastic relocation among disparate locations to generate a role for banks. Furthermore, following Smith (2002) [21] we use aggregate uncertainty about the stochastic relocation to allow bank failures in equilibrium.\(^9\) Finally, in this paper we take Noyes’ (1909) [16] notion of bank failures; that is, in a failure there is

\(^8\)Once banks are introduced, this scrapping option can be interpreted as banks calling back loans.

\(^9\)See also Peck and Shell (2001) for an analysis of an economy where banks face stochastic withdrawal demand in which current withdrawals are history dependent and equilibrium runs may indeed be optimal [17].
suspension of cash payments to depositors, the depletion of cash reserves, the emergence of currency premiums, and the use of “emergency expedients” to provide additional media of exchange.

At the beginning of each period, each young agent is born in one of the two locations. At this point in time, agents cannot move between or communicate across locations. Thus goods and asset market transactions occur autarkically within each location at the beginning of each period. After trade and saving decisions have occurred at time $t$, some randomly selected fraction $\pi_t$ of young agents from location 1 is chosen to move to location 2 and vice versa, so that the two locations remain symmetric. The fraction $\pi_t$ is itself a random variable with a density function $f(\pi_t)$, which is common knowledge, with finite support in the interval $[0, 1]$. We further assume that stored goods are not transportable because transporting them is too costly. Moreover, as in Townsend (1987) [23], Champ, Smith and Williamson (1996) [3], and Schreft and Smith (1997) [19], limited communication implies that relocated agents cannot transact with privately issued liabilities in their new location.\(^{10}\) In effect, then, relocation constitutes a physical story about which goods are “cash goods” (those purchased by relocated agents) and which are “credit goods” (those purchased by non-relocated agents).\(^{11}\)

The need to transact in cash, if one is relocated, implies that agents who move between locations must liquidate all other assets prior to moving, and convert them into currency. If currency is dominated in the rate of return—the situation we focus on throughout the paper—then the necessity of converting higher yielding assets into lower yielding currency constitutes an adverse shock—in effect a “liquidity preference shock”—that agents will wish to be insured against. Furthermore, after holding voluntary cash reserves if liquidity needs are not met, storage investments will be scrapped, incurring then in even greater output losses.\(^{12}\) As in Diamond and Dybvig (1983) [5], such insurance can be provided by banks.

C. Sources of Seasonality

Our ultimate intention is to consider how seasonal fluctuations affect this economy, in particular the banking system. Here we consider three different potential sources of exogenously driven seasonality. One might derive from changes in production conditions that induce sea-

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\(^{10}\)See also Mitsui and Watanabe (1989) [15] or Hornstein and Krusell (1993) [9].

\(^{11}\)Here we are implicitly considering state contingent goods. We distinguish between consumption when relocated and when not.

\(^{12}\)Banks have to allocate resources before the relocation parameter is realized, thus the portfolio decision is based on the average value of the random variable.
sonal variability in real rates of return. We can capture seasonal variations that affect real rates of return by allowing the rate of return on this linear investment technology to vary in the following deterministic manner: \( R_t = R_e(R_o); \) for \( t \) even (odd). This difference across seasons may capture the possibility of different returns on farming activities throughout the year.

Second, there may be seasonal variations in the general availability of resources. We can capture this type of seasonality by allowing endowments to satisfy \( w_t = w_e(w_o); \) for \( t \) even (odd).

Finally, seasonal fluctuations may also derive from deterministic variations in the scrapping value of storage investments. We can induce seasonality in the scrapping value, \( r_t \), by allowing: \( r_t = r_e(r_o); \) for \( t \) even (odd). This seasonal difference may reflect the fact that collecting the proceeds of farming before they are due may vary across seasons. To fix ideas we are going to further assume that \( R_e > R_o, w_e > w_o \) and \( r_e > r_o \).

### D. The Government

In order to model a monetary authority during the late National Banking era, we need to know how money was created and injected into the economy. According to Friedman and Schwartz (1963), during the period 1863-1913 all bank notes were fully backed by government bonds and hence being \textit{de facto} liabilities of the government. A private bank could not issue notes against general assets, and its reserve of federal government bonds constrained its note issue. In order for private banks to print money they first had to buy bonds. As a result, when banks created new money the availability of credit was directly affected.

Friedman and Schwartz document that during 1863-1913 period there was the growing tendency of the Treasury to intervene frequently and regularly in the money market. The central banking activities of the Treasury were being converted from emergency measures to a fairly regular and predictable function. Furthermore, during this period there was no

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13Seasonality that affects real rates of return is very similar to that considered by Sargent and Wallace (1982), and Champ, Smith and Williamson (1996).

14This seasonal difference may reflect the possibility that the wage that farmers obtain from farming activities varies throughout the year.

15Throughout the rest of the paper we will interpret this scrapping activity as liquidating part of the storage investment.

16Note that this money is not really fiat money since it is partly backed by assets.

17Note that we can interpret this situation as the government indirectly printing money through bond issue.

18According to Friedman and Schwartz (1963), the Treasury’s debt management powers were then and are now comparable to the Federal Reserve System’s ability to conduct open market operations. The power
explicit objective of the Treasury to smooth nominal rates of interest through bond issuing.

In this paper we make the simplifying assumption that the government prints money directly, we abstract from bond issuing since our objective is to understand how changes in the availability of resources destined to credit are affected by fluctuations in the nominal rates of interest and how they subsequently affect the fragility of the banking system.\textsuperscript{19} In order to capture the historically standard notion,\textsuperscript{20} that monetary policy works, in part, by affecting the supply of reserves to banks; we further assume that all required injections or withdrawals of money are accomplished through lump-sum tax/transfers to young agents. In particular, let $M_t$ be the value of the time $t$ nominal money stock, let $p_t$ be the time $t$ price level, and let $\tau_t$ be the real value of lump-sum transfers made to young agents at $t$. Then the government’s budget constraint is given by,

$$\tau_{t+1} = \frac{M_{t+1} - M_t}{p_{t+1}}. \quad (1)$$

Finally, we further assume that the Treasury has an inflation target allowing the nominal rates of interest to freely fluctuate seasonally.\textsuperscript{21}

E. Banks

The fact that agents are subject to stochastic relocation –or to liquidity preference shocks– implies that there is a role for banks to provide liquidity in this economy. As in Diamond and Dybvig (1983) \cite{DiamondDybvig1983}, all savings will be intermediated at each date. Hence, at the beginning of period $t$, each young agent deposits the after-tax endowment (savings), $w_t + \tau_t$, with a bank. The bank uses these deposits to acquire the primary assets in the model: Money and storage. Money is held by banks as cash reserves to pay relocated agents.

Taking into account $\tau_t$ is itself a random variable with a known probability distribution, the bank has to decide how to allocate its resources before this stochastic relocation parameter is realized. Let $z_t$ and $k_t$ denote the per depositor quantity of real balances and storage investment, respectively, acquired by a bank at time $t$. The gross real rate of return on real
to deposit or withdraw funds from commercial banks is equivalent to the existing possibility of affecting commercial bank reserves by transferring government deposits from commercial to Federal Reserve Banks \cite{FriedmanSchwartz1963}.

\textsuperscript{19}The particular operating procedure for monetary policy of printing money is somewhat equivalent to forcing banks to hold bonds in order to issue notes.

\textsuperscript{20}See, for instance, Skidelsky’s (1992) description of Keynes’ view of monetary policy \cite{Skidelsky1992} and Friedman and Schwart (1963).

\textsuperscript{21}Here we have some degree of freedom when choosing a monetary rule since the Treasury did not have a clear stated objective when designing monetary policy. Although the policy rule considered has to be consistent with the fact that the nominal rates of interest should be able to fluctuate seasonally in an endogenous fashion.
balances between $t$ and $t+1$ is $p_t/p_{t+1}$; the gross real rate of return on storage is $R_t$, and $r_t$ the associated return of scrapping the storage investment. A situation where scrapping occurs can be interpreted as the bank “calling” back a loan, forcing farmers to collect the proceeds from farming before they are due. Since this process is usually costly, we further assume that $r_t < p_t/p_{t+1}$.

Banks are competitive in asset markets, taking these returns as given. At the same time, banks offer two gross real rates of return to depositors: $\rho_t^m(\pi_t)$ per unit deposited to agents who are relocated, and $\rho_t(\pi_t)$ per unit deposited to agents who are not relocated.

The banks face different constraints. First, banks’ assets must not exceed their liabilities, so that

$$z_t + k_t \leq w_t + \tau_t$$  \hspace{1cm} (2)

must hold. Second, payments to relocated agents at time $t$, $\pi_t(w_t + \tau_t)\rho_t^m(\pi)$, cannot exceed the time $t+1$ value of the bank’s cash reserves, since relocated agents require currency to transact. Hence,

$$\pi_t (w_t + \tau_t) \rho_t^m(\pi) \leq \alpha(\pi_t) z_t \frac{p_t}{p_{t+1}} + \delta(\pi_t) r_t k_t.$$  \hspace{1cm} (3)

The presence of the term $\frac{p_t}{p_{t+1}}$ in this expression reflects any depreciation of currency values that occurs between the time when currency is acquired ($t$) and the time when it is spent ($t+1$). Furthermore, $\alpha(\pi_t)$ captures the fact that there is aggregate uncertainty, so banks could in principle be willing to hold precautionary reserves. Finally, $\delta(\pi_t)$ takes into consideration the fact that in certain states of the world, storage investments may have to be scrapped in order to meet the liquidity needs of early withdrawers.

Finally, payments to non-relocated agents, $(1-\pi_t)(w_t + \tau_t)\rho_t(\pi_t)$, cannot exceed the income from the bank’s capital investments and excess cash holdings. Thus,

$$(1-\pi_t)(w_t + \tau_t)\rho_t(\pi_t) \leq (1-\alpha(\pi_t)) z_t \frac{p_t}{p_{t+1}} + (1-\delta(\pi_t)) R_t k_t.$$  \hspace{1cm} (4)

As a coalition of ex ante identical depositors, the bank maximizes depositor expected utility, which is given by

$$\int_{0}^{1} [\pi_t \ln(\rho_t^m(\pi_t)) + (1-\pi_t) \ln(\rho_t(\pi_t))] f(\pi_t) d\pi_t$$

by choice of $z_t, k_t, \rho_t^m(\pi), \rho_t(\pi), \delta(\pi_t)$ and $\alpha(\pi_t)$ subject to the constraints just described.\footnote{Note that the constraints can be simplified somewhat if we consider the reserve to deposit ratio which is defined $\gamma_t = z_t/(w_t + \tau_t)$.}
Proposition 1. The solution to this problem can be described by the following optimal return schedule:

\[
[\rho_t^+(\pi_t), \rho_t(\pi_t)] = \begin{cases} 
\rho_t^+(\pi_t) = \rho_t(\pi) = \gamma_t \frac{p_t}{p_t + 1} + (1 - \gamma_t) R_t & \text{if } \pi_t < \pi_t^* , \\
\rho_t^+(\pi_t) = \frac{\gamma_t}{\pi_t^*} p_t^+ & \text{if } \pi_t \in (\pi_t^*, \hat{\pi}_t) , \\
\rho_t(\pi_t) = \frac{1 - \gamma_t}{1 - \pi_t} R_t & \text{if } \pi_t \in (\pi_t^*, \hat{\pi}_t) , \\
\rho_t^-(\pi_t) = \frac{\gamma_t}{\pi_t} p_t^+ + \frac{\gamma_t R_t}{\pi_t R_t} \frac{\pi_t - \hat{\pi}_t}{\pi_t} & \text{if } \pi_t > \hat{\pi}_t , \\
\rho_t(\pi_t) = \frac{1 - \gamma_t}{1 - \pi_t} \left( 1 - \frac{R_t}{\pi_t} \right) R_t & \text{if } \pi_t > \hat{\pi}_t .
\end{cases}
\]

and the optimal reserve to deposit ratio, \( \gamma_t = z_t / (w_t + \tau_t) \), which satisfies the following equation:

\[
G(\gamma_t, I_t) \equiv \frac{1 - \frac{\gamma_t + I_t}{R_t} I_t}{\gamma_t + (1 - \gamma_t) \frac{\tau_t + I_t}{R_t}} = \frac{1}{\gamma_t(1 - \gamma_t)} \int_{\hat{\pi}_t}^{\pi_t^*} F(\pi) \, d\pi_t = 0 ;
\]

where \( \pi_t^* = \frac{\gamma_t}{1 + \frac{\gamma_t}{\pi_t^*} I_t} \) represents the maximum fraction of relocated agents for which complete insurance is possible, \( \hat{\pi}_t = \frac{1}{1 + \frac{\gamma_t}{\pi_t^*} I_t} \) denotes the largest fraction of relocated agents for which the scrapping option is not executed, \( F(\pi) \) is the corresponding cumulative distribution function, and \( I_t = \frac{p_t + 1}{R_t} R_t \) is the nominal rate of interest.

Proof: See the appendix.

As we can see from Proposition 1, the bank will provide complete insurance against the event of being relocated in states of the world where there are no bank failures. On the other hand, the bank will exhaust its cash reserves if and only if withdrawal demand exceeds \( \pi_t^* \). This event is associated with a bank failure. In this situation, a bank treats all movers alike. In other words, there is no “sequential” constraint in effect. Furthermore, not all bank failures have associated large output losses. In particular, if \( \pi_t \in (\pi_t^*, \hat{\pi}_t) \) we find banking failures but banks do not liquidate any storage investments. Loans are not called back, there is no extra decrease in output associated with banking failures. On the other hand, if \( \pi_t > \hat{\pi}_t \) we have banking failures and banks do liquidate storage investments. As a result, loans are called back and there is a decrease in output associated with a banking failure. Finally, when the source of seasonal fluctuations derives from changes in the rate of return on storage and the scrapping value, the seasonal difference in the optimal cash to deposit ratio is the largest.

Lemma 1. If the probability distribution of early withdrawals is uniform, then the optimal cash to deposit ratio is unique.

Note that in this situation precautionary reserves are binding; i.e, \( \alpha(\pi_t) \) will be binding whenever withdrawal demand exceeds \( \pi_t^* \).

Note that this sequential constraint would prevent any further bank panics.

Note that in this situation \( \delta(\pi_t) \) will be binding whenever withdrawal demand exceeds \( \hat{\pi}_t \).

This is consistent with Kemmerer’s findings that not all banking crises are equal.
Proof: See the appendix

As we can see from this Lemma, we can not establish uniqueness of the optimal cash to deposit ratio without providing more structure on the probability distribution of the underlying relocation parameter.

**Proposition 2.** The optimal cash to deposit ratio, \( \gamma_t(I_t) \) that satisfies \( G(\gamma_t, I_t) = 0 \), has the following properties: \( \gamma_t(1) = 1, \gamma_t(R_t/r_t) = 0 \), if \( I_t \in (1, R_t/r_t) \) then \( \gamma_t(I_t) \in (0, 1) \) and \( \gamma'_t(I_t) < 0 \).

Proof: See the appendix

Propositions 1 and 2, show that the endogenous thresholds faced by the bank are potentially affected by the operating procedure for monetary policy. In particular, these propositions suggest that \( \partial \hat{\pi}_t / \partial I_t < 0 \) and \( \partial \pi^*_t / \partial I_t < 0 \). Therefore, the probability of experiencing banking failures as well as their associated output losses increases with the nominal rates of interest. Thus, the fragility of the banking system decreases as nominal rates of interest increase.

Finally, we note that there exists a currency premium when bank failures occur.\(^{27}\) If either relocated and non-relocated agents can withdraw their deposits at date \( t \), they receive a payment with a time \( t+1 \) value of \( \rho^m_t(\pi) \) in state \( \pi_t \). This payment is obtained in currency, for which movers have use and non-relocated have not. Thus non-relocated agents will exchange any currency they obtain for claims to the deposits of movers. They will be indifferent between doing so and leaving their deposit until \( t+1 \) if and only if they obtain a real value of \( \rho_t(\pi) \) per unit withdrawn. Similarly, relocated agents are indifferent between obtaining currency from their bank or from non-relocated agents if they can obtain a real value of \( \rho^m_t(\pi) \) at \( t+1 \) in exchange for deposits with a real value of \( \rho_t(\pi) \) at \( t+1 \). Thus after \( \pi_t \) is realized, deposits must exchange for currency at a rate \( q_t(\pi) = \frac{\rho^m_t(\pi)}{\rho_t(\pi)} \), thus a currency premium will exist if \( \epsilon_t(\pi) = q_t(\pi) - 1 < 0 \). Note that \( \epsilon_t(\pi) = 0 \) if \( \pi_t \leq \pi_t^* \). On the other hand, if \( \pi_t \in (\pi_t^*, \hat{\pi}_t) \) we have that

\[
\epsilon_t(\pi) = \frac{2\gamma_t}{\pi_t^*} \frac{p_t}{p_{t+1}} + \frac{\gamma_t R_t (\pi_t - \hat{\pi}_t)}{\pi_t} \frac{\pi_t}{\pi_t^*} - 1 < 0
\]

and if \( \pi_t \geq \hat{\pi}_t \) the currency premium is given by

\[
\epsilon_t(\pi) = \frac{2\gamma_t}{\pi_t} \frac{p_t}{p_{t+1}} + \frac{\gamma_t R_t (\pi_t - \hat{\pi}_t)}{\pi_t} \frac{\pi_t}{\pi_t^*} - 1 < 0.
\]

Thus, whenever banking failures, with or without output losses, occur currency premiums will be observed.

\(^{27}\) A currency premium exists if currency is worth more than deposits having an equal face value.
3 General Equilibrium

The basic condition of equilibrium for this economy is that the supply of real money balances and its demand be equal. Since all savings are intermediated, all beginning of period asset demands derive from banks. It follows that the money market clears at each date $t$ if:

$$z_t = \gamma_t(I_t) \left( w_t + \tau_t \right);$$  \hspace{1cm} (9)

where $z_t \equiv \frac{M_t}{p_t}$ is the outstanding stock of real balances.

When the government follows an inflation rate targeting rule, we have that the return on real balances is given by:

$$\frac{p_t}{p_{t+1}} = \frac{1}{\eta};$$  \hspace{1cm} (10)

where $\eta$ is the targeted inflation rate. It then follows from the government budget constraint that injections/withdrawals of reserves from the government are given by:

$$\tau_t = z_t - \frac{z_{t-1}}{\eta}.$$  \hspace{1cm} (11)

Combining equation (11) and the money market clearing condition, we obtain the following equilibrium law of motion for $z_t$:

$$z_t = \frac{\gamma_t(I_t)}{1 - \gamma_t(I_t)} \left( w_t - \frac{z_{t-1}}{\eta} \right).$$  \hspace{1cm} (12)

3.1 Steady State Equilibria

We now examine how monetary policy affects the magnitude of seasonal fluctuations and the stability of the banking system when the government follows a policy of maintaining an inflation rate target. Under this targeting rule we have that the nominal rates of interest fluctuate seasonally. In particular, we have that $I_t = I_e = R_e \eta$ ($I_o = R_o \eta$) for even (odd) periods.\(^{28}\)

When income, real rates of return and the return on scrapping fluctuate seasonally, the optimal return schedule, the optimal reserve to deposits, currency premiums and the loan to deposit ratios are going to be different across seasons. Furthermore, since $\pi^*_t$ and $\tilde{\pi}_t$ explicitly depend on the different sources of seasonal fluctuations as well as the nominal rates of interest, the probability of experiencing banking failures, with or without output associated losses, also differs across seasons. In particular, if $R_e > R_o$ we have that $\gamma_e(I_e) < \gamma_o(I_o)$. In other words, in seasons when nominal rates of interest are high, the optimal cash to deposit ratio is smaller, reflecting the higher opportunity cost of holding reserves. Furthermore, we have that $\pi^*_e < \pi^*_o$.

\(^{28}\)Notice that we are implicitly assuming that the quarters are symmetric.
and $\hat{\pi}_e < \hat{\pi}_o$, suggesting that the probability of bank failures, with or without output losses, is larger in even than in odd periods. This model also suggests seasonal currency premiums since they directly depend on the probability of experiencing banking failures.\(^{29}\) In particular, currency premiums are going to be greater in even than in odd periods.

Once the government sets the inflation rate target, the demand for real balances is determined. We restrict our attention to periodic equilibria, so that the demand for real balances for even and odd periods are given by $z_e$ and $z_o$ respectively. In particular, in a periodic equilibrium, equation (12) implies the following demand for real money balances,

$$z_e = \frac{\gamma_e(I_e)}{1 - \gamma_e(1) I_e} \left( w_e - \frac{w_o R_e}{I_o} \frac{\gamma_o(I_o)}{1 - \gamma_o(1) I_o} \right),$$

symmetrically for $z_o$.

As we can see, depending on the different sources of seasonal fluctuations the variation in the demand for real money balances across seasons can be quite different. Furthermore, it is easy to show that $\partial z_e / \partial w_e > 0$; that is, when nominal rates of interest and income are high, an increase in this period’s income increases the demand for real money balances in the same period. On the other hand, $\partial z_e / \partial w_o < 0$; that is, when nominal rates of interest and income are high, an increase in next period’s income (which is lower than today’s income) decreases the demand for real money balances in the same period.\(^{30}\)

Once the demand for real money balances is determined for the different seasons, the transfers that young agents receive are completely characterized. In particular, we have that the transfers for even periods are given by,

$$\tau_e = z_e - \frac{R_o}{I_o};$$

similarly for $\tau_o$. It is now easy to show that $\partial \tau_e / \partial w_e > 0$ and $\partial \tau_e / \partial w_o > 0$; i.e, when nominal rates of interest and income are high, an increase in this period’s income increases transfers that young agents receive today. On the other hand, $\partial \tau_e / \partial w_o < 0$ and $\partial \tau_e / \partial w_o > 0$; i.e, when nominal rates of interest and income are high, an increase in next period’s income (which is lower than today’s income) reduces the transfers that young agents receive today. As we can see, money is injected into the economy by transferring resources from “better” times to the “worst” ones. In some sense, allowing monetary policy to affect the credit market is

\(^{29}\)These finding are consistent with the observations of Sprague, Laughlin and Kemmerer that bank failures tended to occur seasonally and that they are not equal.

\(^{30}\)Throughout this analysis we assume that in even periods there are more resources than in odd ones. In particular, we have that $w_e > w_o$. 

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providing some extra insurance that banks are not able to provide in all states of the world. Since banks are an \textit{ex ante} coalition of agents, they can not transfer resources from future young generations since there is not a credible repayment procedure between old and young generations. As a result, such a private insurance scheme is not possible. On the other hand, the government is able to make intertemporal reallocation of resources through subsidies. Thus, when seasonals are present the operating procedure for monetary policy that affects the supply of reserves in the banking system is indeed a desirable policy.

We now would like to explore how an increase in the nominal rates of interest affect the rates of return that depositors receive. It is easy to show that when there are no bank failures, \(\pi_t < \pi_t^*\), depositors obtain the same return, which decreases as the nominal rates of interest rate increase. Therefore, the returns that depositors receive are greater in even than odd periods since \(I_e > I_o\). When bank failures are observed, \(\pi_t \in (\pi_t^*, \hat{\pi}_t)\), depositors face different returns. In particular, it is easy to show that \(\partial \rho^m_t / \partial I_t < 0\). Thus early withdrawers receive a lower return in even than in odd periods. On the other hand, we have that \(\partial \rho_t / \partial I_t > 0\), so that patient depositors receive a higher rate of return in even than in odd periods. As we can see, in these states of the world, banks tend to reduce any incentives for bank runs that patient depositors may have. Finally, when bank failures and output losses are possible, \(\pi_t > \hat{\pi}_t\), depositors also face different rates of return. The effect of increasing the nominal rates of return does not have a clear effect on the rates of return that depositors receive. The effect depends on the realization of the relocation probability, the fragility of the banking system as well as the interest rate elasticity of excess reserves. In particular, if we verify that

\[
(\pi_t - \hat{\pi}_t) \left( \frac{\gamma'_t}{I_t} - \frac{\gamma_t}{I_t^2} \right) < \frac{\gamma_t}{I_t} \frac{\partial \hat{\pi}_t}{\partial I_t} \rightarrow \left\{ \begin{array}{l} \frac{\partial \rho^m_t}{\partial I_t} < 0 \\ \frac{\partial \rho_t}{\partial I_t} > 0 \end{array} \right. 
\]

If this condition is satisfied early withdrawers receive a lower return in even than in odd periods and patient depositors receive a higher rate of return in even than odd periods. Once again, the bank tends to reduce the incentives of patient depositors to rush to the bank.

As we can see, a policy of lowering the nominal rates of interest can not be an optimal policy in all states of the world, since the rates of return that different depositors face react quite differently to increases in the nominal rates of interest; i.e, \(\partial \rho^m_t / \partial I_t < 0\) and \(\partial \rho_t / \partial I_t > 0\). This finding then suggests that having positive inflation rates, thus positive nominal rates of interest, may be indeed desirable since large unfavorable relocation shocks are possible and in these states of the world money is valuable.
3.2 Some Testable Predictions

We now briefly consider the extent to which our predictions about the banking system during the late National Banking are reflected in the data.\(^{31}\)

Our analysis predicts that not all banking failures are not equal. In particular, experiencing a mild banking failure is more likely than a severe one. Furthermore, the probability of experiencing a banking failure, with or without associated output losses, increases with the nominal rates of interest and these are seasonal. In order to find evidence related to these propositions, we use Kemmerer’s dating and classification on financial failures. In particular, we find that the probability of experiencing mild and severe banking crises are 0.0519 and 0.0207, respectively. Thus, we have that \(1 - \pi^* = 0.0519\) and \(\hat{\pi} - \pi^* = 0.0207\). These findings are consistent with our theoretical predictions that banking crises associated with major output losses are less likely than the ones with minor losses; i.e., \(\pi^*_t < \hat{\pi}_t\).

In order to investigate how banking failures depend on the nominal rates of interest, we consider conditional probabilities. In particular, we compute the probability of experiencing mild and severe banking failures conditional on the nominal rates of interest being smaller than 3\%.\(^{32}\) These conditional probabilities are 0.0263 and 0 respectively. We then condition on nominal rates of interest being smaller than 4\%, which yields 0.0410 and 0.0051 respectively, which corroborates our theoretical predictions. To further control for possible factors, other than nominal rates of interest (INT), that might affect the probability of experiencing banking failures, we estimate a logit model. In particular, we consider a constant \((C)\), a seasonal dummy \((DUM)\), an index of industrial production \((PRO)\), exchange rates \((EXC)\) and an interaction term between the nominal rates of interest and seasonal dummies \((SES)\). We first consider all failures, we then consider separate specifications according to the strength of the banking failures according to Kemmerer’s dating and classification, and finally we consider all failures. What we find is that indeed seasonal nominal rates of interest are important in explaining banking failures. If minor failures are taken into account, the nominal rates of interest are less significant. Finally, when major failures are considered, we also find that seasonal nominal rates of interest are crucial in explaining banking crises. For specific values of the estimates we refer to Table 1 in the appendix.

\(^{31}\)The seasonal data used for this section was obtained from the NBER Macrohistory database and Champ et al (1996). In particular, our data set contains monthly observations that cover the period from 1890 to 1913.

\(^{32}\)The value used to separate events is arbitrary and our results do not qualitatively change when reasonable values are considered.
Our model has also some predictions about the optimal cash to deposit ratio. In particular, the cash to deposit ratio decreases as the nominal rates of interest increase. In order to find some preliminary evidence related to this proposition, we compute the correlation between the cash to deposit ratio and the nominal rates of interest. The resulting correlation is -0.498 which is consistent with our initial prediction. In order to control for other factors that might affect the cash to deposit ratio, we consider a linear regression with additional controls. In particular, we consider a constant \((c)\), a stock market index \((\text{IND})\) and a seasonal dummy \((\text{SES})\). The regression shows that the coefficient corresponding to the nominal rates of interest has the predicted sign at standard significant levels. Further details of the linear estimation can be found on Table 2 in the appendix.

These findings are then consistent with our prediction that the cash to deposit ratio decreases as the nominal rates of interest increase. In particular, when nominal rates of interest fluctuate seasonally, we find that the cash to deposit ratio is greater in some seasons than in others.

At least based on this casual evidence, we can conclude that there is some support for the notion that seasonal variation in the nominal rates of interest can have direct effects on the fragility of the banking system as well as the optimal cash to deposit ratio that banks want to hold.

### 3.3 Non-Periodic Equilibria

We now briefly examine the scope for equilibria that are not purely periodic to exist. When monetary policy maintains a constant inflation rate, equation (12) implies the following law of motion for the demand for real balances when \(t\) is even,

\[
z_t = \frac{\gamma_e(I_e)}{1 - \gamma_e(I_e)} w_e - \frac{\gamma_o(I_o)}{1 - \gamma_o(I_o)} w_o + \frac{1}{\eta^2} \frac{\gamma_e(I_e)}{1 - \gamma_e(I_e)} 1 - \gamma_o(I_o) z_{t-2};
\]

symmetrically for \(t\) odd. Evidently, there is a unique periodic equilibrium if the density of the relocation parameter is uniform. And, this equilibrium is asymptotically stable if the following sufficient condition is satisfied:

\[
\left( \frac{\gamma_o(I_o)}{1 - \gamma_o(I_o)} \right)^2 < \eta^2.
\]

Thus, if this condition holds, there is a continuum of non-stationary equilibria, all of which converge to the periodic equilibrium.\(^{33}\) In short, a policy of targeting the nominal rate

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\(^{33}\)This condition is empirically plausible. In particular, if the average cash to deposit ratio in odd periods is smaller than 0.5 and there are positive inflation rates, then the condition is trivially satisfied. For the period
of interest according to different seasons leads to dynamic indeterminacy. Moreover, all non-
stationary equilibria will display seasonal fluctuations in real balances exceeding those that
would be observed in a purely periodic equilibria.\textsuperscript{34}

Why does targeting the inflation rate create so much scope for multiplicity of equilibria
exhibiting endogenously generated volatility? The answer is that, if agents demand high
levels of real balances at $t$, the government must—in order to prevent the inflation rate from
decreasing—inject reserves into the banking system. This reserve injection, in turn, increases
the volume of deposits and, consequently, the demand for reserves. As a result, actions that the
government takes to hold the inflation rate constant across seasons validate the endogenously
high level of demand for real balances at date $t$. Moreover, in order for this process not to
be destabilizing, the central bank must withdraw reserves at $t + 1$, implying a low level of
deposits and a low level of demand for real balances. Thus any periodic equilibria where real
balances deviate from their steady state level must display oscillation.

Finally, we observe that, along dynamical equilibrium paths where $z_t$ fluctuates, economic
volatility is not confined to fluctuations in real balances. The variability of the government
reserve injections (lump-sum transfers) will also translate into variability in investment activity
(storage).

4 Conclusions

In this paper, we consider a model in which spatial separation and limited communication
create a transactions role for currency, and in which aggregate uncertainty associated with
agents’ patterns of movement, creates a role for banks and allows the possibility of banking
failures. In this context, we introduce deterministic seasonal fluctuations. Within this envi-
ronment, we study how the banking system during the later National Banking era is affected
by seasonal fluctuations in the nominal rates of interest.

This paper shows that the source of seasonal impulses is important when considering
the optimal portfolio allocation of banks. We also find that allowing the nominal rates of
interest to fluctuate seasonally may have adverse consequences on the fragility of the banking
system. Furthermore, when nominal rates fluctuate seasonally, the probability of experiencing
banking failures, the cash to deposit ratio, the demand for real balances, currency premiums
considered in this paper, the average odd period cash to deposit is 0.27 and the average gross inflation rate is
1.01.

\textsuperscript{34}Symmetric laws of motion and sufficient conditions for dynamic indeterminacy are found when $t$ is odd.
and injections withdrawals to the banking system will also fluctuate seasonally.

This paper also shows that the operating procedure for monetary policy of affecting the supply of reserves in the banking system is indeed a desirable policy when seasonals are present. We also find that a policy of letting the nominal rates of interest to zero is not optimal in all states of the world. Finally, the policy of targeting the inflation rate is an operating procedure for monetary policy that is conducive to dynamic indeterminacies.

References


Appendix

Proof of Proposition 1.

Recall that $\gamma_t$ denotes the optimal cash to deposit ratio, it is easy to show then that the optimal choice of $\alpha(\pi_t)$ satisfies

$$\rho_t(\pi_t) = \frac{1 - \alpha(\pi_t)}{1 - \pi_t} \frac{p_t}{p_{t+1}} \gamma_t + \frac{1 - \delta(\pi_t)}{1 - \pi_t} (1 - \gamma_t) R_t \geq \rho_t^m(\pi_t) = \frac{\alpha(\pi_t)}{\pi_t} \frac{p_t}{p_{t+1}} \gamma_t + \frac{\delta(\pi_t)}{\pi_t} (1 - \gamma_t) R_t$$

with strict equality if $\alpha(\pi_t)<1$. Similarly, the optimal choice of $\delta(\pi_t)$ verifies

$$\rho(\pi_t) = \frac{1 - \alpha(\pi_t)}{1 - \pi_t} \frac{p_t}{p_{t+1}} \gamma_t + \frac{1 - \delta(\pi_t)}{1 - \pi_t} (1 - \gamma_t) R_t \leq \frac{R_t}{r_t} \rho_t^m(\pi_t) = \frac{R_t}{r_t} \left( \frac{\alpha(\pi_t)}{\pi_t} \frac{p_t}{p_{t+1}} \gamma_t + \frac{\delta(\pi_t)}{\pi_t} (1 - \gamma_t) R_t \right)$$

with strict equality if $\delta(\pi_t)<1$. The optimal return follows from substituting the optimal liquidating and scrapping strategies into the bank’s portfolio constraints.

If we now substitute $\rho_t(\pi_t)$ and $\rho_t^m(\pi_t)$ into the bank’s objective function and integrate by parts, one obtains the following objective function

$$\ln \left( \gamma_t \frac{p_t}{p_{t+1}} + R_t(1 - \gamma_t) \right) F(\pi_t^*) + \ln \left( r_t(1 - \gamma_t) + \gamma_t \frac{p_t}{p_{t+1}} \right) (1 - F(\pi_t)) +$$

$$+ \int_{\pi_t^*}^{\hat{\pi}_t} \left( \pi_t \ln \left( \frac{\gamma_t p_t}{\pi_t p_{t+1}} \right) + (1 - \pi_t) \ln \left( R_t \frac{1 - \gamma_t}{1 - \pi_t} \right) \right) f(\pi_t) d\pi_t + \ln \left( \frac{R_t}{r_t} \right) \int_{\pi_t}^{1} F(\pi_t) d\pi_t -$$

$$- \ln \left( \frac{R_t}{r_t} \right) \left( 1 - \pi_t F(\pi_t) \right) + \ln \left( \frac{R_t}{r_t} \right) (1 - F(\pi_t)).$$

Now differentiating this expression with respect to $\gamma_t$ yields the following first order condition

$$\pi_t^* f(\pi_t^*) \left[ \ln \left( \gamma_t \frac{p_t}{p_{t+1}} + R_t(1 - \gamma_t) \right) - \pi_t^* \ln \left( \frac{\gamma_t}{\pi_t^*} \frac{p_t}{p_{t+1}} \right) - (1 - \pi_t^*) \ln \left( \frac{R_t(1 - \gamma_t)}{1 - \pi_t^*} \right) \right]$$

$$- f(\hat{\pi}_t) \hat{\pi}_t \left[ \ln \left( \gamma_t \frac{p_t}{p_{t+1}} + r_t(1 - \gamma_t) \right) + \ln \left( \frac{R_t}{r_t} \right) (1 - \hat{\pi}_t) - \hat{\pi}_t \ln \left( \frac{\gamma_t}{\hat{\pi}_t} \frac{p_t}{p_{t+1}} \right) - (1 - \hat{\pi}_t) \ln \left( \frac{R_t(1 - \gamma_t)}{1 - \hat{\pi}_t} \right) \right]$$

$$+ F(\hat{\pi}_t) \frac{p_t}{p_{t+1}} - R_t \left( \gamma_t \frac{p_t}{p_{t+1}} + R_t(1 - \gamma_t) \right) + (1 - F(\hat{\pi}_t)) \frac{p_t}{p_{t+1}} - r_t(1 - \gamma_t) + \int_{\pi_t}^{\hat{\pi}_t} \frac{\pi_t - \gamma_t}{\gamma_t(1 - \gamma_t)} f(\pi_t) d\pi_t = 0.$$
### Table 1: Logit Estimation

<table>
<thead>
<tr>
<th>Variable</th>
<th>Minor Failures</th>
<th>Major Failures</th>
<th>All Failures</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>-16.587</td>
<td>51.634</td>
<td>-3.501</td>
</tr>
<tr>
<td></td>
<td>(39.689)</td>
<td>(69.925)</td>
<td>(37.204)</td>
</tr>
<tr>
<td>$INT_t$</td>
<td>0.0791*</td>
<td>0.184**</td>
<td>0.146**</td>
</tr>
<tr>
<td></td>
<td>(0.04)</td>
<td>(0.052)</td>
<td>(0.037)</td>
</tr>
<tr>
<td>$PRO_t$</td>
<td>-0.0076</td>
<td>0.013</td>
<td>-0.0039</td>
</tr>
<tr>
<td></td>
<td>(0.012)</td>
<td>(0.022)</td>
<td>(0.0117)</td>
</tr>
<tr>
<td>$EXC$</td>
<td>0.0363</td>
<td>-0.133</td>
<td>-0.013</td>
</tr>
<tr>
<td></td>
<td>(0.094)</td>
<td>(0.167)</td>
<td>(0.0888)</td>
</tr>
<tr>
<td>$DUM$</td>
<td>0.2445</td>
<td>0.083</td>
<td>0.252</td>
</tr>
<tr>
<td></td>
<td>(0.264)</td>
<td>(0.425)</td>
<td>(0.244)</td>
</tr>
<tr>
<td>R-Adj</td>
<td>0.019</td>
<td>0.06</td>
<td>0.06</td>
</tr>
<tr>
<td>Correct Fraction</td>
<td>0.948</td>
<td>.983</td>
<td>0.927</td>
</tr>
<tr>
<td>Schwarz BIC</td>
<td>70.37</td>
<td>35.61</td>
<td>80.8992</td>
</tr>
</tbody>
</table>

Standard errors are in parentheses. *(**) denote 10%(5%) significance level, respectively.

### Table 2: Linear Estimation

<table>
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<th>Variable</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
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</tr>
<tr>
<td></td>
<td>(0.625)</td>
</tr>
<tr>
<td>$INT_t$</td>
<td>-0.463**</td>
</tr>
<tr>
<td></td>
<td>(0.059)</td>
</tr>
<tr>
<td>$IND_t$</td>
<td>-0.098**</td>
</tr>
<tr>
<td></td>
<td>(0.0107)</td>
</tr>
<tr>
<td>$DUM$</td>
<td>-0.055</td>
</tr>
<tr>
<td></td>
<td>(0.327)</td>
</tr>
<tr>
<td>R-Adj</td>
<td>0.438</td>
</tr>
</tbody>
</table>

Standard errors are in parentheses. ** denotes 5% significance level.
Proof of Lemma 1.

If we now assume that the probability distribution of early withdrawers is uniform, the optimal cash to deposit ratio is given by the following fixed point equation:

\[
\frac{1 - \frac{I_t r_t}{R_t}}{\gamma_t + (1 - \gamma_t) \frac{I_t r_t}{R_t}} = \frac{1}{\gamma_t (1 - \gamma_t)} \int_{\pi_t}^{\hat{\gamma}_t} \pi_t d\pi_t.
\]

If we now integrate by parts and rewrite \(\pi_t\) and \(\pi_t\), we obtain the following cubic equation for \(\gamma_t\):

\[
2 \gamma_t^3 \left(1 - \frac{I_t r_t}{R_t}\right) \left(1 + I_t^2 - 2I_t - \frac{I_t r_t}{R_t} - \frac{I_t^3 r_t}{R_t} + 2 \frac{I_t^2 r_t}{R_t}\right) + \gamma_t^2 \left(2 \left(1 - \frac{I_t r_t}{R_t}\right) \left(-2I_t^2 + 2I_t + \frac{I_t r_t}{R_t} + 3 \frac{I_t^3 r_t}{R_t} - 4 I_t^2 r_t}{R_t}\right) - 2I_t \left(1 - \frac{r_t}{R_t}\right) + I_t^2 \left(1 + \frac{r_t}{R_t}\right) \left(1 + \frac{I_t r_t}{R_t}\right)\right) + \gamma_t \left(2 \left(1 - \frac{I_t r_t}{R_t}\right) \left(I_t^2 - 3 \frac{I_t^3 r_t}{R_t} + 2 \frac{I_t^2 r_t}{R_t}\right) - I_t^2 \left(1 - \frac{r_t}{R_t}\right) \left(1 + \frac{r_t}{R_t}\right) + 2 \left(1 - \frac{I_t r_t}{R_t}\right) \frac{I_t^3 r_t}{R_t}\right) = 0
\]

The optimal cash to deposit ratio that satisfies this cubic equation consists of a real and two complex solutions. The only economic relevant solution, the real one, is given by:

\[
\gamma_t = S + T - \frac{a_1}{3}
\]

where \(S=\left(R + (Q^3 + R^2)^{0.5}\right)^{0.5}, Q=(3a_2 - a_1^2)/9, R=(9a_1 a_2 - 27a_3 - 2a_1^3)/54, \)

\[
a_1 = \frac{2 \left(1 - \frac{I_t r_t}{R_t}\right) \left(-2I_t^2 + 2I_t + \frac{I_t r_t}{R_t} + 3 \frac{I_t^3 r_t}{R_t} - 4 I_t^2 r_t}{R_t}\right) - 2I_t \left(1 - \frac{r_t}{R_t}\right) + I_t^2 \left(1 + \frac{r_t}{R_t}\right) \left(1 + \frac{I_t r_t}{R_t}\right)}{2 \left(1 - \frac{I_t r_t}{R_t}\right) \left(1 + I_t^2 - 2I_t - \frac{I_t r_t}{R_t} - \frac{I_t^3 r_t}{R_t} + 2 \frac{I_t^2 r_t}{R_t}\right)},
\]

\[
a_2 = \frac{2 \left(1 - \frac{I_t r_t}{R_t}\right) \left(I_t^2 - 3 \frac{I_t^3 r_t}{R_t} + 2 \frac{I_t^2 r_t}{R_t}\right) - I_t^2 \left(1 - \frac{r_t}{R_t}\right) \left(1 + \frac{r_t}{R_t}\right)}{\left(1 + I_t^2 - 2I_t - \frac{I_t r_t}{R_t} - \frac{I_t^3 r_t}{R_t} + 2 \frac{I_t^2 r_t}{R_t}\right)},
\]

\[
a_3 = \frac{I_t^3 r_t / R_t}{\left(1 + I_t^2 - 2I_t - \frac{I_t r_t}{R_t} - \frac{I_t^3 r_t}{R_t} + 2 \frac{I_t^2 r_t}{R_t}\right)}.
\]

Proof of Proposition 2.

In order to prove the first property of Proposition 2, we note that

\[
\lim_{\gamma_t \to 1} \left[\frac{1}{\gamma_t (1 - \gamma_t)}\right]_{\pi_t}^{\hat{\gamma}_t} \int_{\pi_t}^{\hat{\gamma}_t} F(\pi_t) d\pi_t = I_t \left(1 - \frac{r_t}{R_t}\right).
\]

It is then immediate from \(G(\gamma_t, I_t)=0\) that \(\gamma_t(1) = 1\). In addition, we notice that

\[
\lim_{\gamma_t \to 0} \left[\frac{1}{\gamma_t (1 - \gamma_t)}\right]_{\pi_t}^{\hat{\gamma}_t} \int_{\pi_t}^{\hat{\gamma}_t} F(\pi_t) d\pi_t = 0.
\]
Thus, $G(\gamma_t, I_t)=0$ implies that $\gamma_t(R_t/r_t)=0$ as claimed in the second part of the proposition.

Finally, in order to show that $\gamma'_t(I_t)<0$, we notice that

$$G_1(\gamma_t, I_t)\gamma'_t(I_t) = -G_2(\gamma_t, I_t).$$

Since $G_1(\gamma_t, I_t)<0$ holds at a maximum, the remainder of the proposition then follows if we can show that $G_2(\gamma_t, I_t)<0$. Note that

$$G_2(\gamma_t, I_t) = -\frac{r_t/R_t}{(\gamma_t + (1-\gamma_t)I_t(r_t/R_t))^2} - \frac{1}{\gamma_t(1-\gamma_t)} \left( F(\hat{\pi}_t) \frac{\partial \hat{\pi}_t}{\partial I_t} - F(\pi^*_t) \frac{\partial \pi^*_t}{\partial I_t} \right)$$

where

$$\frac{\partial \hat{\pi}_t}{\partial I_t} = -\frac{(r_t/R_t)\gamma_t(1-\gamma_t)}{(\gamma_t + (1-\gamma_t)I_t(r_t/R_t))^2} < 0$$

$$\frac{\partial \pi^*_t}{\partial I_t} = -\frac{\gamma_t(1-\gamma_t)}{(\gamma_t + (1-\gamma_t)I_t)^2} < 0.$$